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Smooth Solutions of a Nonhomogeneous Iterative Functional Differential Equation with Variable Coefficients

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Using the fixed point theorems of Banach and Schauder, in this paper we discuss the existence, uniqueness, and stability of smooth solutions of a nonhomogeneous iterative functional differential equation with variable coefficients.

Key Words: iterative functional differential equation; smooth solutions; fixed point theorem

Since the publication of Jack Hale's work [1], there have been a lot of monographs and research articles published on functional differential equations. Most of them paid attention to the case in which the deviating arguments depend only on the time itself. But another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are concerned with iterative differential equations of the form

$$x'(t) = H\left(x^{[0]}(t), x^{[1]}(t), \ldots, x^{[m]}(t)\right),$$

where $x^{[0]}(t) = t$, $x^{[1]}(t) = x(t)$, $x^{[k]}(t) = x(x^{[k-1]}(t))$, $k = 2, 3, \ldots, m$. More specifically, Eder [2] considers the functional differential equation

$$x'(t) = x^{[2]}(t)$$

and proves that every solution either vanishes identically or is strictly monotonic. Feckǎn [3] studies the equation

$$x'(t) = f\left(x^{[2]}(t)\right),$$

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where $f \in C^1(R)$, and obtains an existence theorem for solutions satisfying $x(0) = 0$. Wang [4] studies the same equation, but $f\colon R \to R$ is continuous, monotone, $f(0) = 0$, and $|f(z)| \geq \lambda|z|$ for some $\lambda > 0$. Staněk [5] studies the equation

$$x'(t) = x(t) + x^{[2]}(t)$$

and shows that every solution either vanishes identically or is strictly monotonic. In [6], the authors consider the equation

$$x'(t) = x^{[m]}(t)$$

and establish sufficient conditions for the existence of analytic solutions.

In this paper, we will be concerned with a class of nonhomogeneous iterative functional differential equations with variable coefficients of the form

$$x'(t) = \sum_{j=1}^{m} a_j(t) x^{[j]}(t) + F(t), \tag{1}$$

where $a_j(t)$ and $F(t)$ are known real functions. When $m = 1$, Eq. (1) has the unique solution

$$x(t) = e^{\int_{\xi_0}^{t} a_1(s)\, ds} \left(\xi_0 + \int_{\xi_0}^{t} F(s) e^{-\int_{\xi_0}^{s} a_1(r)\, dr}\, ds \right),$$

which satisfies the side condition

$$x(\xi) = \xi_0, \qquad \xi, \xi_0 \in R.$$

Thus when the coefficient functions $a_j(t)$ and the forcing function $F(t)$ are smooth, (1) has a unique smooth solution. We will show that certain features of this statement remain true when m is an arbitrary positive integer. In particular, we will show that a smooth local solution exists that depends continuously on the smooth functions $a_j(t)$ and $F(t)$. The case in which the a_j's are constant was considered in [7].

In this note, a smooth function is taken to mean one that has a number of continuous derivatives and for which the highest continuous derivative is also Lipschitz. As usual, we write $f \in C^n$ if $f, f', \dots, f^{(n)}$ are continuous. The set of all C^n functions, each of which maps a closed interval I into I, will be denoted by $C^n(I, I)$. Explicitly, $C^n(I, I)$ is a Banach space with the norm $\|\cdot\|_n$, where

$$\|x\|_n = \sum_{k=0}^{n} \|x^{(k)}\|, \quad \|x\| = \max_{t \in I} \{|x(t)|\}.$$

For given constants $M_i > 0$ $(i = 1, 2, \dots, n+1)$, let

$$\Omega(M_1, \dots, M_{n+1}; I) = \{ f \in C^n(I, I) \colon |f^{(i)}(t)| \le M_i,\ i = 1, 2, \dots, n; \\ |f^{(n)}(t_1) - f^{(n)}(t_2)| \le M_{n+1}|t_1 - t_2|, t, t_1, t_2 \in I \}.$$

For convenience, we take the following notations:

$$x_{ij}(t) = x^{(i)}(x^{[j]}(t)),$$

and

$$x_{*jk}(t) = (x^{[j]}(t))^{(k)},$$

where i, j, and k are nonnegative integers. Let $I \subset R$ be a closed interval. By induction, we may prove that

$$x_{*jk}(t) = P_{jk}(x_{10}(t), \dots, x_{1,j-1}(t); \dots; x_{k0}(t), \dots, x_{k,j-1}(t)), \quad (2)$$

where P_{jk} is a uniquely defined multivariate polynomial with nonnegative coefficients. The proof can also be found in [7].

