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Classifying quadratic quantum $\mathbb{P}^2$s by using graded skew Clifford algebras

Manizheh Nafari [1], Michaela Vancliff [*,2], Jun Zhang

Department of Mathematics, P.O. Box 19408, University of Texas at Arlington, Arlington, TX 76019-0408, United States

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ABSTRACT

We prove that quadratic regular algebras of global dimension three on degree-one generators are related to graded skew Clifford algebras. In particular, we prove that almost all such algebras may be constructed as a twist of either a regular graded skew Clifford algebra or of an Ore extension of a regular graded skew Clifford algebra of global dimension two. In so doing, we classify all quadratic regular algebras of global dimension three that have point scheme either a nodal cubic curve or a cuspidal cubic curve in $\mathbb{P}^2$.

Introduction

In [2], M. Artin and W. Schelter introduced a notion of regularity for a non-commutative graded algebra on degree-one generators. To such an algebra one may associate some geometry via certain graded modules over the algebra, as discussed by M. Artin, J. Tate and M. Van den Bergh in [3]. In this spirit, one may describe such an algebra using a certain scheme (called the point scheme in [9]) and, in [1], M. Artin introduced the language of "quantum $\mathbb{P}^2$" for a regular algebra of global dimension three on degree-one generators. Generic quantum $\mathbb{P}^2$s were classified in [2–4].

* Corresponding author.
E-mail addresses: manizheh@uta.edu (M. Nafari), vancliff@uta.edu (M. Vancliff), zhangjun19@gmail.com (J. Zhang).
URL: http://uta.edu/math/vancliff (M. Vancliff).

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In [5], a new relatively simple method was given for constructing some quadratic regular algebras of finite global dimension, and quadratic regular algebras produced with that method are called regular graded skew Clifford algebras (see Definition 1.8). At this time, it is unclear how useful this method will be in helping resolve the open problem of classifying all quadratic regular algebras of global dimension four. A first step in this direction is to determine how useful graded skew Clifford algebras are in classifying quadratic quantum $\mathbb{P}^2$s. In this article, we prove that most quadratic quantum $\mathbb{P}^2$s may be constructed as a twist of either a regular graded skew Clifford algebra or of an Ore extension of a regular graded skew Clifford algebra of global dimension two. For the precise results, see Theorem 1.10, Corollaries 2.3, 3.2, Lemmas 4.1, 4.2 and 4.3. The only algebras we did not relate in this way to graded skew Clifford algebras are some that have point scheme an elliptic curve; specifically those of type E in [2] (up to isomorphism and anti-isomorphism, there is at most one such algebra) and an open subset of those of type A in [2] (although a weak relationship is discussed for these algebras in Remark 4.4). It is still open whether or not similar results hold for quadratic regular algebras of global dimension four.

Henceforth, let A denote a quadratic quantum $\mathbb{P}^2$. The work in this article is partitioned across sections according to the point scheme of A. Those A whose point scheme contains a line are discussed in Section 1; is a nodal cubic curve in Section 2; cuspidal cubic curve in Section 3; and, finally, elliptic curve in Section 4. Those A whose point scheme is either a nodal cubic curve or a cuspidal cubic curve in $\mathbb{P}^2$ are not specifically discussed in [3], as they are not generic. In Sections 2 and 3 of this article, we classify all such algebras (see Theorems 2.2 and 3.1). Indeed, up to isomorphism, there is at most a one-parameter family of quadratic quantum $\mathbb{P}^2$s with point scheme a nodal cubic curve, whereas there is only one such algebra with point scheme a cuspidal cubic curve.

1. Quantum $\mathbb{P}^2$s with reducible or non-reduced point scheme

Throughout the article, $\Bbbk$ denotes an algebraically closed field and $\Bbbk^\times = \Bbbk \setminus \{0\}$. It is well known that a quadratic regular algebra of global dimension $n+1$ with point scheme given by $\mathbb{P}^n$ is a twist, by an automorphism, of the polynomial ring on $n+1$ variables (cf., [4, p. 378] and Remark 1.4(iii)). We prove a generalization of this result in this section for certain quadratic quantum $\mathbb{P}^2$s. Precisely, we prove (Theorem 1.7) that a quadratic quantum $\mathbb{P}^2$ that has a reducible or non-reduced point scheme is either a twist by an automorphism of a skew polynomial ring or is a twist, by a twisting system, of an Ore extension of a regular algebra of global dimension two. This result is restated in terms of graded skew Clifford algebras in Theorem 1.10 where $\mathrm{char}(\Bbbk) \neq 2$.

As in the Introduction, let A denote a quadratic quantum $\mathbb{P}^2$ with point scheme given by a cubic divisor $C \subset \mathbb{P}^2$. (Here, $\mathbb{P}^2$ may be identified with $\mathbb{P}(A_1^*)$ where A_1 denotes the span of the homogeneous degree-one elements of A, and A_1^* denotes the vector-space dual of A_1.) Throughout this section, we assume that C is reducible or non-reduced, so that C is either the union of a nondegenerate conic and a line, or the union of three distinct lines, or the union of a double line and a line, or is a triple line. The automorphism encoded by the point scheme will be denoted by $\sigma \in \mathrm{Aut}(C)$. There are two cases to consider: either C contains a line that is invariant under σ or it does not. Both cases use the notion of twisting a graded algebra by an automorphism, which is defined in [4, §8]; in the case of a quadratic algebra, it is defined as follows.

Definition 1.1. (See [4, §8].) Let D denote a quadratic algebra, let D_1 denote the span of the homogeneous degree-one elements of D and let ϕ be a graded degree-zero automorphism of D. The twist D^ϕ of D by ϕ is a quadratic algebra that has the same underlying vector space as D, but has a new multiplication $*$ defined as follows: if $a, b \in D_1 = (D^\phi)_1$, then $a * b = a\phi(b)$, where the right-hand side is computed using the original multiplication in D.

In general, twisting by an automorphism is a reflexive and symmetric operation, but not a transitive operation; in fact, twisting an algebra D twice yields a twist of D by a *twisting system*, and that notion is defined in [10].

1.1. Case 1: C contains a line invariant under σ

Suppose $L \subset C$ is a line that is invariant under σ, and let $x \in A_1$ be such that L is the zero locus, $\mathcal{V}(x)$, of x. By [4, Theorem 8.16(i)(ii) and Corollary 8.6], x is normal in A, and one may twist A, by an automorphism, to obtain a quadratic regular algebra B in which the image of x is central.

