Banach spaces which are somewhat uniformly noncreasy

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Abstract

We consider a family of spaces wider than r-UNC spaces and we give some fixed point results in the setting of these spaces.

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1. Introduction

Let \((X, \| \cdot \|)\) be a Banach space. Let \(C\) be a nonempty subset of \(X\). A mapping \(T : C \to C\) is said to be nonexpansive whenever \(\|Tx - Ty\| \leq \|x - y\|\) for all \(x, y \in C\).

We say that a Banach space \(X\) has the fixed point property (FPP in short) if every nonexpansive self mapping \(T\) on any nonempty bounded, closed, convex subset \(C \subset X\) has a fixed point. Since 1965, Browder [1], Göhde [5], Kirk [9], and other authors have established that, under various conditions of a geometric kind on the norm of \(X\), the FPP is guaranteed.

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A classical result by Turett [14] states that if the characteristic of convexity of $X$ is less than 1 (i.e., $\epsilon_0(X) < 1$), then both $X$ and $X^*$ are super-reflexive and have the FPP. In this sense, in [2,7] the classical coefficient $\epsilon_0(X)$ was generalized to $\tilde{\epsilon}_0^k(X)$ (see definition below), where $\tilde{\epsilon}_0^k(X)$ is an extension to higher finite dimensions of that coefficient and it was shown that if $\tilde{\epsilon}_0^k(X) < 1$, then $X$ has the FPP. However, it remains unknown whether the FPP holds for super-reflexive Banach spaces, in particular for every uniformly nonsquare Banach space (i.e., $\epsilon_0(X) < 2$).

On the other hand, in 1996 Prus [12,13] introduced the uniformly noncreasy (UNC) Banach spaces, and showed that they are super-reflexive and they have the FPP. The property UNC is an ingenious three-dimensional generalization of uniform convexity and uniform smoothness properties, which does not imply normal structure.

Recently, in [3] the authors introduced the class of $r$-UNC Banach spaces (see definition below) and they showed that these Banach spaces are super-reflexive and moreover, when $r \leq 1$, they also enjoy the FPP. $r$-UNC Banach spaces include uniformly nonsquare Banach spaces and also Banach spaces with $\epsilon_0(X) < 1$; uniformly noncreasy Banach spaces are $r$-UNC Banach spaces with $r \leq 2$.

In this paper we study a class of Banach spaces which is larger than both $r$-UNC and $\tilde{\epsilon}_0^k(X) < 2$ and we establish some fixed point results for this class of spaces.

2. Preliminaries

Let $X$ be a Banach space. By $X^*$, $B_X$, and $S_X$ we denote the dual space, the unit ball, and the unit sphere of $X$, respectively, and diam $A$ denotes the diameter of a bounded set $A \subset X$.

In [2] the authors gave the following definition: Let $(X, \| \cdot \|)$ be a Banach space and $k \in \mathbb{N}$. Denote by $s_k(X)$ the supremum of the set of numbers $\epsilon \in [0, 2]$ for which there exist points $x_1, \ldots, x_{k+1}$ in $B_X$ with $\min\{\|x_i - x_j\| : i \neq j\} \geq \epsilon$.

Define the function $\tilde{\delta}^k : [0, s_k(X)) \to [0, 1]$ by

$$\tilde{\delta}^k(\epsilon) = \inf\left\{ 1 - \frac{k+1}{\sum_{i=1}^{k+1} x_i} : x_i \in B_X, i = 1, \ldots, k+1, \min_{i \neq j} \|x_i - x_j\| \geq \epsilon \right\}$$

and let $\tilde{\epsilon}_0^k(X)$ be the number $\tilde{\epsilon}_0^k(X) = \sup\{\epsilon \in [0, s_k(X)) : \tilde{\delta}^k(\epsilon) = 0\}$.

Given two functionals $x^*, y^* \in S_X^*$, and a scalar $\delta \in [0, 1]$, we put $S(x^*, \delta) := \{x \in B_X : x^*(x) \geq 1 - \delta\}$ and

$$S(x^*, y^*, \delta) := S(x^*, \delta) \cap S(y^*, \delta).$$

Let $r \in (0, 2]$. Following [3], we say that a Banach space $X$ is $r$-uniformly noncreasy ($r$-UNC in short) provided that there exist $\epsilon \in (0, r)$ and $\delta > 0$ such that if $x^*, y^* \in S_X^*$ and $\|x^* - y^*\| \geq \epsilon$, then diam $S(x^*, y^*, \delta) \leq \epsilon$.