It is obvious that there are positive constants N_{uv}^{jk} for $1 \le u \le k$ and $0 \le v \le j-1$, such that the polynomials P_{jk} satisfy the condition that

$$\left| P_{jk}(\bar{\lambda}_{10}, \dots, \bar{\lambda}_{k,j-1}) - P_{jk}(\tilde{\lambda}_{10}, \dots, \tilde{\lambda}_{k,j-1}) \right| \\ \le \sum_{u=1}^{k} \sum_{v=0}^{j-1} N_{uv}^{jk} |\bar{\lambda}_{uv} - \tilde{\lambda}_{uv}| \quad (3)$$

on compact sets $\Lambda_{jk} = [0, 1]^j \times [0, M_2]^j \times \cdots \times [0, M_k]^j$.

To seek a solution $x(t)$ of (1) in $C^n(I, I)$ such that ξ is a fixed point of the function $x(t)$, i.e., $x(\xi) = \xi$, it is natural to seek an interval I of the form $[\xi - \delta, \xi + \delta]$ with $\delta > 0$, and in view of (1), it is also natural to require that $a_j(t)$, $F(t) \in C^{n-1}(I, I)$. There are other natural requirements also. More precisely, let $a_j(t) \in \Omega(L_{j1}, \dots, L_{jn}; I)$ for $j = 1, 2, \dots, m$ and $F(t) \in \Omega(N_1, \dots, N_n; I)$. If $x(t)$ is a solution of (1) on I, then we must have

$$x'(\xi) = \xi \sum_{j=1}^{m} a_j(\xi) + F(\xi),$$

and

$$x^{(k)}(\xi) = \sum_{j=1}^{m} \sum_{s=0}^{k-1} C_{k-1}^{s} a_j^{(k-1-s)}(\xi) \times P_{js}\Big(\overbrace{x'(\xi),\dots,x'(\xi)}^{j\text{ terms}};\dots;\overbrace{x^{(k)}(\xi),\dots,x^{(k)}(\xi)}^{j\text{ terms}}\Big) + F^{(k-1)}(\xi)$$

for $k = 2, 3, \dots, n$. For this reason, we define

$$\Psi(\xi;\eta_0,\dots,\eta_{n-1};N_1,\dots,N_n;I) = \{f \in \Omega(N_1,\dots,N_n;I) : f^{(i)}(\xi) = \eta_i,\ i = 0,1,\dots,n-1\}$$

and

$$X(\xi;\xi_0,\dots,\xi_n;1,M_2,\dots,M_{n+1};I) = \{x \in \Omega(1,M_2,\dots,M_{n+1};I) : x(\xi) = \xi_0 = \xi,\ x^{(i)}(\xi) = \xi_i,\ i = 1,2,\dots,n\}.$$

Moreover, we take the notations

$$\beta_{jk} = P_{jk}\Big(\overbrace{x'(\xi),\dots,x'(\xi)}^{j\text{ terms}};\dots;\overbrace{x^{(k)}(\xi),\dots,x^{(k)}(\xi)}^{j\text{ terms}}\Big)$$

and

$$H_{jk} = P_{jk}\Big(\overbrace{1,\dots,1}^{j\text{ terms}};\overbrace{M_2,\dots,M_2}^{j\text{ terms}};\dots;\overbrace{M_k,\dots,M_k}^{j\text{ terms}}\Big).$$

THEOREM 1. *Let $I = [\xi - \delta, \xi + \delta]$, where ξ and δ satisfy*

$$|\xi| \le \frac{\sqrt{1+4m} - 1}{2m}, \tag{4}$$

and

$$0 < \delta < \frac{\sqrt{1+4m} - 1 - 2m|\xi|}{2m}. \tag{5}$$

Suppose that

$$F \in \Psi(\xi; \eta_0, \dots, \eta_{n-1}; N_1, \dots, N_n; I)$$

and

$$a_j \in \Psi(\xi; \zeta_{j0}, \dots, \zeta_{j,n-1}; L_{j1}, \dots, L_{jn}; I)$$

for $j = 1, 2, \dots, m$. *Then* (1) *has a solution in*

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I),$$

provided the following conditions hold:

(i)

$$\xi_1 = \xi \sum_{j=1}^{m} \zeta_{j0} + \eta_0, \tag{6}$$

$$\xi_k = \sum_{j=1}^{m} \sum_{s=0}^{k-1} C_{k-1}^{s} \zeta_{j,k-1-s} \beta_{js} + \eta_{k-1}, \qquad k = 2, \dots, n, \tag{7}$$

(ii)

$$\sum_{j=1}^{m} \sum_{s=0}^{k-1} C_{k-1}^{s} L_{j,k-1-s} H_{js} + N_{k-1} \le M_k, \qquad k = 2, \dots, n, \tag{8}$$

(iii)

$$\begin{aligned} &\sum_{j=1}^{m} \left(\frac{\sqrt{1+4m} - 1}{2m} L_{jn} + L_{j,n-1} \right) \\ &\quad + \sum_{j=1}^{m} \sum_{s=1}^{n-1} C_{n-1}^{s} \left(L_{j,n-s} H_{js} + \sum_{u=1}^{s} \sum_{v=0}^{j-1} L_{j,n-1-s} N_{uv}^{js} M_{u+1} \right) \\ &\quad + N_n \le M_{n+1}. \end{aligned} \tag{9}$$

Proof. We will employ Schauder's fixed point theorem to accomplish the proof.