Proposition 1.2. *In the above notation, the twist B of A is an Ore extension of the polynomial ring on two variables.*

Proof. Let x' denote the image of x in B. Since $B/\langle x'\rangle$ is a regular algebra of global dimension two, it is isomorphic to either $\Bbbk\langle Y,Z\rangle/\langle YZ-ZY-Y^2\rangle$ or $\Bbbk\langle Y,Z\rangle/\langle ZY-qYZ\rangle$ where $q \in \Bbbk^\times$ [2, p. 172]. It follows from [4, Theorem 8.16(iii)] that B is a $\Bbbk$-algebra on generators x', y, z with defining relations

$$x'y = yx', \qquad x'z = zx', \qquad h = 0,$$

where either

$$\begin{aligned} &\text{(i)} \quad h = zy - yz + y^2 + x'\big(\alpha x' + \beta y + \gamma z\big) \quad \text{or} \\ &\text{(ii)} \quad h = zy - qyz + x'\big(\alpha x' + \beta y + \gamma z\big), \end{aligned}$$

for some $\alpha, \beta, \gamma \in \Bbbk$. Let $B' = \Bbbk[x', y]$, and define $\phi \in \operatorname{Aut}(B')$ and a left ϕ-derivation $\delta : B' \to B'$ as follows:

$$\phi\big(x'\big) = x', \qquad \phi(y) = sy - \gamma x', \qquad \delta\big(x'\big) = 0, \qquad \delta(y) = -ty^2 - \alpha\big(x'\big)^2 - \beta x'y,$$

where $(s,t) = (1,1)$ if h is given by (i) and $(s,t) = (q,0)$ if h is given by (ii). In both cases, $B = B'[z; \phi, \delta]$. □

Corollary 1.3. *If the point scheme of A contains a line that is invariant under σ, then A is a twist, by an automorphism, of an Ore extension of the polynomial ring on two variables.*

Proof. Combining Proposition 1.2 with the preceding discussion proves the result. □

1.2. Case 2: No line in C is invariant under σ

Suppose that no line in C is invariant under σ. It follows that C is the union of three distinct lines that are cyclically permuted by σ. Such lines have the property that either no point lies on all three lines or the three lines meet at exactly one point.

Remark 1.4. (i) Let D denote a quadratic quantum $\mathbb{P}^2$. Let $V = D_1$ and let $W \subset V \otimes_\Bbbk V$ denote the span of the defining relations of D, and let $\mathcal{V}(W)$ denote the zero locus, in $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$, of the elements of W, where $\mathbb{P}(V^*)$ is identified with $\mathbb{P}^2$. The Koszul dual $D^\star$ of D is the quotient of the free algebra on V^* by the ideal generated by $W^\perp \subset V^* \otimes_\Bbbk V^*$. If $(p,q) \in \mathcal{V}(W)$, then $\Bbbk p \subset V^*$, $\Bbbk q \subset V^*$ and $p \otimes q \in W^\perp$, and conversely. This provides a method of passing between $\mathcal{V}(W)$ and the relations of $D^\star$.

(ii) With notation as in (i), by [3], $\mathcal{V}(W)$ is the graph of an automorphism τ of a subscheme $\mathcal{P}$ of $\mathbb{P}(V^*) = \mathbb{P}^2$.

(iii) With notation as in (i) and (ii), if τ may be extended to an automorphism of $\mathbb{P}(V^*) = \mathbb{P}^2$, then, since D is regular, D is a twist, by an automorphism, of the polynomial ring and $\mathcal{P} = \mathbb{P}(V^*)$. This is because the homogeneous degree-two forms that vanish on the graph of τ have the form $\tau(u)v - \tau(v)u$ for all $u, v \in D_1$.

Lemma 1.5. *Suppose C is the union of three distinct lines that are cyclically permuted by σ. If no point lies on all three lines, then A is a twist, by an automorphism, of a skew polynomial ring.*

Proof. By hypothesis, there exist linearly independent elements $x, y, z \in A_1$ such that $C = \mathcal{V}(xyz)$ and $\sigma : \mathcal{V}(x) \to \mathcal{V}(y) \to \mathcal{V}(z) \to \mathcal{V}(x)$. Since $\sigma(1,0,0) \in \mathcal{V}(x,z)$ and $\sigma(0,1,0) \in \mathcal{V}(x,y)$ and $\sigma(0,0,1) \in \mathcal{V}(y,z)$, it follows that

$$\sigma(0,\beta,\gamma) = (d\gamma, 0, \beta), \qquad \sigma(\alpha,0,\gamma) = (\gamma, e\alpha, 0), \qquad \sigma(\alpha,\beta,0) = (0,\alpha, f\beta),$$

for some $d, e, f \in \mathbb{k}^\times$, for all $(\beta,\gamma), (\alpha,\gamma), (\alpha,\beta) \in \mathbb{P}^1$. By Remark 1.4(ii), this implies that A is a $\mathbb{k}$-algebra on generators x, y, z with defining relations:

$$yx = dz^2, \qquad zy = ex^2, \qquad xz = fy^2.$$

Define $\tau \in \mathrm{Aut}(A)$ by $\tau(x) = \lambda_1 y$, $\tau(y) = \lambda_2 z$ and $\tau(z) = \lambda_3 x$, where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{k}^\times$ satisfy $d\lambda_1\lambda_3^2 = e\lambda_1^2\lambda_2 = f\lambda_2^2\lambda_3$. Twisting by τ yields a $\mathbb{k}$-algebra on x, y, z with defining relations:

$$yz + czy = 0, \qquad zx + cxz = 0, \qquad xy + cyx = 0,$$

for some $c \in \mathbb{k}^\times$, and such an algebra is a skew polynomial ring. $\square$

Proposition 1.6. *Suppose C is the union of three distinct lines that are cyclically permuted by σ. If all three lines intersect at exactly one point, then A is either* (a) *an Ore extension of a regular algebra of global dimension two, or* (b) *a twist, by a twisting system, of an Ore extension of the polynomial ring on two variables*; *if* $\mathrm{char}(\mathbb{k}) \neq 3$, *then A is described by* (b).