Let $\mathcal{U}$ be a free ultrafilter on the set of natural numbers.

Consider the closed linear subspace of $\ell_\infty(X)$,

$$\mathcal{N} = \left\{ (x_n) \in \ell_\infty(X) : \lim_{n \to \mathcal{U}} \|x_n\| = 0 \right\}.$$
The ultrapower \( \tilde{X} \) of the space \( X \) is defined as the quotient space \( \ell_\infty(X)/N \). Given an element \( x = (x_n) \in \ell_\infty(X) \), \( \tilde{x} \) stands for the equivalence class of \( x \). The quotient norm in \( \tilde{X} \) verifies \( \|\tilde{x}\| = \lim_{n \to U} \|x_n\| \).

If \( f = (x_n^*) \) is a bounded sequence of functionals in \( X^* \), the expression \( \tilde{f}(\tilde{x}) = \lim_{n \to U} x_n^*(x_n) \) for \( x = (x_n) \in \ell_\infty(X) \) defines an element in the dual space of \( \tilde{X} \) with \( \|\tilde{f}\| = \lim_{n \to U} \|x_n^*\| \). (For more details about the construction of an ultrapower of a Banach space \( X \) see, for example, [10].)

Suppose that \( C \) is a weakly compact convex subset of a Banach space \( X \), and \( T : C \to C \) is a nonexpansive mapping. The set \( C \) contains a weakly compact convex subset \( K \) which is minimal for \( T \). That means that \( T(K) \) is contained in \( K \) and no strictly smaller weakly compact convex subset of \( K \) is invariant under \( T \). If \( K \) contains only one point then \( T \) has a fixed point. Otherwise we can assume that \( \text{diam}(K) > 0 \). It is easy to see that \( K \) contains a sequence \( (x_n) \) with \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \) (such a sequence is called an approximate fixed point sequence (afps) for \( T \)).

A well-known property of minimal sets is the following Goebel–Karlovitz lemma (see [4,8]).

**Lemma 1** (GK). Let \( K \) be a minimal weakly compact convex subset for a nonexpansive mapping \( T \) and let \( (x_n) \) be an afps sequence for \( T \) in \( K \). Then for all \( x \in K \)
\[
\lim_{n \to \infty} \|x_n - x\| = \text{diam}(K).
\]

The following result was proved by Maurey in [11].

**Theorem 2** (Maurey). Let \( \delta \in (0, 1) \). Let \( K \) be a minimal weakly compact convex set for a nonexpansive mapping \( T \) which does not have a fixed point. If \( x = (x_n), y = (y_n) \) are afps for \( T \) in \( K \), then there exists an afps \( w = (w_n) \) in \( K \) such that
\[
\|\tilde{w} - \tilde{x}\| = \delta \|\tilde{x} - \tilde{y}\| \text{ and } \|\tilde{w} - \tilde{y}\| = (1 - \delta) \|\tilde{x} - \tilde{y}\|.
\]

3. SUNC Banach spaces. Examples

In order to be able to define the concept we are interested in, we need the following generalization of the diameter of a set.

**Definition 3.** Let \( X \) be a Banach space. Let \( A \) be a bounded subset of \( X \). For every \( k \in \mathbb{N} \) we define
\[
\beta^k(A) := \sup \{ r : \exists x_1, x_2, \ldots, x_{k+1} \in A \text{ with } \|x_i - x_j\| \geq r \text{ for } i \neq j \}.
\]

We propose the following weakening of the notion \( r \)-UNC.

**Definition 4.** Let \( k \in \mathbb{N} \) and \( r \in (0, 2] \). A Banach space \( X \) is \( (r, k) \)-SUNC in short, if there exist \( \epsilon \in (0, r) \) and \( \delta > 0 \) such that if \( x^*, y^* \in S_{X^*} \) and \( \|x^* - y^*\| \geq \epsilon \), then \( \beta^k(S(x^*, y^*, \delta)) \leq \epsilon \).
Definition 5. A Banach space $X$ is said to be somewhat uniformly noncreasy (SUNC in short), if there exist $k \in \mathbb{N}$ and $r \in (0, 2]$ such that $X$ is $(r, k)$-SUNC.