First of all, we assert that for all $x(t), y(t)$ in X, the inequalities in (10)–(12) hold:

$$|x^{[j]}(t_1) - x^{[j]}(t_2)| \le |t_1 - t_2|, \quad t_1, t_2 \in I, \qquad j = 0, 1, \dots, m, \tag{10}$$

$$\|x^{[j]} - y^{[j]}\| \le j\|x - y\|, \qquad j = 1, \dots, m, \tag{11}$$

and

$$\|x - y\| \leq \delta^n \|x^{(n)} - y^{(n)}\|. \tag{12}$$

In fact, note first that (10) follows from

$$|x^{[j]}(t_1) - x^{[j]}(t_2)| \leq |x^{[j-1]}(t_1) - x^{[j-1]}(t_2)| \leq \cdots \leq |t_1 - t_2|.$$

Next, it is clear that the inequality in (11) holds for $j = 1$. Assume by induction that the inequality holds for $j = 2, \dots, s$. Then

$$\begin{aligned}
&\|x^{[s+1]} - y^{[s+1]}\| \\
&\quad = \max_{t \in I} |x^{[s+1]}(t) - y^{[s+1]}(t)| \\
&\quad \leq \max_{t \in I} \left\{ |x(x^{[s]}(t)) - x(y^{[s]}(t))| + |x(y^{[s]}(t)) - y(y^{[s]}(t))| \right\} \\
&\quad \leq \|x^{[s]} - y^{[s]}\| + \|x - y\| \leq (s + 1)\|x - y\|,
\end{aligned}$$

as required. Finally, (12) follows from

$$\begin{aligned}
|x(t) - y(t)| &= |(x(t) - y(t)) - (x(\xi) - y(\xi))| \\
&= |x'(\tau_1) - y'(\tau_1)|\,|t - \xi| \\
&= |(x'(\tau_1) - y'(\tau_1)) - (x'(\xi) - y'(\xi))|\,|t - \xi| \\
&= |x''(\tau_2) - y''(\tau_2)|\,|\tau_1 - \xi|\,|t - \xi| = \cdots \\
&= |x^{(n)}(\tau_n) - y^{(n)}(\tau_n)|\,|\tau_{n-1} - \xi| \cdots |\tau_1 - \xi|\,|t - \xi| \\
&\leq \delta^n \|x^{(n)} - y^{(n)}\|,
\end{aligned}$$

where the equalities are obtained from the mean value theorem and the inequality from $|t - \xi|, |\tau_1 - \xi|, \dots, |\tau_{n-1} - \xi| \leq \delta$.

Define an operator T from X into $C^n(I, I)$ by

$$(Tx)(t) = \xi + \sum_{j=1}^{m} \int_{\xi}^{t} a_j(s) x^{[j]}(s)\, ds + \int_{\xi}^{t} F(s)\, ds, \qquad x \in X. \tag{13}$$

We will prove that for any $x \in X$, $Tx \in X$.

To see this, note that

$$\begin{aligned}|(Tx)(t)-\xi| &\le \sum_{j=1}^{m}\left|\int_{\xi}^{t}|a_j(s)|\,|x^{[j]}(s)|\,ds\right|+\left|\int_{\xi}^{t}|F(s)|\,ds\right| \\ &\le \sum_{j=1}^{m}(|\xi|+\delta)^2|t-\xi|+(|\xi|+\delta)|t-\xi| \\ &= \big(m(|\xi|+\delta)^2+(|\xi|+\delta)\big)|t-\xi|<\delta,\end{aligned}$$

where the second inequality is from $x(I)\subseteq I$, $a_j(I)\subset I(j=1,\dots,m)$ and $F(I)\subseteq I$, and the third is from (5). Thus $(Tx)(I)\subseteq I$.

From (13), it is easy to see that

$$(Tx)'(t)=\sum_{j=1}^{m}a_j(t)x^{[j]}(t)+F(t),$$

and for $k=2,\dots,n$,

$$\begin{aligned}(Tx)^{(k)}(t) &= \sum_{j=1}^{m}\left(a_j(t)x^{[j]}(t)\right)^{(k-1)}+F^{(k-1)}(t) \\ &= \sum_{j=1}^{m}\sum_{s=0}^{k-1}C_{k-1}^{s}a_j^{(k-1-s)}(t)\left(x^{[j]}(t)\right)^{(s)}+F^{(k-1)}(t).\end{aligned}$$

Next, note that $(Tx)(\xi)=\xi$. Furthermore, by (6) and (7), we have

$$(Tx)'(\xi)=\sum_{j=1}^{m}a_j(\xi)x^{[j]}(\xi)+F(\xi)=\xi\sum_{j=1}^{m}\zeta_{j0}+\eta_0=\xi_1,$$

and

$$\begin{aligned}(Tx)^{(k)}(\xi) &= \sum_{j=1}^{m}\sum_{s=0}^{k-1}C_{k-1}^{s}a_j^{(k-1-s)}(\xi)x_{*js}(\xi)+F^{(k-1)}(\xi) \\ &= \sum_{j=1}^{m}\sum_{s=0}^{k-1}C_{k-1}^{s}\zeta_{j,k-1-s}\beta_{js}+\eta_{k-1}=\xi_k,\ k=2,\dots,n,\end{aligned}$$

respectively. Thus $(Tx)^{(k)}(\xi)=\xi_k$ for $k=0,1,\dots,n$.