Proof. Suppose all three lines of C intersect at only one point $p = (0,0,1)$, and write $C = \mathcal{V}(xy(x-y))$, where $x, y \in A_1$. We may assume $\sigma : \mathcal{V}(x) \to \mathcal{V}(y) \to \mathcal{V}(x-y) \to \mathcal{V}(x)$. Since $\sigma(p) = p$, we have

$$\sigma(0,\beta,\gamma) = (\beta, 0, a\beta + b\gamma), \qquad \sigma(\alpha,0,\gamma) = (\alpha,\alpha, c\alpha + d\gamma), \qquad \sigma(\beta,\beta,\gamma) = (0,\beta, e\beta + f\gamma),$$

for some $a, c, e \in \mathbb{k}$, $b, d, f \in \mathbb{k}^\times$, for all $(\beta,\gamma), (\alpha,\gamma) \in \mathbb{P}^1$. In A_1^*, let $\{X, Y, Z\}$ denote the dual basis to $\{x, y, z\}$. By Remark 1.4(i), the following relations hold in the Koszul dual of A:

$$YX + aYZ = 0, \qquad Y^2 + XY + eXZ + eYZ = 0, \qquad X^2 + XY + cXZ = 0,$$
$$Z^2 = 0, \qquad ZY + ZX + dXZ = 0, \qquad ZX + bYZ = 0, \qquad ZY + fYZ + fXZ = 0.$$

Since A is a quantum $\mathbb{P}^2$, these relations span at most a 6-dimensional space. As the first four relations span a 4-dimensional space, and the last three relations are linearly independent of the first four relations, it follows that the span of the last three relations has dimension at most two. This implies that $d = f = -b$, so that we may write

$$\sigma(0,\beta,\gamma) = (\beta, 0, a\beta + b\gamma), \qquad \sigma(\alpha,0,\gamma) = (\alpha,\alpha, c\alpha - b\gamma), \qquad \sigma(\beta,\beta,\gamma) = (0,\beta, e\beta - b\gamma).$$

By Remark 1.4(iii), since $C \neq \mathbb{P}^2$, σ cannot be extended to $\mathbb{P}^2$, from which we obtain $c \neq e + a$. Suppose $b^2 + b + 1 \neq 0$, and define

$$\tau = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ k & g & 1 \end{bmatrix} \in \mathrm{Aut}(C),$$

for some $g, k \in \Bbbk$. Thus, $\tau : \mathcal{V}(x) \to \mathcal{V}(x-y) \to \mathcal{V}(y) \to \mathcal{V}(x)$. In order for σ and τ to commute on C, we require g and k to satisfy

$$bg - k = a - e \quad \text{and} \quad g + (b+1)k = -a - c.$$

Since $b^2 + b + 1 \neq 0$, these equations have a unique solution, so we may choose g and k so that σ and τ commute on C. By [4, Proposition 8.8], τ induces an automorphism τ' of A. Twisting A by τ' produces an algebra B whose point scheme is C with automorphism $\tau \circ \sigma \in \operatorname{Aut}(C)$. Since each line of C is invariant under $\tau \circ \sigma$, it follows, by Corollary 1.3, that B is a twist, by an automorphism, of an Ore extension D of the polynomial ring on two variables. Hence, A is a twist by an automorphism of a twist by an automorphism of D. Since twisting by an automorphism need not be transitive, we can at most conclude that A is a twist of D by a twisting system.

Instead, suppose $b^2 + b + 1 = 0$. If $\operatorname{char}(\Bbbk) \neq 3$, then the Koszul dual of A has Hilbert series $H(t) = 1 + 3t + 3t^2$, giving that A is not a quantum $\mathbb{P}^2$, which is a contradiction. However, if $\operatorname{char}(\Bbbk) = 3$, then $b = 1$, and A is generated by x, y, z with defining relations:

$$\begin{aligned} xy &= x^2 + y^2, \\ zy &= -xz + cx^2 + ey^2, \\ zx &= (y - x)z - ayx + cx^2. \end{aligned} \tag{$*$}$$

In this case, $A = B[z; \phi, \delta]$ where $B = \Bbbk\langle x, y\rangle / \langle x^2 + y^2 - xy\rangle$ and $\phi \in \operatorname{Aut}(B)$ is given by $\phi(x) = y - x$, $\phi(y) = -x$, and δ is the left ϕ-derivation of B given by $\delta(x) = cx^2 - ayx$, $\delta(y) = cx^2 + ey^2$. Mapping $x \mapsto r_2$ and $y \mapsto r_1 - r_2$ yields that $B \cong \Bbbk\langle r_1, r_2\rangle / \langle r_1 r_2 - r_2 r_1 - r_1^2\rangle$ (since $\operatorname{char}(\Bbbk) = 3$), which is a regular algebra of global dimension two, so, by [6, Theorem 4.2], any algebra with the relations $(*)$ is regular. □

Summarizing our work in this section yields the next result.

Theorem 1.7. *If the point scheme of a quadratic quantum $\mathbb{P}^2$ is reducible or non-reduced, then the algebra is either* (a) *an Ore extension of a regular algebra of global dimension two, or* (b) *a twist, by an automorphism, of a skew polynomial ring, or* (c) *a twist, by a twisting system, of an Ore extension of the polynomial ring on two variables*; *if* $\operatorname{char}(\Bbbk) \neq 3$, *then the algebra is described by* (b) *or* (c)*.*

Proof. Combine Corollary 1.3, Lemma 1.5 and Proposition 1.6. □

If $\operatorname{char}(\Bbbk) \neq 2$, then skew polynomial rings are graded skew Clifford algebras, so, in this setting, Theorem 1.7 may be rephrased as Theorem 1.10 below. We first recall the definition of a graded skew Clifford algebra and of some terms used in its definition. For the definition, we assume $\operatorname{char}(\Bbbk) \neq 2$.

Definition 1.8. (See [5].) For $\{i, j\} \subset \{1, \dots, n\}$, let $\mu_{ij} \in \Bbbk^\times$ satisfy $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$, and write $\mu = (\mu_{ij}) \in M(n, \Bbbk)$. A matrix $M \in M(n, \Bbbk)$ is called *μ-symmetric* if $M_{ij} = \mu_{ij} M_{ji}$ for all $i, j = 1, \dots, n$. Henceforth, suppose $\mu_{ii} = 1$ for all i, and fix μ-symmetric matrices $M_1, \dots, M_n \in M(n, \Bbbk)$. A *graded skew Clifford algebra* associated to μ and $M_1, \dots, M_n$ is a graded $\Bbbk$-algebra on degree-one generators $x_1, \dots, x_n$ and on degree-two generators $y_1, \dots, y_n$ with defining relations given by

(a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$, and
(b) the existence of a normalizing sequence $\{r_1, \dots, r_n\}$ of homogeneous elements of degree two that span $\Bbbk y_1 + \cdots + \Bbbk y_n$.