Clearly, if $A$ is a bounded subset of a Banach space $X$, then $\beta^k(A) \leq \text{diam}(A)$ and thus if a Banach space $X$ is $r$-UNC, then the space also is $(r, k)$-SUNC. In fact, we may notice that $(r, 1)$-SUNC is exactly the same as $r$-UNC. Also, if $A$ is a bounded convex subset of $X$, then $\text{diam}(A) \leq k \beta^k(A)$; so if $X$ is $(r, k)$-SUNC, we have that $X$ is $rk$-UNC.

It is also clear from the definition that, if $X$ is $(r, k)$-SUNC and $s \geq r$, then $X$ is $(s, k)$-SUNC.

In order to measure the degree of SUNC-ness of $X$, we define the following modulus (see [3] to find a similar modulus concerning $r$-UNC Banach spaces).

Definition 6. Given $k \in \mathbb{N}$, for any $\epsilon \in [0, S_k(X))$ we define

$$\delta^k(X, \epsilon) := \inf \left\{ 1 - \frac{(x^* + y^*)}{2} \left( \sum_{i=1}^{k+1} \frac{x_i}{k+1} \right) : x_i \in B_X, \quad i = 1, \ldots, k+1, \quad \min_{i \neq j} \|x_i - x_j\| \geq \epsilon, \quad x^*, y^* \in S_X^*, \quad \|x^* - y^*\| \geq \epsilon \right\}$$

and

$$\epsilon_{k,SUNC}(X) := \sup \{ \epsilon \in [0, S_k(X)) : \delta^k(X, \epsilon) = 0 \}.$$

Theorem 7. Let $X$ be a Banach space, $k \in \mathbb{N}$, and $r \in (0, 2]$. $X$ is $(r, k)$-SUNC if and only if $\epsilon_{k,SUNC}(X) < r$.

Proof. Observe first that $\epsilon_{k,SUNC}(X) < r$ if and only if there exists $\epsilon < r$ such that $\delta^k(X, \epsilon) > 0$.

Suppose there is $\epsilon_0 \in (0, r)$ with $\delta_0 = \delta^k(X, \epsilon_0) > 0$ and let $\delta \in (0, \delta_0)$. Let $x^*, y^* \in S_X^*$ with $\|x^* - y^*\| \geq \epsilon_0$ and $x_1, \ldots, x_{k+1} \in S(x^*, y^*, \delta)$. Then

$$\frac{x^* + y^*}{2} \left( \sum_{i=1}^{k+1} \frac{x_i}{k+1} \right) = \frac{1}{2(k+1)} \sum_{i=1}^{k+1} \left( x^*(x_i) + y^*(x_i) \right) \geq 1 - \delta > 1 - \delta_0,$$

that is,

$$1 - \frac{x^* + y^*}{2} \left( \sum_{i=1}^{k+1} \frac{x_i}{k+1} \right) < \delta_0,$$

and hence there exist $i$ and $j$ with $i \neq j$ so that $\|x_i - x_j\| < \epsilon_0$.

Consequently $\beta^k(S(x^*, y^*, \delta)) \leq \epsilon_0$ and $X$ is $(r, k)$-SUNC.

Suppose now that $X$ is $(r, k)$-SUNC. Then there exist $\epsilon_0 \in (0, r)$ and $\delta_0 > 0$ so that for every $x^*, y^* \in S_X^*$ with $\|x^* - y^*\| \geq \epsilon_0$, we have

$$\beta^k(S(x^*, y^*, \delta_0)) \leq \epsilon_0.$$
Let $\epsilon \in (\epsilon_0, r)$, $x_1, \ldots, x_k \in B_X$ with $\|x_i - x_j\| \geq \epsilon$ for $i \neq j$ and $x^*, y^* \in S_X^*$ with $\|x^* - y^*\| \geq \epsilon$. Since $\beta^k((x_1, \ldots, x_{k+1})) \geq \epsilon$, we obtain that $\{x_1, \ldots, x_{k+1}\}$ is not contained in $S(x^*, y^*, \delta_0)$ and this means that there exists some $i \in \{1, \ldots, k + 1\}$ such that $x_i \notin S(x^*, y^*, \delta_0)$. Then either $x^*(x_i) \leq 1 - \delta_0$ or $y^*(x_i) \leq 1 - \delta_0$. In both cases we have

$$\frac{x^* + y^*}{2} \left( \sum_{i=1}^{k+1} x_i \right) \leq \frac{2k + 1 + 1 - \delta_0}{2(k + 1)} = 1 - \frac{\delta_0}{2(k + 1)},$$

and thus

$$1 - \frac{x^* + y^*}{2} \left( \sum_{i=1}^{k+1} x_i \right) \geq \frac{\delta_0}{2(k + 1)},$$

therefore we may conclude that $\delta_X^{k-SUNC}(\epsilon) \geq \delta_0/(2(k + 1)) > 0$. 