Next, we have

$$\begin{aligned}|(Tx)'(t)| &\le \sum_{j=1}^{m}|a_j(t)x^{[j]}(t)|+|F(t)| \\ &\le m(|\xi|+\delta)^2+(|\xi|+\delta)<1=M_1,\end{aligned}$$

where the last inequality follows from (5). In view of (8), we also have

$$
\begin{aligned}
|(Tx)^{(k)}(t)| &\leq \sum_{j=1}^{m}\sum_{s=0}^{k-1} C_{k-1}^{s}|a_j^{(k-1-s)}(t)|\,|\big(x^{[j]}(t)\big)^{(s)}| + |F^{(k-1)}(t)| \\
&\leq \sum_{j=1}^{m}\sum_{s=0}^{k-1} C_{k-1}^{s} L_{j,k-1-s} H_{js} + N_{k-1} \leq M_k \qquad k = 2,\dots,n.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&|(Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2)| \\
&\quad\leq \sum_{j=1}^{m}\sum_{s=0}^{n-1} C_{n-1}^{s}|a_j^{(n-1-s)}(t_1)\big(x^{[j]}(t_1)\big)^{(s)} - a_j^{(n-1-s)}(t_2)\big(x^{[j]}(t_2)\big)^{(s)}| \\
&\qquad +|F^{(n-1)}(t_1) - F^{(n-1)}(t_2)| \\
&\quad\leq \sum_{j=1}^{m}\Big\{|a_j^{(n-1)}(t_1) - a_j^{(n-1)}(t_2)|\,|x^{[j]}(t_1)| \\
&\qquad\qquad +|a_j^{(n-1)}(t_2)|\,|x^{[j]}(t_1) - x^{[j]}(t_2)|\Big\} \\
&\qquad + \sum_{j=1}^{m}\sum_{s=1}^{n-1} C_{n-1}^{s}\Big\{|a_j^{(n-1-s)}(t_1) - a_j^{(n-1-s)}(t_2)| \\
&\qquad\qquad \times|\big(x^{[j]}(t_1)\big)^{(s)}|\,|a_j^{(n-1-s)}(t_2)| \\
&\qquad\qquad \times|P_{js}\big(x_{10}(t_1),\dots,x_{s,j-1}(t_1)\big) \\
&\qquad\qquad -P_{js}\big(x_{10}(t_2),\dots,x_{s,j-1}(t_2)\big)|\Big\} + N_n|t_1 - t_2| \\
&\quad\leq \sum_{j=1}^{m}\big(L_{jn}(|\xi| + \delta) + L_{j,n-1}\big)|t_1 - t_2| \\
&\qquad + \sum_{j=1}^{m}\sum_{s=1}^{n-1} C_{n-1}^{s}\Bigg\{L_{j,n-s}H_{js}|t_1 - t_2| \\
&\qquad\qquad +L_{j,n-1-s}\sum_{u=1}^{s}\sum_{v=0}^{j-1} N_{uv}^{js}|x_{uv}(t_1) - x_{uv}(t_2)|\Bigg\} \\
&\qquad +N_n|t_1 - t_2|.
\end{aligned}
$$

Since

$$
|x_{uv}(t_1) - x_{uv}(t_2)| \leq M_{u+1}|x^{[v]}(t_1) - x^{[v]}(t_2)| \leq M_{u+1}|t_1 - t_2|
$$

holds by means of the mean value theorem and (10), we see that

$$
\begin{aligned}
&|(Tx)^{(n)}(t_1) - (Tx)^{(n)}(t_2)| \\
&\quad \le \left(\sum_{j=1}^{m} \left(\frac{\sqrt{1+4m}-1}{2m} L_{jn} + L_{j,n-1} \right) + \sum_{j=1}^{m} \sum_{s=1}^{n-1} C_{n-1}^{s} \right. \\
&\qquad \left. \times \left(L_{j,n-s} H_{js} + \sum_{u=1}^{s} \sum_{v=0}^{j-1} L_{j,n-1-s} N_{uv}^{js} M_{u+1} \right) + N_n \right) |t_1 - t_2| \\
&\quad \le M_{n+1} |t_1 - t_2|.
\end{aligned}
$$

Therefore, T is an operator from X into itself.