One should note that if $\mu_{ij} = 1$ for all i, j, and if the y_k are all central in the algebra, then the algebra is a graded Clifford algebra. Moreover, polynomial rings, and skew polynomial rings, on finitely-many generators are graded skew Clifford algebras. Although graded skew Clifford algebras need not be quadratic nor regular in general, a simple geometric criterion was established in [5, Theorem 4.2] for determining when such an algebra is quadratic and regular. We refer the reader to [5,8] for results on graded Clifford algebras and graded skew Clifford algebras.

Example 1.9. Suppose that $\operatorname{char}(\Bbbk) \neq 2$ and that the μ_{ij} are given as in Definition 1.8. Fix $\alpha_1, \alpha_2, \alpha_3 \in \Bbbk$ and define

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & \alpha_1 \\ 0 & \mu_{32}\alpha_1 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 0 & \alpha_2 \\ 0 & 2 & 0 \\ \mu_{31}\alpha_2 & 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & \alpha_3 & 0 \\ \mu_{21}\alpha_3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

These μ-symmetric matrices, with values for α_1, α_2, α_3 and the μ_{ij} given below, yield a regular graded skew Clifford algebra of global dimension three on generators x_1, x_2, x_3 with defining relations

$$x_i x_j + \mu_{ij} x_j x_i = \alpha_k x_k^2,$$

for all distinct i, j, k, with point scheme isomorphic to a subscheme $\mathcal{P}$ (given below) of $\mathbb{P}^2$.

(i) $\alpha_i = 0$ for all i, $\mu_{13} + \mu_{12}\mu_{23} = 0$, $\mathcal{P} = \mathbb{P}^2$;
(ii) $\alpha_i = 0$ for all i, $\mu_{13} + \mu_{12}\mu_{23} \neq 0$, $\mathcal{P} = \mathcal{V}(x_1x_2x_3)$, which is a "triangle";
(iii) $\alpha_1 = 0 = \alpha_2 \neq \alpha_3$, $\mu_{13}\mu_{23} = 1$, $\mu_{13} + \mu_{12}\mu_{23} = 0$, $\mathcal{P} = \mathcal{V}(x_3^3)$, which is a triple line;
(iv) $\alpha_1 = 0 = \alpha_2 \neq \alpha_3$, $\mu_{13}\mu_{23} = 1$, $\mu_{13} + \mu_{12}\mu_{23} \neq 0$, $\mathcal{P} = \mathcal{V}(((\mu_{13} + \mu_{12}\mu_{23})x_1x_2 + \alpha_3 x_3^2)x_3)$, which is the union of a nondegenerate conic and a line;
(v) $\alpha_1 = 0 \neq \alpha_2\alpha_3$, $\mu_{12} = \mu_{23}$, $\mu_{12}^3 = 1$, $\mu_{13} = \mu_{12}^2$, $\mathcal{P} = \mathcal{V}(\alpha_2 x_2^3 + \alpha_3\mu_{13}x_3^3 + 2\mu_{12}x_1x_2x_3)$, which is a nodal cubic curve;
(vi) $\alpha_1\alpha_2\alpha_3 \neq 0$, $\mu_{12} = \mu_{23}$, $\mu_{12}^3 = 1$, $\mu_{13} = \mu_{12}^2$, $\alpha_1\alpha_2\alpha_3 + \mu_{12}^2 \neq 0$, $\mathcal{P} = \mathcal{V}(\alpha_1 x_1^3 + \alpha_2\mu_{12}x_2^3 + \alpha_3 x_3^3 + (2\mu_{12}^2 - \alpha_1\alpha_2\alpha_3)x_1x_2x_3)$, which is an elliptic curve if and only if $\alpha_1\alpha_2\alpha_3 \neq 8\mu_{12}^2$.

The method to find the above values uses [5, Theorem 4.2]. For other values of the α_i that yield a regular algebra, see [7]. This example highlights the wide variety of point schemes that can be obtained directly from regular graded skew Clifford algebras of global dimension three.

Theorem 1.10. *Suppose* $\operatorname{char}(\Bbbk) \neq 2$. *If the point scheme of a quadratic quantum* $\mathbb{P}^2$ *is reducible or non-reduced, then either the algebra is a twist, by an automorphism, of a graded skew Clifford algebra, or the algebra is a twist, by a twisting system, of an Ore extension of a regular graded skew Clifford algebra of global dimension two.*

Proof. This is a restatement of Theorem 1.7 in terms of graded skew Clifford algebras. □

The subsequent sections of the article focus on the case where the cubic divisor C is reduced and irreducible. It is straightforward to prove that if C contains two or more singular points, then C contains a line. Thus, in the remaining sections, C has at most one singular point. Indeed, by Lemma 2.1, C will be either a nodal cubic curve, a cuspidal cubic curve, or an elliptic curve.

2. Quantum $\mathbb{P}^2$s with point scheme a nodal cubic curve

In this section, we classify those quadratic quantum $\mathbb{P}^2$s whose point scheme is a nodal cubic curve in $\mathbb{P}^2$, and prove that, up to isomorphism, there is at most a one-parameter family of such algebras (Theorem 2.2). Moreover, if $\operatorname{char}(\Bbbk) \neq 2$, we show, in Corollary 2.3, that such algebras are Ore extensions of regular graded skew Clifford algebras of global dimension two, and, under certain conditions, are even graded skew Clifford algebras themselves.

Throughout this section, we use x, y and z for homogeneous degree-one linearly independent (commutative) coordinates on $\mathbb{P}^2$. Our next result shows that a nodal cubic curve and a cuspidal cubic curve are the only irreducible cubic divisors in $\mathbb{P}^2$ with a unique singular point; for lack of a suitable reference, we include its simple proof.

Lemma 2.1. *Let C denote an irreducible cubic divisor in $\mathbb{P}^2$ with a unique singular point. Up to isomorphism, $C = \mathcal{V}(f)$, where either* (a) $f = x^3 + y^3 + xyz$, *or* (b) $f = y^3 + x^2z$, *or* (c) $f = y^3 + x^2z + xy^2$; *if* $\operatorname{char}(\Bbbk) \neq 3$, *then f is given by* (a) *or* (b).

Proof. By rechoosing x, y and z if needed, we may assume that $\mathcal{V}(x, y)$ is the unique singular point on C and that $C = \mathcal{V}(f)$, where $f = s_1 + s_2xz$, where $s_1 = \alpha_1x^3 + \alpha_2x^2y + \alpha_3xy^2 + y^3$, $s_2 \in \{x, y\}$ and $\alpha_i \in \Bbbk$ for all i. Moreover, if $s_2 = y$, then $\alpha_1 \neq 0$, as C is irreducible.