As a trivial consequence of Theorem 7 we obtain the following relationship between Banach spaces with $\tilde{\delta}_0^\beta(X) \leq r$ and the $(r, k)$-SUNC property.

**Corollary 8.** If $\tilde{\delta}_0^\beta(X) < r \leq 2$, then $X$ is a $(r, k)$-SUNC Banach space.

Now we will give an example of a space which is $(2, k - 1)$-SUNC but is not 2-UNC.

**Theorem 9.** For $k > 2$, let $X_k$ be $\mathbb{R}^k$ endowed with the norm

$$\|x_1, \ldots, x_k\| = \max_j \sum_{i \in \{1, \ldots, k\}, i \neq j} |x_i|.$$

Then there exists $\phi < 2$ such that $\tilde{\delta}_X^{k-1}(\phi) > 0$.

**Proof.** Assume that $k > 2$ and take $x, y$ such that $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| \geq 2 - \epsilon$ and $\|x + y\| \geq 2 - \epsilon$, where $0 < \epsilon < 1/(4(k - 1)^2)$. Then $\min\{\|x\|, \|y\|\} \geq 1 - \epsilon$. Let $x = (a_1, \ldots, a_k)$ and $y = (b_1, \ldots, b_k)$. We will show that $\min\{|a_i|, |b_i|\} \leq 2\epsilon$ for every $i = 1, \ldots, k$.

Let $|a_p + b_p| = \min\{|a_i + b_i|, P = \{1, \ldots, k\} \setminus \{p\}, |a_m - b_m| = \min\{|a_i - b_i|, M = \{1, \ldots, k\} \setminus \{m\}$. Since $k > 2$, the set $A = P \cap M$ is nonempty.

Observe now that

$$|a + (-1)^s b| = |a| + |b| - 2\min\{|a|, |b|\}$$

for all real numbers $a, b$ with $(-1)^s ab \leq 0$, where $s = 1, 2$. It follows that

$$2 - \epsilon \leq \|x + y\| = \sum_{j \in P} |a_j| + \sum_{j \in P} |b_j| = 2 \sum_{j \in P} \min\{|a_j|, |b_j|\}$$

$$\leq 2 - 2 \sum_{j \in P} \min\{|a_j|, |b_j|\}.$$
where \( P_1 = \{ j \in P: a_j b_j \leq 0 \} \). Hence \( \min(\{a_j, b_j\}) \leq \epsilon/2 \) for every \( j \in P_1 \). Similarly, \( \min(\{a_j, b_j\}) \leq \epsilon/2 \) for every \( j \in M_1 \), where \( M_1 = \{ j \in M: a_j b_j \geq 0 \} \). Consequently,
\[
\min(\{a_j, b_j\}) \leq \epsilon/2
\]
for every \( j \in A \).

It remains to show that \( \min(\{a_j, b_j\}) \leq 2\epsilon \) for \( j = p, m \). Suppose that \( \min(\{a_p, b_p\}) > 2\epsilon \). Fix \( i \in A \) and put \( P_2 = P \setminus \{ i \} \). We have \( \min(\{a_i, b_i\}) \leq \epsilon/2 \). If \( |a_i| \leq \epsilon/2 \), then
\[
2 - \epsilon \leq \|x + y\| \leq \sum_{j \in P} |a_j| + 1 \leq \sum_{j \in P_2} |a_j| + \frac{\epsilon}{2} + 1.
\]

Hence
\[
\sum_{j \in P_2} |a_j| \geq 1 - \frac{3\epsilon}{2}.
\]
But
\[
\sum_{j \in P_2} |a_j| + 2\epsilon < \sum_{j \in P_2} |a_j| + |a_p| \leq 1,
\]
which is a contradiction. The remaining cases are similar.