Now we will show that T is continuous. Let $x, y \in X$; then

$$
\begin{aligned}
&\|Tx - Ty\|_n \\
&\quad = \|Tx - Ty\| + \|(Tx)' - (Ty)'\| + \sum_{k=2}^{n} \|(Tx)^{(k)} - (Ty)^{(k)}\| \\
&\quad = \max_{t \in I} \left| \sum_{j=1}^{m} \int_{\xi}^{t} a_j(s) \left(x^{[j]}(s) - y^{[j]}(s) \right) ds \right| \\
&\qquad + \max_{t \in I} \left| \sum_{j=1}^{m} a_j(t) \left(x^{[j]}(t) - y^{[j]}(t) \right) \right| \\
&\qquad + \sum_{k=2}^{m} \max_{t \in I} \left| \sum_{j=1}^{m} \left\{ a_j^{(k-1)}(t) \left(x^{[j]}(t) - y^{[j]}(t) \right) \right. \right. \\
&\qquad\qquad + \sum_{s=1}^{k-1} C_{k-1}^{s} a_j^{(k-1-s)}(t) \left[P_{js}\left(x_{10}(t), \dots, x_{s,j-1}(t) \right) \right. \\
&\qquad\qquad \left. \left. \left. - P_{js}\left(y_{10}(t), \dots, y_{s,j-1}(t) \right) \right] \right\} \right| \\
&\quad \le \delta(|\xi| + \delta) \sum_{j=1}^{m} \|x^{[j]} - y^{[j]}\| + (|\xi| + \delta) \sum_{j=1}^{m} \|x^{[j]} - y^{[j]}\| \\
&\qquad + \sum_{k=2}^{m} \sum_{j=1}^{m} \left\{ L_{j,k-1} \|x^{[j]} - y^{[j]}\| \right. \\
&\qquad\qquad \left. + \sum_{s=1}^{k-1} C_{k-1}^{s} L_{j,k-1-s} \sum_{u=1}^{s} \sum_{v=0}^{j-1} N_{uv}^{js} \max_{t \in I} |x_{uv}(t) - y_{uv}(t)| \right\}
\end{aligned}
$$

$$\leq \delta(|\xi| + \delta) \sum_{j=1}^{m} j\|x - y\| + (|\xi| + \delta) \sum_{j=1}^{m} j\delta^n \|x^{(n)} - y^{(n)}\|$$

$$+ \sum_{k=2}^{n} \sum_{j=1}^{m} jL_{j,k-1}\|x - y\|$$

$$+ \sum_{k=2}^{m} \sum_{j=1}^{m} \sum_{s=1}^{k-1} \sum_{u=1}^{s} \sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js} \max_{t\in I} |x_{uv}(t) - y_{uv}(t)|,$$

where we have made use of (11) and (12) in obtaining the last inequality. Since the mean value theorem and (11) imply

$$\begin{aligned} &|x_{uv}(t) - y_{uv}(t)| \\ &\quad\leq |x^{(u)}(x^{[v]}(t)) - y^{(u)}(x^{[v]}(t))| + |y^{(u)}(x^{[v]}(t)) - y^{(u)}(y^{[v]}(t))| \\ &\quad\leq \|x^{(u)} - y^{(u)}\| + M_{u+1}\|x^{[v]} - y^{[v]}\| \\ &\quad\leq \|x^{(u)} - y^{(u)}\| + M_{u+1} v\|x - y\|, \end{aligned}$$

and since

$$\sum_{k=2}^{n} \sum_{j=1}^{m} \sum_{s=1}^{k-1} \sum_{u=1}^{s} \sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js} = \sum_{u=1}^{n-1} \sum_{s=u}^{n-1} \sum_{k=s+1}^{n} \sum_{j=1}^{m} \sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js},$$

we see that

$$\begin{aligned} &\|Tx - Ty\|_n \\ &\quad\leq \Bigg\{ \frac{(1+m)\sqrt{1+4m} - 1}{4} \delta \\ &\qquad + \sum_{k=2}^{n} \sum_{j=1}^{m} \Bigg(jL_{j,k-1} + \sum_{s=1}^{k-1} \sum_{u=1}^{s} \sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js} M_{u+1} v \Bigg) \Bigg\} \|x - y\| \\ &\qquad + \sum_{u=1}^{n-1} \Bigg\{ \sum_{s=u}^{n-1} \sum_{k=s+1}^{n} \sum_{j=1}^{m} \sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js} \Bigg\} \|x^{(u)} - y^{(u)}\| \\ &\qquad + \Bigg\{ \frac{(1+m)\sqrt{1+4m} - 1}{4} \delta^n \Bigg\} \|x^{(n)} - y^{(n)}\| \\ &\quad\leq \Gamma\|x - y\|_n. \end{aligned} \tag{14}$$

where

$$\Gamma = \max\Bigg\{ \frac{(1+m)\sqrt{1+4m} - 1}{4}\delta + \sum_{k=2}^{n}\sum_{j=1}^{m}\left(jL_{j,k-1} + \sum_{s=1}^{k-1}\sum_{u=1}^{s}\sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js} M_{u+1} v \right); \max_{1 \le u \le n-1}\left\{ \sum_{s=u}^{n-1}\sum_{k=s+1}^{n}\sum_{j=1}^{m}\sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} N_{uv}^{js} \right\} \Bigg\}. \tag{15}$$

This completes the proof of the fact that T is continuous.