If $s_2 = y$, then $f \mapsto x^3 + y^3 + xyz$ by mapping $x \mapsto \beta^{-1}x$, $y \mapsto y$ and $z \mapsto \beta z - \alpha_2\beta^{-1}x - \alpha_3y$, where $\beta \in \Bbbk$ is any solution of the equation $\beta^3 = \alpha_1$.

On the other hand, suppose $s_2 = x$. If $\operatorname{char}(\Bbbk) \neq 3$, then $f \mapsto y^3 + x^2z$ by mapping $x \mapsto x$, $y \mapsto y - (\frac{\alpha_3}{3})x$ and $z \mapsto z + (\frac{\alpha_2\alpha_3}{3} - \alpha_1 - \frac{2\alpha_3^3}{27})x + (\frac{\alpha_3^2}{3} - \alpha_2)y$. If $\operatorname{char}(\Bbbk) = 3$ and $\alpha_3 = 0$, then $f \mapsto y^3 + x^2z$ by mapping $x \mapsto x$, $y \mapsto y$ and $z \mapsto z - \alpha_1x - \alpha_2y$. If $\operatorname{char}(\Bbbk) = 3$ and $\alpha_3 \neq 0$, then $f \mapsto y^3 + x^2z + xy^2$ by mapping $x \mapsto \alpha_3^{-1}x$, $y \mapsto y$ and $z \mapsto \alpha_3^2z - \alpha_1\alpha_3^{-1}x - \alpha_2y$. $\square$

In Lemma 2.1, if C is given by (a), we refer to C as a nodal cubic curve; otherwise, we refer to C as a cuspidal cubic curve. For the rest of this section, C denotes a nodal cubic curve.

In order to classify those quadratic quantum $\mathbb{P}^2$ whose point scheme is given by a nodal cubic curve C in $\mathbb{P}^2$, we will classify all such algebras whose defining relations vanish on the graph of an automorphism of C (see Remark 1.4(ii)).

Theorem 2.2. *Let A denote a quadratic quantum $\mathbb{P}^2$. If the point scheme of A is a nodal cubic curve in $\mathbb{P}^2$, then A is isomorphic to a quadratic algebra on three generators x_1, x_2, x_3 with defining relations*:

$$\lambda x_1x_2 = x_2x_1, \qquad \lambda x_2x_3 = x_3x_2 - x_1^2, \qquad \lambda x_3x_1 = x_1x_3 - x_2^2,$$

where $\lambda \in \Bbbk$ and $\lambda(\lambda^3 - 1) \neq 0$. Moreover, for any such λ, any quadratic algebra with these defining relations is a quadratic quantum $\mathbb{P}^2$ with point scheme given by a nodal cubic curve in $\mathbb{P}^2$.

Proof. By Lemma 2.1, we may write $C = \mathcal{V}(x^3 + y^3 + xyz)$ for the nodal cubic curve. It follows that

$$C = \{(a^2, a, -a^3 - 1): a \in \Bbbk\},$$

and the unique singular point of C is $p = (0, 0, 1)$. Thus if $\sigma \in \operatorname{Aut}(C)$, then

$$\sigma(a^2, a, -a^3 - 1) = (f(a)^2, f(a), -f(a)^3 - 1),$$

for all $a \in \Bbbk$, where f is a rational function of one variable. Since σ is invertible, f is also, so $f(a) = (\lambda_1a + \lambda_2)/(\lambda_3a + \lambda_4)$, where $\lambda_i \in \Bbbk$ for all i. However, $\sigma(p) = p$ implies that $f(0) = 0$, and the domain of f is $\Bbbk$, so $\lambda_2 = 0 = \lambda_3$. Thus, there exists $\lambda \in \Bbbk^\times$ such that $f(a) = \lambda a$ for all $a \in \Bbbk$. It follows that

$$\sigma(a^2, a, -a^3 - 1) = (\lambda^2a^2, \lambda a, -\lambda^3a^3 - 1),$$

for all $a \in \Bbbk$. Since σ may be extended to $\mathbb{P}^2$ if $\lambda^3 = 1$, by Remark 1.4(iii), we may assume $\lambda^3 \neq 1$.

Let $\{x_1, x_2, x_3\}$ be a basis for A_1, let $\{z_1, z_2, z_3\}$ be the dual basis for A_1^*, and let W and $W^\perp$ be as in Remark 1.4(i). We may produce some elements of $W^\perp$ from the graph of σ as follows. Firstly, suppose that $\operatorname{char}(\Bbbk) \neq 3$ and fix $\omega \in \Bbbk$ such that $\omega^2 - \omega + 1 = 0$. This yields the following six elements in $W^\perp$ corresponding to the given distinct values of a:

$$\begin{aligned}
a=0\colon &\quad z_3^2,\\
a=-1\colon &\quad (z_1-z_2)\big(\lambda^2 z_1-\lambda z_2+\big(\lambda^3-1\big)z_3\big),\\
a=\omega\colon &\quad \omega(\omega z_1+z_2)\big(\lambda^2\omega^2 z_1+\lambda\omega z_2+\big(\lambda^3-1\big)z_3\big),\\
a=-\lambda^{-1}\colon &\quad \big(\lambda^{-2}z_1-\lambda^{-1}z_2-\big(1-\lambda^{-3}\big)z_3\big)(z_1-z_2),\\
a=\omega\lambda^{-1}\colon &\quad \big(\omega^2\lambda^{-2}z_1+\omega\lambda^{-1}z_2-\big(1-\lambda^{-3}\big)z_3\big)(\omega z_1+z_2)\omega,\\
a=-\omega^2\colon &\quad -\omega(z_1+\omega z_2)\big(-\lambda^2\omega z_1-\lambda\omega^2 z_2+\big(\lambda^3-1\big)z_3\big).
\end{aligned}$$

Taking linear combinations of these six elements yields the following basis for $W^{\perp}$:

$$\begin{aligned}
&z_3^2, && z_1z_2+\lambda z_2z_1,\\
&\lambda z_2^2+\big(\lambda^3-1\big)z_1z_3, && z_2z_3+\lambda z_3z_2,\\
&\lambda z_1^2+\big(\lambda^3-1\big)z_3z_2, && z_3z_1+\lambda z_1z_3.
\end{aligned}$$

It follows that W is the span of the elements:

$$\lambda x_1x_2-x_2x_1,\qquad \lambda(\lambda x_2x_3-x_3x_2)+\big(\lambda^3-1\big)x_1^2,\qquad \lambda(\lambda x_3x_1-x_1x_3)+\big(\lambda^3-1\big)x_2^2,$$

if $\operatorname{char}(\Bbbk)\neq 3$. Moreover, these three linearly independent elements vanish on the graph of σ even if $\operatorname{char}(\Bbbk)=3$, so W is the span of these three elements even in this case. Furthermore, since $\lambda(\lambda^3-1)\neq 0$, we may map $x_1\mapsto x_1$, $x_2\mapsto x_2$ and $x_3\mapsto \lambda^{-1}(\lambda^3-1)x_3$, so A is isomorphic to the algebra in the statement of the theorem.

