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NOTE

Finite Doubly Transitive Affine Planes

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In this article, we classify the finite affine planes admitting doubly transitive collineation groups.

1. INTRODUCTION

In the late 1950s and early 1960s, Ostrom and Wagner studied projective and affine planes which admit collineation groups acting either doubly transitively or flag transitively. The most successful classification result during this period is the celebrated Ostrom–Wagner Theorem [7] which states that if a finite projective plane admits a collineation group which acts doubly transitively on the points then the plane is Desarguesian. Attempts at proving the affine version of this result have not been entirely sucessful although it follows from arguments of Ostrom–Wagner that a finite affine plane admitting a collineation group acting doubly transitively on points is a translation plane. More generally, Wagner [8] showed that a finite affine plane admitting a collineation group which acts transitively on the affine points and transitively on the infinite points is a translation plane. Furthermore, this latter condition is equivalent to having a group acting transitively on the flags of the affine plane.

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However, it has been an open question whether a finite affine plane admitting a doubly transitive group acting on the points is necessarily Desarguesian. In fact, it is not as the Hall plane of order 9 was seen to admit such collineation groups.

Let π be a finite affine plane admitting a collineation group G which acts doubly transitive on the affine points. We shall call such a plane a doubly transitive affine plane. When the group G is solvable, there is a complete classification due to Foulser.

THEOREM 1.1 [2] *If π is a solvable doubly transitive affine plane then π is either Desarguesian or the Hall plane of order* 9.

The only known nonsolvable doubly transitive non-Desarguesian affine planes are the Hering plane of order 3^3 which admits $SL(2, 13)$ and the Hall plane of order 9. We note that the full collineation group induced on the line at infinity of the Hall plane of order 9 is S_5 (see e.g. [6] p. 36).

In this article, we show that the only doubly transitive affine planes are the known ones. That is, our main result is

THEOREM 1.2. *Let π be a finite doubly transitive affine plane. Then π is one of the following types of planes*:

(1) *Desarguesian*,
(2) *Hall of order* 9 *or*
(3) *Hering of order* 27.

2. THE MAIN THEOREM

Recently, with a few exceptions, the authors classified all finite translation planes that admit a collineation group inducing a nonsolvable doubly transitive group on a set of line size. It turns out that the line size set is either a line or a Baer subplane or the translation plane is a generalized twisted field plane for which the group is actually solvable and is hence eliminated from consideration (see Jha, Johnson [4]). The above mentioned results depend upon the fundamental results of Hering [3] on transitive linear groups.

THEOREM 2.1 [3]. *Let $(V, +)$ be a $GF(p)$-vector space of order p^n for p a prime. Let G be a subgroup of $GL(V, +)$ which acts transitively on the non-zero vectors of V and let K be a maximal field among all the subrings of $End(V, +)$ that are closed with respect to inversion and conjugation by G so that G is a subgroup of $\Gamma L(V, K)$. Let the dimension of V over K be n^* and K isomorphic to $GF(q)$ for $q = p^t$ so that the cardinality of the vector space is $q^{n^*} = p^{tn^*} = p^n$ and $tn^* = n$.*

If G is non-solvable then one of the following occur:

(1) *$G^{(\infty)}$ is isomorphic to $SL(2, 5)$ (the last term of the derived series), $n^* = 2$ and the order of K is* 9, 11, 19, 29 *or* 59,

(2) *G contains a perfect normal subgroup H which acts transitively on $V - \{0\}$ and one of the following possibilities occur:*

(a) *H is isomorphic to $SL(n^*, q)$,*

(b) *V as a K-Space contains a symplectic form * such that H is the corresponding symplectic group $S_p(n^*, q)$,*

(c) *$K \simeq GF(2^m)$, $n^* = 6$ and H is the Chevalley group $G_2(2^m)$,*

(d) *$K \simeq GF(2)$, $n^* = 4$ and $H = G \simeq A_6$ or A_7.*

(e) *$K \simeq GF(2)$, $n^* = 6$ and $H = G \simeq PSU(3, 3^2)$,*

(f) *$K \simeq GF(3)$, $n^* = 6$ and $H = G \simeq SL(2, 13)$.*

Assumption. G is a collineation group of a translation plane of order p^r in the translation complement $\Gamma L(2r, p)$ which acts transitively on the nonzero vectors of the underlying vector space V. We note that the semidirect product of the translation group T with G acts doubly transitively and conversely any doubly transitive group acting on the vectors is a group of this type. Hence, we are considering a nonsolvable collineation group which fixes the zero vector and which acts transitively on the nonzero vectors of a 2r-dimensional $GF(p)$-vector space. Hence, in the above theorem $n = 2r$, $n^* = n/t = 2r/t$ where $K \simeq GF(q = p^t)$ and $q^{n^*} = p^{2r}$.