Take now vectors \( x_1, \ldots, x_k \) such that \( \|x_i\| \leq 1 \) for every \( i = 1, \ldots, k \) and \( \|x_i - x_j\| \geq 2 - \epsilon \) whenever \( i \neq j \). If \( \|x_i + x_j\| < 2 - \epsilon \) for some \( i \neq j \), then
\[
\frac{1}{k} \left\| \sum_{n=1}^{k} x_n \right\| \leq \frac{1}{k} \left( k - 2 + \|x_i + x_j\| \right) < 1 - \frac{\epsilon}{k}.
\]

Assume now that \( \|x_i + x_j\| \geq 2 - \epsilon \) whenever \( i \neq j \). Let \( x_i = (x_i^1, \ldots, x_i^k) \). For each \( j \) there is at most one \( i \) such that \( |x_i^j| > 2\epsilon \). Therefore
\[
\left| \sum_{i=1}^{k} x_i^j \right| \leq 1 + 2(k - 1)\epsilon.
\]

Consequently,
\[
\frac{1}{k} \left\| \sum_{i=1}^{k} x_i \right\| \leq \frac{1}{k} (k - 1)(1 + 2(k - 1)\epsilon) = 1 - \frac{1}{k} (1 - 2(k - 1)^2\epsilon) < 1 - \frac{1}{2k}
\]
and thus we may conclude that \( \delta^{k-1}(2 - \epsilon) > \epsilon/k > 0 \). \( \square \)

If \( k = 2 \), then the space \( X_k \) is just \( \mathbb{R}^2 \) with the maximum norm, so it is UNC in a trivial way. From this, Theorem 9 and Corollary 8, we deduce the following corollary immediately.

**Corollary 10.** For \( k \geq 2 \), the Banach space \( X_k \) is \((2, k - 1)\)-UNC.

Next we characterize the dual space of \( X_k \).
Theorem 11. $X^*_k = \mathbb{R}^k$ equipped with the norm

$$
\| (y_1, \ldots, y_k) \| = \max \left\{ \max_{1 \leq i \leq k} |y_i|, \frac{|y_1| + \cdots + |y_k|}{k - 1} \right\}.
$$

Proof. Let $f = (y_1, \ldots, y_k) \in X^*_k$ and $x = (x_1, \ldots, x_k) \in S_{X^*_k}$. If we take $\text{sgn} x_i = \text{sgn} y_i$, we have that

$$
f(x_1, \ldots, x_k) = \sum_{i=1}^{k} |x_i| |y_i|,
$$

thus in order to calculate $\| f \|$ we will assume that $x_i \geq 0$ and $y_i \geq 0$ for $i = 1, 2, \ldots, k$.

Further we will suppose that $x_1 \geq \cdots \geq x_k$. Then $1 = \| x \| = \sum_{i=1}^{k-1} x_i$.

Let

$$
A = \{ x = (x_1, \ldots, x_k) \in X : x_1 \geq \cdots \geq x_k, \| x \| = 1 \}
$$

and

$$
z_1 = (1, 0, \ldots, 0), \quad z_2 = \left( \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 \right), \quad \ldots,
$$

$$
z_{k-1} = \left( \frac{1}{k-1}, \frac{1}{k-1}, \ldots, \frac{1}{k-1}, 0 \right), \quad z_k = \left( \frac{1}{k-1}, \frac{1}{k-1}, \ldots, \frac{1}{k-1}, \frac{1}{k-1} \right).
$$

Then $x \in A$ if and only if $x$ is a convex combination of $z_1, \ldots, z_k$. In fact, if $x = \sum_{i=1}^{k} \mu_i z_i$ with $\sum_{i=1}^{k} \mu_i = 1$ and $\mu_i \geq 0$ for $i = 1, \ldots, k$, we have that

$$
x_i = \sum_{j=i}^{k-1} \frac{1}{k-1} \mu_j + \frac{1}{k-1} \mu_k \quad \text{for } i = 1, \ldots, k-1, \quad x_k = \frac{1}{k-1} \mu_k.
$$

Thus $x_1 \geq \cdots \geq x_k$ and

$$
\| x \| = \mu_k + \sum_{j=1}^{k-1} \frac{1}{k-1} \mu_j = \mu_k + \sum_{j=1}^{k-1} \frac{1}{k} \mu_j = \sum_{i=1}^{k} \mu_i = 1.
$$

Consequently $x \in A$.