If $\lambda\in\Bbbk$ where $\lambda(\lambda^3-1)\neq 0$, then an algebra with the given relations is an Ore extension $B[x_3;\phi,\delta]$ of the algebra $B=\Bbbk\langle x_1,x_2\rangle/\langle\lambda x_1x_2-x_2x_1\rangle$ using $\phi\in\operatorname{Aut}(B)$ and δ a left ϕ-derivation of B where

$$\phi(x_1)=\lambda^{-1}x_1,\qquad \phi(x_2)=\lambda x_2,\qquad \delta(x_1)=-\lambda^{-1}x_2^2,\qquad \delta(x_2)=x_1^2.$$

Since B is a regular algebra of global dimension two, it follows, by [6, Theorem 4.2], that such an Ore extension of B is a regular algebra of global dimension three.

The point scheme of the algebra with the defining relations in the theorem is given by $\mathcal{V}(\lambda x^3+\lambda y^3+(\lambda^3-1)xyz)$, which is indeed a nodal cubic curve. □

Corollary 2.3. *Suppose* $\operatorname{char}(\Bbbk)\neq 2$. *If* $\lambda^3=-1$, *then the algebra in Theorem* 2.2 *is a graded skew Clifford algebra*; *if* $\lambda^3\notin\{0,1\}$, *then the algebra is an Ore extension of a regular graded skew Clifford algebra.*

Proof. Let S denote the quadratic algebra on generators z_1, z_2 and z_3 with defining relations

$$z_1z_2+\lambda z_2z_1=0,\qquad z_2z_3+\lambda z_3z_2=0,\qquad z_3z_1+\lambda z_1z_3=0.$$

If $\lambda^3=-1$, then the set $X=\{z_3^2, z_2^2+z_1z_3, z_1^2+z_3z_2\}$ is a normalizing sequence in S. In the free algebra on z_1, z_2, z_3, let Y denote the span of the defining relations of S, and let $\hat{X}$ denote the span of any preimage of X. The zero locus in $\mathbb{P}^2\times\mathbb{P}^2$ of $\hat{X}+Y$ is the empty set. By [5, Theorem 4.2], since $\operatorname{char}(\Bbbk)\neq 2$, it follows that the Koszul dual of $S/\langle X\rangle$ is a regular graded skew Clifford algebra; by construction, this algebra is isomorphic to the algebra in Theorem 2.2.

If $\lambda(\lambda^3-1)\neq 0$, then the proof of Theorem 2.2 shows that the algebra therein is an Ore extension of a regular algebra B of global dimension two, and B is a graded skew Clifford algebra by [5, Corollary 4.3]. □

3. Quantum $\mathbb{P}^2$s with point scheme a cuspidal cubic curve

In this section, we prove that, up to isomorphism, there is a unique quadratic quantum $\mathbb{P}^2$ whose point scheme is a cuspidal cubic curve in $\mathbb{P}^2$ (Theorem 3.1). Moreover, this algebra exists if and only if $\operatorname{char}(\Bbbk) \neq 3$. We also prove in Corollary 3.2 that, if $\operatorname{char}(\Bbbk) \neq 2$, then such an algebra is an Ore extension of a regular graded skew Clifford algebra of global dimension two.

As in Section 2, we continue to use x, y and z for homogeneous degree-one linearly independent (commutative) coordinates on $\mathbb{P}^2$. By Lemma 2.1, we may assume that the cuspidal cubic curve is given by $C = \mathcal{V}(y^3 + x^2z)$ or $C = \mathcal{V}(y^3 + x^2z + xy^2)$, with the second case occurring only if $\operatorname{char}(\Bbbk) = 3$.

Theorem 3.1. *Let A denote a quadratic quantum $\mathbb{P}^2$. If $\operatorname{char}(\Bbbk) = 3$, then the point scheme of A is not a cuspidal cubic curve in $\mathbb{P}^2$. If $\operatorname{char}(\Bbbk) \neq 3$ and if the point scheme of A is a cuspidal cubic curve in $\mathbb{P}^2$, then A is isomorphic to a quadratic algebra on three generators x_1, x_2, x_3 with defining relations*:

$$x_1x_2 = x_2x_1 + x_1^2, \qquad x_3x_1 = x_1x_3 + x_1^2 + 3x_2^2, \qquad x_3x_2 = x_2x_3 - 3x_2^2 - 2x_1x_3 - 2x_1x_2.$$

Moreover, any quadratic algebra with these defining relations is a quadratic quantum $\mathbb{P}^2$; it has point scheme given by a cuspidal cubic curve in $\mathbb{P}^2$ if and only if $\operatorname{char}(\Bbbk) \neq 3$.

Proof. Suppose first that the cuspidal cubic curve is $C = \mathcal{V}(y^3 + x^2z)$. It follows that

$$C = \{(0,0,1)\} \cup \{(1, b, -b^3)\colon b \in \Bbbk\},$$

and that the unique singular point of C is $p = (0,0,1)$. Thus, if $\sigma \in \operatorname{Aut}(C)$, then $\sigma(p) = p$ and

$$\sigma(1, b, -b^3) = (1, f(b), -f(b)^3),$$

for all $b \in \Bbbk$, where f is a rational function of one variable. Since σ is invertible, f is also, so $f(b) = (\lambda_1 b + \lambda_2)/(\lambda_3 b + \lambda_4)$, where $\lambda_i \in \Bbbk$ for all i. However, the domain of f is $\Bbbk$, so $\lambda_3 = 0$ and $\lambda_1, \lambda_4 \in \Bbbk^\times$. Writing the points of C in the form $(a^3, a^2b, -b^3)$ for all $(a,b) \in \mathbb{P}^1$ and rechoosing the λ_i, we may write

$$\sigma(a^3, a^2b, -b^3) = (a^3, \lambda_1 a^2(b + \lambda_2 a), -\lambda_1^3(b + \lambda_2 a)^3)$$

for all $(a,b) \in \mathbb{P}^1$, where $\lambda_1 \in \Bbbk^\times$ and $\lambda_2 \in \Bbbk$. If $\lambda_2 = 0$ or if $\operatorname{char}(\Bbbk) = 3$, then σ may be extended to $\mathbb{P}^2$, so, by Remark 1.4(iii), we may assume $\lambda_2 \neq 0$ and $\operatorname{char}(\Bbbk) \neq 3$.