It is easy to see that X is closed and convex. We now show that X is a relatively compact subset of $C^n(I, I)$. For any $x = x(t)$ in X,

$$\|x\|_n \le \|x\| + \sum_{k=1}^{n} \|x^{(k)}\| \le |\xi| + \delta + 1 + \sum_{k=2}^{n} M_k.$$

Hence X is bounded in $C^n(I, I)$. Next, for any $x = x(t)$ in X and any $t_1, t_2 \in I$, we have

$$|x(t_1) - x(t_2)| \le |t_1 - t_2|.$$

This shows that X is equicontinuous on I. By means of the Arzela–Ascoli theorem, we see that X is relatively compact in $C^n(I, I)$.

By Schauder's fixed point theorem, we conclude that

$$g(t) = \xi + \sum_{j=1}^{m} \int_{\xi}^{t} a_j(s) g^{[j]}(s)\, ds + \int_{\xi}^{t} F(s)\, ds$$

for some $g(t)$ in X. By differentiating both sides of the above equality, we see that g is the desired solution of (1). This completes the proof.

THEOREM 2. *Let* $I = [\xi - \delta, \xi + \delta]$, *where* ξ *and* δ *satisfy* (4) *and* (5), *respectively. Let*

$$F \in \Psi(\xi; \eta_0, \dots, \eta_{n-1}; N_1, \dots, N_n; I)$$

and

$$a_j \in \Psi(\xi; \zeta_{j0}, \dots, \zeta_{j,n-1}; L_{j1}, \dots, L_{jn}; I)$$

for $j = 1, 2, \dots, m$. *Then* (1) *has a unique solution in*

$$X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I),$$

provided the conditions (7)–(10) *hold and* $\Gamma < 1$ *in* (15).

Proof. Since $\Gamma < 1$, we see that T defined by (13) is contraction mapping on the close subset X of $C^n(I, I)$. Thus the fixed point g in the proof of Theorem 1 must be unique. This completes the proof.

THEOREM 3. *The unique solution obtained in Theorem* 2 *depends continuously on the given functions* F *and* a_j $(j = 1, 2, \dots, m)$.

Proof. Under the assumptions of Theorem 2, if $G = G(t)$ and $H = H(t)$ are any two functions in $\Psi(\xi; \eta_0, \dots, \eta_{n-1}; N_1, \dots, N_n; I)$; $a_j(t)$ and $b_j(t)$ are any functions in $\Psi(\xi; \zeta_{j0}, \dots, \zeta_{j,n-1}; L_{j1}, \dots, L_{jn}; I)$ for $j = 1, 2, \dots, m$, then there are two unique corresponding functions $g = g(t)$ and $h = h(t)$ in $X(\xi; \xi_0, \dots, \xi_n; 1, M_2, \dots, M_{n+1}; I)$ such that

$$g(t) = \xi + \sum_{j=1}^{m} \int_{\xi}^{t} a_j(s) g^{[j]}(s)\, ds + \int_{\xi}^{t} G(s)\, ds$$

and

$$h(t) = \xi + \sum_{j=1}^{m} \int_{\xi}^{t} b_j(s) h^{[j]}(s)\, ds + \int_{\xi}^{t} H(s)\, ds.$$

We have

$$\begin{aligned}
\|g - h\|_n &= \|g - h\| + \|g' - h'\| + \sum_{k=2}^{n} \|g^{(k)} - h^{(k)}\| \\
&\le \max_{t \in I} \left\{ \sum_{j=1}^{m} \left| \int_{\xi}^{t} |a_j(s) g^{[j]}(s) - b_j(s) h^{[j]}(s)| ds \right| \right\} \\
&\quad + \max_{t \in I} \left\{ \sum_{j=1}^{m} |a_j(t) g^{[j]}(t) - b_j(t) h^{[j]}(t)| \right\} \\
&\quad + \sum_{k=2}^{m} \max_{t \in I} \left\{ \sum_{j=1}^{m} \left[|a_j^{(k-1)}(t) g^{[j]}(t) - b_j^{(k-1)}(t) h^{[j]}(t)| \right.\right. \\
&\qquad + \sum_{s=1}^{k-1} C_{k-1}^{s} |a_j^{(k-1-s)}(t) P_{js}\big(g_{10}(t), \dots, g_{s,j-1}(t)\big) \\
&\qquad \left.\left. - b_j^{(k-1-s)}(t) P_{js}\big(h_{10}(t), \dots, h_{s,j-1}(t)\big)| \right] \right\} \\
&\quad + \max_{t \in I} \left| \int_{\xi}^{t} |G(s) - H(s)|\, ds \right| + \max_{t \in I} |G(t) - H(t)| \\
&\quad + \sum_{k=2}^{n} \max_{t \in I} |G^{(k-1)}(t) - h^{(k-1)}(t)|
\end{aligned}$$