On the other hand, if $x \in A$, let

$$
\mu_i = \begin{cases} 
   i (x_i - x_{i+1}) & \text{if } i \leq k - 1, \\
   (k-1)x_k & \text{if } i = k.
\end{cases}
$$

Then $\mu_i \geq 0$,

$$
\sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k-1} i (x_i - x_{i+1}) + (k-1)x_k = \sum_{i=1}^{k-1} x_i + x_k = 1
$$

and

$$
\sum_{i=1}^{k} \mu_i z_i = \sum_{i=1}^{k-1} i (x_i - x_{i+1}) z_i + (k-1)x_k z_k = x.
$$
Hence, if \( f(x) = \sum_{i=1}^{k} x_i y_i \), \( f \) attains its maximum in \( A \) at one of the extreme points \( z_1, \ldots, z_k \). But

\[
f(z_i) = \begin{cases} 
\frac{y_i + \cdots + y_i}{y_i y_{k-1}} & \text{if } 1 \leq i \leq k-1, \\
\frac{y_i - \cdots - y_i}{y_i} & \text{if } i = k.
\end{cases}
\]

Suppose \( \max_{1 \leq i \leq k} y_i = y_r \). Then \( f(0, 0, \ldots, y_r, \ldots, 0) = y_r \). Thus

\[
\|f\| = \max \left\{ \max_{1 \leq i \leq k} |y_i|, \frac{|y_1| + \cdots + |y_k|}{k - 1} \right\}.
\]

From above we derive that being \((2, k-1)\)-SUNC is not the same as being \((2, k-3)\)-SUNC.

**Theorem 12.** Let \( k \geq 4 \). The space \( X_k \) is not \((2, k-3)\)-SUNC.

**Proof.** Let \( x^* = e_1 + e_2 + \cdots + e_{k-1} \) and \( y^* = e_1 + e_2 + \cdots - e_{k-1} \). By Theorem 11, \( \|x^*\| = \|y^*\| = 1 \) and \( \|x^* - y^*\| = 2 \). On the other hand, \( x^*(e_i) = y^*(e_i) = 1 \) for \( i = 1, 2, \ldots, k - 2 \) and \( \|e_i - e_j\| = 2 \) for \( i \neq j \). Then \( e_1, \ldots, e_k \in S(x^*, y^*, 0) \) and \( \beta^{k-3}(S(x^*, y^*, 0)) = 2 \).

Finally, by the previous theorems, we may conclude that there exists a Banach space which is \((2, 3)\)-SUNC but is not \(2\)-UNC.

**Corollary 13.** Let \( k = 4 \). The space \( X_4 \) is not \(2\)-UNC but is \((2, 3)\)-SUNC.

4. Super-reflexivity

To study super-reflexivity in SUNC Banach spaces, we recall the following result due to James [6].

**Lemma 14.** Let \( X \) be a Banach space. \( X \) is super-reflexive if and only if \( X \) does not satisfy the following condition: For every \( n \in \mathbb{N} \) and for every \( \rho \in (0, 1) \) there exist \( x_1, \ldots, x_n \in B_X \) such that

\[
\left| \sum_{i=1}^{j} x_k + \sum_{i=j+1}^{n} x_k \right| > \rho n
\]

for every \( j = 1, \ldots, n \).

We can now state the following result.

**Theorem 15.** Let \( X \) be a Banach space. If \( X \) is SUNC, then \( X \) is super-reflexive.

**Proof.** Since \( X \) is SUNC, there exists \( k \in \mathbb{N} \) such that \( X \) is \((2, k)\)-SUNC.
Let \( \epsilon_0 \in (0, 2) \) and \( \delta_0 > 0 \) so that for every \( x^*, y^* \in S_{X^*} \) with \( \|x^* - y^*\| \geq \epsilon_0 \), we have \( \beta^k(S(x^*, y^*, \delta_0)) \leq \epsilon_0 \) and suppose that \( X \) is not super-reflexive. Let
\[
0 < \delta < \min \left( \frac{\delta_0}{k + 2}, \frac{2 - \epsilon_0}{2(k + 2)} \right).
\]
By Lemma 14, there exist \( x_1, \ldots, x_k+2 \in B_X \) such that
\[
\|x_1 + \cdots + x_k+2\| > (1 - \delta)(k + 2),
\]
\[
\|x_1 + \cdots + x_{k+1} - x_{k+2}\| > (1 - \delta)(k + 2),
\]
\[
\vdots
\]
\[
\|x_1 - x_2 - \cdots - x_{k+2}\| > (1 - \delta)(k + 2).
\]
Let \( x^*, y^* \in S_{X^*} \) such that \( x^*(x_1 + \cdots + x_{k+2}) = \|x_1 + \cdots + x_{k+2}\| \) and \( y^*(x_1 - x_2 - \cdots - x_{k+2}) = \|x_1 - x_2 - \cdots - x_{k+2}\| \). Then
\[
(k + 1) + \min_{1 < i, j \leq k+2} x^*(x_i) > x^*(x_1) + \cdots + x^*(x_{k+2}) > 1 - \delta)(k + 2)
\]
and thus
\[
\min_{1 \leq i \leq k+2} x^*(x_i) \geq 1 - (k + 2)\delta > 1 - \delta_0.
\]
Similarly we obtain that
\[
\min_{2 \leq i, j \leq k+2} \{y^*(x_1), -y^*(x_j)\} > 1 - \delta_0.
\]
Hence \( x_i \in S(x^*, -y^*, \delta_0) \) for every \( i = 2, \ldots, k + 2 \).