Using the method and notation in the proof of Theorem 3.1, we find that $W^\perp$ has basis:

$$\begin{aligned}
&z_3^2, && z_2^2 - 3\lambda_1^2\lambda_2 z_1z_3 - 3\lambda_1^2\lambda_2^2 z_2z_3,\\
&z_3z_2 + \lambda_1^2 z_2z_3, && z_3z_1 + \lambda_1^3 z_1z_3 + 2\lambda_1^3\lambda_2 z_2z_3,\\
&z_1^2 + \lambda_1\lambda_2 z_1z_2 - \lambda_1^3\lambda_2^3 z_1z_3, && z_2z_1 + \lambda_1 z_1z_2 + 2\lambda_1^3\lambda_2^3 z_2z_3.
\end{aligned}$$

(Alternatively, the reader may simply verify that the dual elements to these elements vanish on the graph of σ.) Mapping $z_1 \mapsto z_1$, $z_2 \mapsto z_2/\lambda_2$ and $z_3 \mapsto z_3/\lambda_2^3$ allows us to take $\lambda_2 = 1$. It follows that the Hilbert series of the Koszul dual of A is $H(t) = (1+t)^3$ if and only if $\lambda_1 = 1$ (since $\operatorname{char}(\Bbbk) \neq 3$). If $\lambda_1 = 1$, then A is the algebra given in the statement of the theorem, where $\{x_1, x_2, x_3\}$ is the dual basis to $\{z_1, z_2, z_3\}$.

To prove the relations in the statement determine a regular algebra, we write the algebra as an Ore extension of the regular algebra $B = \Bbbk\langle x_1, x_2\rangle/\langle x_1x_2 - x_2x_1 - x_1^2\rangle$ using $\phi \in \operatorname{Aut}(B)$ and δ a left ϕ-derivation of B where

$$\phi(x_1) = x_1, \qquad \phi(x_2) = x_2 - 2x_1, \qquad \delta(x_1) = x_1^2 + 3x_2^2, \qquad \delta(x_2) = -2x_1x_2 - 3x_2^2.$$

It follows, from [6, Theorem 4.2], that such an Ore extension of B is a regular algebra of global dimension three. The point scheme of such an Ore extension is given by $\mathcal{V}(3(y^3 + x^2z))$, which is a cuspidal cubic curve if and only if $\mathrm{char}(\Bbbk) \neq 3$.

By Lemma 2.1, we henceforth assume $C = \mathcal{V}(y^3 + x^2z + xy^2)$ and $\mathrm{char}(\Bbbk) = 3$. It follows that

$$C = \{(0, 0, 1)\} \cup \{(1, b, -b^2 - b^3): b \in \Bbbk\},$$

and that the unique singular point of C is $p = (0, 0, 1)$. In this setting, if $\sigma \in \mathrm{Aut}(C)$, then $\sigma(p) = p$ and

$$\sigma(1, b, -b^2 - b^3) = (1, f(b), -f(b)^2 - f(b)^3),$$

for all $b \in \Bbbk$, where f is a rational function of one variable. As before, the invertibility of σ implies that f is invertible, and that $f(b) = (\lambda_1 b + \lambda_2)/(\lambda_3 b + \lambda_4)$, where $\lambda_i \in \Bbbk$ for all i. Since the domain of f is $\Bbbk$, $\lambda_3 = 0$ and $\lambda_1, \lambda_4 \in \Bbbk^\times$. Thus, writing the points of C in the form $(a^3, a^2b, -ab^2 - b^3)$ for all $(a, b) \in \mathbb{P}^1$ and rechoosing the λ_i, we may write

$$\sigma(a^3, a^2b, -ab^2 - b^3) = (a^3, a^2(\lambda_1 b + \lambda_2 a), -a(\lambda_1 b + \lambda_2 a)^2 - (\lambda_1 b + \lambda_2 a)^3)$$

for all $(a, b) \in \mathbb{P}^1$, where $\lambda_1 \in \Bbbk^\times$ and $\lambda_2 \in \Bbbk$. By Remark 1.4(iii), we further assume that $\lambda_1 \neq 1$, since $\lambda_1 = 1$ if and only if σ may be extended to $\mathbb{P}^2$ (since $\mathrm{char}(\Bbbk) = 3$).

Using the notation in the proof of Theorem 3.1, and using x_1, x_2, x_3 as generators for A, and by seeking homogeneous degree-two elements that vanish on the graph of σ, we find that a basis for W is

$$\begin{aligned}
&x_1x_2 - \lambda_1 x_2x_1 - \lambda_2 x_1^2,\\
&x_1x_3 - \lambda_1^3 x_3x_1 + \lambda_1(1 - \lambda_1)x_2^2 + \lambda_1\lambda_2(1 + \lambda_1)x_2x_1 + \lambda_2^2(1 + \lambda_2)x_1^2,\\
&x_2x_3 - \lambda_1^2 x_3x_2 + \lambda_1^2(\lambda_1 + \lambda_2 - 1)x_3x_1 + (\lambda_1^2 - \lambda_1 + 2\lambda_2)x_2^2 + \lambda_2(\lambda_2^2 - \lambda_2 - \lambda_1^2 + \lambda_1)x_2x_1.
\end{aligned}$$

Since $\lambda_1(\lambda_1 - 1) \neq 0$, it follows that the Hilbert series of A is $H(t) = 1 + 3t + 6t^2 + 9t^3 + \cdots$, which contradicts A being a quadratic quantum $\mathbb{P}^2$. □

Corollary 3.2. *If* $\mathrm{char}(\Bbbk) \neq 2$, *then the algebra in Theorem* 3.1 *is an Ore extension of a regular graded skew Clifford algebra.*

Proof. The proof of Theorem 3.1 shows that the algebra is an Ore extension of a regular algebra B of global dimension two, and B is a graded skew Clifford algebra by [5, Corollary 4.3]. □

4. Quantum $\mathbb{P}^2$s with point scheme an elliptic curve

It remains to consider quadratic quantum $\mathbb{P}^2$s with point scheme an elliptic curve. In [2], such algebras are classified into four types, A, B, E and H, where some members of each type might not have an elliptic curve as their point scheme, but a generic member does. We show, in Lemmas 4.1, 4.2 and 4.3, that all regular algebras of types B and H that have point scheme an elliptic curve, and some regular algebras of type A that have point scheme an elliptic curve are given by graded skew Clifford algebras, or twists thereof. Up to isomorphism and anti-isomorphism, type E consists of at most one algebra and it appears not to be directly related to a graded skew Clifford algebra (but this issue is still open), so this type is only discussed in Remark 4.4 regarding a weak relationship to a graded skew Clifford algebra.