$$
\begin{aligned}
\le \max_{t\in I}\Bigg\{ \sum_{j=1}^{m} \bigg| \int_{\xi}^{t} \Big[|a_j(s) - b_j(s)|\,|g^{[j]}(s)| \\
+ |b_j(s)|\,|g^{[j]}(s) - h^{[j]}(s)| \Big]\, ds \bigg| \Bigg\} \\
+ \max_{t\in I}\Bigg\{ \sum_{j=1}^{m} \Big[|a_j(t) - b_j(t)|\,|g^{[j]}(t)| \\
+ |b_j(t)|\,|g^{[j]}(t) - h^{[j]}(t)| \Big] \Bigg\} \\
+ \sum_{k=2}^{n} \max_{t\in I}\Bigg\{ \sum_{j=1}^{m} \Big[|a_j^{(k-1)}(t) - b_j^{(k-1)}(t)|\,|g^{[j]}(t)\big)| \\
+ |b_j^{(k-1)}(t)|\,|g^{[j]}(t) - h^{[j]}(t)| \Big] \\
+ \sum_{j=1}^{m} \sum_{s=1}^{k-1} C_{k-1}^{s} \Big[|a_j^{(k-1-s)}(t) - b_j^{(k-1-s)}(t)| \\
\times |P_{js}\big(g_{10}(t), \ldots, g_{s,j-1}(t)\big)| \\
+ |b_j^{(n-1-s)}(t)|\,|P_{js}\big(g_{10}(t), \ldots, g_{s,j-1}(t)\big) \\
- P_{js}\big(h_{10}(t), \ldots, h_{s,j-1}(t)\big)| \Big] \Bigg\} \\
+ \delta\|G - H\| + \|G - H\| + \sum_{k=2}^{n} \|G^{(k-1)} - H^{(k-1)}\| \\
\le \sum_{j=1}^{m} \delta N_j \|a_j - b_j\| + \sum_{j=1}^{m} N_j \|a_j - b_j\| \\
+ \sum_{k=2}^{n} \sum_{j=1}^{m} \Bigg[N_j \|a_j^{(k-1)} - b_j^{(k-1)}\| \\
+ \sum_{s=1}^{k-1} C_{k-1}^{s} H_{js} \|a_j^{(k-1-s)} - b_j^{(k-1-s)}\| \Bigg] \\
+ \sum_{j=1}^{m} \delta(|\xi| + \delta) j \|g - h\| + \sum_{j=1}^{m} (|\xi| + \delta) j \delta^{n} \|g^{(n)} - h^{(n)}\| \\
+ \sum_{k=2}^{n} \sum_{j=1}^{m} j L_{j,k-1} \|g - h\|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=2}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k-1}\sum_{u=1}^{s}\sum_{v=0}^{j-1} C_{k-1}^{s} L_{j,k-1-s} \\
&\times N_{uv}^{js}\big(\|g^{(u)}-h^{(u)}\| + M_{u+1} v \|g-h\|\big) \\
&+(\delta+1)\|G-H\|_{n-1} \\
\le{}& \left\{\sum_{j=1}^{m}\left[N_j(\delta+1) + \sum_{s=1}^{n-1}\sum_{k=s+1}^{n} C_{k-1}^{s} H_{js}\right]\right\}\|a_j-b_j\| \\
&+\sum_{i=1}^{n-1}\left\{\sum_{j=1}^{m}\left[N_j + \sum_{s=1}^{n-1-i}\sum_{k=s+1}^{n} C_{k-1}^{s} H_{js}\right]\right\}\|a_j^{[i]}-b_j^{[i]}\| \\
&+\Gamma\|g-h\|_n + (\delta+1)\|G-H\|_{n-1} \\
\le{}& \sum_{j=1}^{m}\Gamma_j\|a_j-b_j\|_{n-1} + \Gamma\|g-h\|_n + (\delta+1)\|G-H\|_{n-1},
\end{aligned}
$$

where Γ is defined in (15) by means of the same reasoning used in obtaining (14), and

$$
\Gamma_j = \max\left\{N_j(\delta+1) + \sum_{s=1}^{n-1}\sum_{k=s+1}^{n} C_{k-1}^{s} H_{js};\ \max_{1\le i\le n-1}\left\{N_j + \sum_{s=1}^{n-1-i}\sum_{k=s+1}^{n} C_{k-1}^{s} H_{js}\right\}\right\}
$$

for $j = 1, 2, \ldots, m$. Thus we have

$$
\|g-h\|_n \le \sum_{j=1}^{m}\frac{\Gamma_j}{1-\Gamma}\|a_j-b_j\|_{n-1} + \frac{\delta+1}{1-\Gamma}\|G-H\|_{n-1}.
$$

We may now conclude that the solution of (1) depends continuously on the functions F and $a_j (j = 1, 2, \ldots, m)$. This completes the proof.