On the other hand,
\[
\|x^* - (-y^*)\| \geq (x^* + y^*)(x_1) \geq 2(1 - (k + 2)\delta) > \epsilon_0
\]
and for \( i, j > 1, i < j \), we have
\[
\|x_i - x_j\| \geq \|x_1 + \cdots + x_i - x_{i+1} - \cdots - x_j - \cdots - x_{k+2}\| - \sum_{r \neq i, j} \|x_r\|
\]
\[
\geq (1 - \delta)(k + 2) - k = 2 - (k + 2)\delta > \epsilon_0.
\]
Then \( \beta^k(S(x^*, -y^*, \delta_0)) \geq \beta^k(\{x_i\}_{i=2}^{k+2}) > \epsilon_0 \) and this contradiction proves the theorem. \( \square \)

**Remark.** In [7] the author asks whether a Banach space \( X \) with \( \tilde{c}_0^X(X) < 2 \) is reflexive. This question was fully answered in [3] since this condition implies even super-reflexivity. However, the proof given in [3] does not work when \( \tilde{c}_0^X(X) < 2 \) with \( k > 2 \). As a consequence of the above theorem and of Corollary 8 we have obtained that if \( X \) is a Banach space with \( \tilde{c}_0^X(X) < 2 \) for some \( k \in \mathbb{N} \), then \( X \) is SUNC and therefore super-reflexive.

Recall that a Banach space \( Y \) is said to be finitely representable in a Banach space \( X \) if for every \( \epsilon > 0 \) and every finite-dimensional subspace \( Z \) of \( Y \) there is a linear isomorphism \( T : Z \to X \) for which
\[
(1 - \epsilon)\|y\| \leq \|Ty\| \leq (1 + \epsilon)\|y\|
\]
for all $y \in Z$. Next we will see that “being $(r,k)$-SUNC” is a super-property. We will omit the proof of the following result since it is practically the same as the one of Theorem 2 in [3].

**Theorem 16.** Let $r \in (0,2]$ and $k \in \mathbb{N}$. If $X$ is a $(r,k)$-SUNC Banach space and $Y$ is another Banach space finitely representable in $X$, then $Y$ is also $(r,k)$-SUNC.

Since an ultrapower $\tilde{X}$ of a Banach space $X$ is always finitely representable in $X$, the following result easily follows.

**Proposition 17.** A Banach space $X$ is $(r,k)$-SUNC if and only if $\tilde{X}$ is $(r,k)$-SUNC.

### 5. Fixed point results

In [3] the authors showed that 1-UNC Banach spaces have the FPP. The following theorem generalizes this result.

**Theorem 18.** If a Banach space $X$ is $(1,k)$-SUNC, then it has the FPP.

**Proof.** Suppose that $X$ is a $(1,k)$-SUNC space lacking FPP. Then there is a weakly compact convex subset $K$ of $X$ which is minimal for a nonexpansive fixed point free mapping $T : K \to K$. We can assume that $\text{diam}(K) = 1$ and that $K$ contains an afps weakly null sequence $(x_n)$ in $K$, with

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} x_n - x_m = 1 \quad \text{for each } m \in \mathbb{N}. \quad (1)$$

For each index $n$ we choose a functional $x^*_n \in S_{X^*}$ such that $x^*_n(x_n) = \|x_n\|$. Since $(x_n)$ is a weakly null sequence, we can suppose, passing to subsequences if necessary, that

$$\lim_{n \to \infty} x^*_n(x_{n+1}) = 0. \quad (2)$$

Let us consider $\tilde{x} = (\tilde{x}_n)$, $\tilde{y} = (\tilde{x}_{n+1})$ and $\tilde{f}$ the functional on $\tilde{X}$ corresponding to the sequence $(x^*_n)$ in $X^*$.