We may apply the theorem of Hering to realize the possible groups that can act on translation planes. The key to establishing that most of the groups do not act in this way is a result due to Jha and Kallaher who provide a bound on the order of a planar p-group.

THEOREM 2.2. [5, result 4]. *If P is a planar p-group of a translation plane of order p^r then $|P| \leqslant p^{r-1}$, and equality occurs only when $r = 2$ or $p^* = 16$.*

We consider the various cases in turn.

2.1. *Case* 1. $G^{(\infty)}$ is isomorphic to $SL(2, 5)$ (the last term of the derived series), $n^* = 2$ and the order of K is 9, 11, 19, 29 or 59. Note that $n^* = 2r/t = 2$ which implies that $r = t$ so that the order of the plane is 9, 11, 19, 29, or 59 so the plane is either Desarguesian or Hall of order 9.

2.2. *Case* 2. (a) $SL(n^*, q)$. The Sylow p-subgroups of H are of order $q^{n^*(n^*-1)/2} = p^{r(2r/t-1)}$ acting on a translation plane of order p^r. Since there are $p^r + 1$ components, a Sylow p-subgroup S must leave a component invariant. The stabilizer of a second component has order divisible by $p^{r(2r/t-1)}/p^r$. Any planar p-group has order $< p^r$ by Jha and Kallaher [5]. Since any p-group which fixes two components must fix a third, it follows that there is a planar p-group of order at least $p^{r(2r/t-1)}/p^r < p^r$. It follows that we must have $2r/t - 1 < 2$ or equivalently that $r \leqslant t$. Hence, $t = r$ and the group is $SL(2, q)$ acting on a translation plane of order q which forces the plane to be Desarguesian by Lüneburg [6].

2.3. *Case* 2. (b) V as a K-space contains a symplectic form * such that H is the corresponding symplectic group $Sp(n^*, q)$. In this situation, $n^* = 2r/t$ and n^* must be even so that t divides r and the order of a Sylow p-subgroup is $q^{(r/t)^2} = p^{t(r/t)^2} = p^{r^2/t} < p^{2r}$ by the proof of case (2) (a). Hence, $r/t < 2$ so that it can only be that $r = t$. But, $S_p(2, q)$ is $SL(2, q)$, the order of the plane is q so that the previous note shows that the plane is Desarguesian.

2.4. *Case* 2. (c) $K \simeq GF(2^m)$, $n^* = 6$ and H is the Chevalley group $G_2(2^m)$. The order of a Sylow 2-subgroup is q^6 where $q = 2m$ and the order of the plane is q^3. The previous argument provides the contradiction that $q^6 < q^{2.3=6}$.

We now consider the sporadic cases:

2.5. *Case* 2. (d) $K \simeq GF(2)$, $n^* = 4$ and $H = G \simeq A_6$ or A_7. Here the plane is of order 4 so the plane is Desarguesian.

2.6. *Case* 2. (e) $K \simeq GF(2)$, $n^* = 6$ and $H = G \simeq PSU(3, 3^2)$. In this case, the plane is of order 2^3 so is also Desarguesian which is a contradiction as the group indicated does not act on a Desarguesian plane.

2.7. *Case* 2. (f) $K \simeq GF(3)$, $n^* = 6$ and $H = G \simeq SL(2, 13)$. Since the order of the plane is 3^3, we may apply the results in Barriga and Pomareda [1]. Hence, it follows that the only possibility is the Hering plane of order 3^3.

This completes the proof of the main result stated in the Introduction.

Note added in proof. A more general proof (for 1-designs) using the classification theorem of finite simple groups is given in W. M. Kantor, Homogeneous designs and geometric lattices, *J. Comb. Theory Ser. A* **38** (1985), 66–74.

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