We now show that the conditions in Theorem 1 do not self-contradict, by means of an example. Consider the following equation:

$$
x'(t) = a_1(t)x(t) + a_2(t)x(x(t)) + F(t), \tag{16}
$$

where $F(t) = (t-\xi)^{14/3} + \xi$, $a_1(t) = (t-\xi)^5 + \xi$, $a_2(t) = (t-\xi)^{24/5} + \xi$, and $|\xi| < 1/2$. Obviously, (4) is satisfied. We pick δ such that

$$
0 < \delta < \tfrac{1}{2} - |\xi|;
$$

then (5) is also satisfied. The numbers ξ and δ now define the interval $I = [\xi - \delta, \xi + \delta]$. Note that

$$F'(t) = \frac{14}{3}(t-\xi)^{11/3}, \quad F''(t) = \frac{154}{9}(t-\xi)^{8/3},$$

$$F'''(t) = \frac{1332}{27}(t-\xi)^{5/3}, \quad F^{(4)}(t) = \frac{6160}{81}(t-\xi)^{2/3},$$

$$a_1'(t) = 5(t-\xi)^4, \quad a_1''(t) = 20(t-\xi)^3,$$

$$a_1'''(t) = 60(t-\xi)^2, \quad a_1^{(4)}(t) = 120(t-\xi),$$

$$a_2'(t) = \frac{24}{5}(t-\xi)^{19/5}, \quad a_2''(t) = \frac{456}{25}(t-\xi)^{14/5},$$

$$a_2'''(t) = \frac{6384}{125}(t-\xi)^{9/5}, \quad a_2^{(4)}(t) = \frac{57456}{625}(t-\xi)^{4/5},$$

thus

$$\begin{aligned} F &\in \Psi(\xi; \eta_0, \dots, \eta_3; N_1, \dots, N_4; I) \\ &= \Psi\left(\xi; \xi, 0, 0, 0; \frac{14}{3}\delta^{11/3}, \frac{154}{9}\delta^{8/3}, \frac{1332}{27}\delta^{5/3}, \frac{6160}{81}\delta^{2/3}; I\right). \\ a_1 &\in \Psi(\xi; \zeta_{10}, \dots, \zeta_{13}; L_{11}, \dots, L_{14}; I) \\ &= \Psi(\xi; \xi, 0, 0, 0; 5\delta^4, 20\delta^3, 60\delta^2, 120\delta; I) \end{aligned}$$

and

$$\begin{aligned} a_2 &\in \Psi(\xi; \zeta_{20}, \dots, \zeta_{23}; L_{21}, \dots, L_{24}; I) \\ &= \Psi\left(\xi; \xi, 0, 0, 0; \frac{24}{5}\delta^{19/5}, \frac{456}{25}\delta^{14/5}, \frac{6384}{125}\delta^{9/5}, \frac{57456}{625}\delta^{4/5}; I\right). \end{aligned}$$

Now take $\xi_0 = \xi$,

$$\begin{aligned} \xi_1 &= (2\xi + 1)\xi, \\ \xi_2 &= (\xi_1 + 1)\xi\xi_1, \\ \xi_3 &= \left(\xi_1^2 + \xi_1 + 1\right)\xi\xi_2, \\ \xi_4 &= \left(\xi_3 + \xi_3\xi_1^3 + 3\xi_2^2\xi_1 + \xi_4\xi_1\right)\xi. \end{aligned}$$

Then (6) and (7) are satisfied. Furthermore, if we take

$$M_1 = 1, \quad M_2 = 1 + \tfrac{1}{2}(L_{11} + L_{21}) + N_1,$$
$$M_3 = 2L_{11} + \tfrac{1}{2}L_{12} + 2L_{21} + \tfrac{1}{2}L_{22} + \tfrac{3}{2}M_2 + N_2,$$
$$M_4 = 3L_{12} + \tfrac{1}{2}L_{13} + 3L_{22} + \tfrac{1}{2}L_{23} + 3(L_{11} + 2L_{21})M_2 + \tfrac{3}{2}M_2^2 + \tfrac{3}{2}M_3 + N_3,$$

then $M_1, \dots, M_4$ are positive, and (8) is satisfied. Finally, if we take

$$M_5 = \sum_{j=1}^{2}\left(\frac{1}{2}L_{j4} + L_{j3}\right) + \sum_{j=1}^{2}\sum_{s=1}^{3} C_3^s\left(L_{j,4-s}H_{js} + \sum_{u=1}^{s}\sum_{v=0}^{j-1} L_{j,3-s}N_{uv}^{js}M_{u+1}\right) + N_4,$$

then $M_5 > 0$ and (9) is satisfied.

We have thus shown that when $\xi_0, \dots, \xi_4$ and $M_1, \dots, M_5$ are defined as above, then there will be a solution of (1) in $X(\xi; \xi_0, \dots, \xi_4; 1, M_2, \dots, M_5; I)$.

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