From (2) and (1) we obtain that $\tilde{x}, \tilde{y} \in S_{\tilde{X}}, \tilde{f}(\tilde{x}) = 1 = \|\tilde{f}\|, \tilde{f}(\tilde{y}) = 0$, and $\|\tilde{x} - \tilde{y}\| = 1$.

Let $\delta$ be a fixed real number of the interval $(0,1)$. Theorem 2 provides an afps $(\tilde{w}_n)$ in $K$ such that, if $\tilde{w} = (\tilde{w}_n)$, then

$$\|\tilde{w} - \tilde{x}\| = \delta \quad \text{and} \quad \|\tilde{w} - \tilde{y}\| = 1 - \delta.$$

We have that

$$\tilde{f}(\tilde{w}) = \tilde{f}(\tilde{x}) - \tilde{f}(\tilde{x} - \tilde{w}) \geq 1 - \|\tilde{w} - \tilde{x}\| = 1 - \delta,$$

and on the other hand

$$\tilde{f}(\tilde{w}) = \tilde{f}(\tilde{w} - \tilde{y}) \leq \|\tilde{w} - \tilde{y}\| = 1 - \delta,$$

that is

$$\tilde{f}(\tilde{w}) = 1 - \delta.$$
Consider now for each index $n$ a functional $y^*_n \in S\mathcal{X}^*$ such that $y^*_n(\omega_n) = \|\omega_n\|$. Denote by $\tilde{g}$ the functional of $\tilde{X}$ corresponding to the sequence $(y^*_n)$. Clearly $\|\tilde{g}\| = 1$. And, since $(\omega_n)$ is an afps, by Lemma 1 we have that $\tilde{g}(\tilde{w}) = \|\tilde{w}\| = 1$. Then

$$
\|\tilde{f} - \tilde{g}\| \geq (\tilde{f} - \tilde{g})\left(\frac{1}{\delta}(x - \tilde{w})\right) = 1 + \frac{1}{\delta}(1 - \tilde{g}(\tilde{x})) \geq 1.
$$

Since $(\omega_n)$ is a weakly null sequence, we know that $\lim_{n \to \infty} f_i(\omega_n) = 0$ and $\lim_{n \to \infty} g_i(\omega_n) = 0$ for every $i \in \mathbb{N}$. Hence we can construct a sequence

$$
\left\{ \left\{ m^k \right\}_{l=1}^{k+1} \right\}_{k=1}^\infty
$$

such that

$$
\max\left( |f_k(\omega_n)|, |g_k(\omega_n)| \right) < \frac{1}{k} \quad \text{for every } n \geq m^k_1
$$

and

$$
\|x_{m^k_j} - x_{m^k_i}\| > 1 - \frac{1}{k+1} \quad \text{for } i, j = 1, \ldots, k + 1, i \neq j.
$$

Let $u_m^k = x_{m+1}^{n+1}$ for $k = 1, \ldots, n - 1$, $u_m^k = x_{m}^{k}$ for $k \geq n$, and define $\bar{z}_n = (u_m^k)_{k=1}^\infty$. Clearly $\|\bar{z}_n - \bar{z}_x\| = \lim_{k \to \infty} \|u_m^k - u_m^1\| = 1$, $\tilde{f}(\bar{z}_n) = \tilde{g}(\bar{z}_n) = 0$, and $\|\tilde{w} - \bar{z}_n\| \leq \text{diam } K = 1$. Thus $\tilde{w}, \tilde{w} - \bar{z}_n \in S(\tilde{f}, \tilde{g}, \delta)$ for every $\delta > 0$ and for every $n$, and $\|(\tilde{w} - \bar{z}_n) - (\tilde{w} - \bar{z}_s)\| = \|\bar{z}_n - \bar{z}_s\| = 1$ for every $n \neq s$. Hence we have that

$$
\beta^k(S(\tilde{f}, \tilde{g}, \delta)) \geq 1
$$

for every $\delta > 0$ and for every $k$; thus $\tilde{X}$ and hence $X$ are not $(1, k)$-SUNC.

As a corollary of the last theorem and Theorem 16 we have

**Corollary 19.** If a Banach space $X$ is $(1, k)$-SUNC, then it has the super-FPP.

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**References**