4.1. Type H

By [2, p. 207], there are at most two regular algebras of type H (up to isomorphism) and they are given by $\Bbbk$-algebras on generators x, y, z with defining relations:

$$y^2 = x^2, \qquad zy = -iyz, \qquad yx - xy = iz^2,$$

where i is a primitive fourth root of unity. In the following result, such an algebra is denoted H; its point scheme is an elliptic curve unless $\operatorname{char}(\Bbbk) = 2$.

Lemma 4.1. *If* $\operatorname{char}(\Bbbk) \neq 2$*, then the algebra H is a regular graded skew Clifford algebra and a twist of a graded Clifford algebra by an automorphism.*

Proof. Suppose $\operatorname{char}(\Bbbk) \neq 2$, and let H_1^* have basis $\{X, Y, Z\}$ dual to $\{x, y, z\}$. Let S denote the $\Bbbk$-algebra on X, Y, Z with defining relations:

$$YX = -XY, \qquad YZ = iZY, \qquad ZX = \nu XZ,$$

where $\nu \in \Bbbk^\times$. For all $\nu \in \Bbbk^\times$, the Koszul dual $H^\star$ to H is the quotient of S by the ideal spanned by the normalizing sequence $\{XZ, iXY - Z^2, X^2 + Y^2\}$. The defining relations of $H^\star$ have empty zero locus in $\mathbb{P}^2 \times \mathbb{P}^2$. By [5, Theorem 4.2], H is a regular graded skew Clifford algebra. If, further, we choose $\nu = i$, then S is a twist of a polynomial ring by an automorphism, so, by [5, Proposition 4.5], H is a twist of a graded Clifford algebra by an automorphism. □

4.2. Type B

By [2, p. 207], the regular algebras of type B that have point scheme an elliptic curve are given by $\Bbbk$-algebras on generators x, y, z with defining relations:

$$xy + yx = z^2 - y^2, \qquad xy + yx = az^2 - x^2, \qquad zx - xz = a(yz - zy),$$

where $a \in \Bbbk$, $a(a-1) \neq 0$. (A sign error in the first relation on p. 207 of [2] has been corrected above.) In the following result, such an algebra is denoted B; its point scheme is an elliptic curve for generic values of a unless $\operatorname{char}(\Bbbk) \in \{2, 3\}$.

Lemma 4.2. *If* $\operatorname{char}(\Bbbk) \neq 2$*, then the algebra B is regular if and only if* $a^2 - a + 1 \neq 0$*; in this case, B is a graded skew Clifford algebra and a twist of a graded Clifford algebra by an automorphism.*

Proof. Suppose $\operatorname{char}(\Bbbk) \neq 2$, and let B_1^* have basis $\{X, Y, Z\}$ dual to $\{x, y, z\}$. Let S denote the $\Bbbk$-algebra on X, Y, Z with defining relations:

$$YX = XY, \qquad YZ = -ZY, \qquad ZX = -XZ.$$

The Koszul dual $B^\star$ to B is the quotient of S by the ideal spanned by the normalizing sequence $\{Z(aX - Y), X^2 + Y^2 - XY, aX^2 + Y^2 + Z^2\}$. The defining relations of $B^\star$ have empty zero locus in $\mathbb{P}^2 \times \mathbb{P}^2$ if and only if $a^2 - a + 1 \neq 0$. In fact, if $a^2 - a + 1 = 0$, then $\dim_\Bbbk(B^\star)$ is infinite, and so B is not regular. On the other hand, suppose $a^2 - a + 1 \neq 0$. It follows from [5, Theorem 4.2] that B is a regular graded skew Clifford algebra, and, by [5, Proposition 4.5], that B is a twist of a graded Clifford algebra by an automorphism, since S is a twist of a polynomial ring by an automorphism. □

4.3. Type A

By [2, p. 207], the regular algebras of type A that have point scheme an elliptic curve are given by $\Bbbk$-algebras on generators x, y, z with defining relations:

$$axy + byx + cz^2 = 0, \qquad ayz + bzy + cx^2 = 0, \qquad azx + bxz + cy^2 = 0, \tag{$*$}$$

where $a, b, c \in \Bbbk$. We denote such an algebra by A'; by [3], its point scheme is an elliptic curve if and only if $abc \neq 0$, $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ and $\operatorname{char}(\Bbbk) \neq 3$. Thus, we assume $abc \neq 0$ and $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ and, with these assumptions, A' is regular unless $a^3 = b^3 = c^3$ [3].

Lemma 4.3. *Suppose* $\operatorname{char}(\Bbbk) \neq 2$. *If $a^3 = b^3 \neq c^3$, then A' is a regular graded skew Clifford algebra and a twist of a graded Clifford algebra by an automorphism. If $b^3 = c^3 \neq a^3$ or if $a^3 = c^3 \neq b^3$, then A' is a twist of a regular graded skew Clifford algebra by an automorphism.*

Proof. Define $\tau \in \operatorname{Aut}(A')$ by $\tau(x) = y$, $\tau(y) = z$ and $\tau(z) = x$. Twisting A' by τ (respectively, by τ^2) yields an algebra on x, y, z with the same defining relations as in $(*)$ except that a, b and c have been cyclically permuted one place (respectively, two places) to the left. Thus, the second part of the result follows from the first part, so it remains to prove the first part.

Let $(A'_1)^*$ have basis $\{X, Y, Z\}$ dual to $\{x, y, z\}$ and let S denote the $\Bbbk$-algebra on X, Y, Z with defining relations:

$$aYX = bXY, \qquad aZY = bYZ, \qquad aXZ = bZX.$$

Suppose $a^3 = b^3$. In this case, $\{cXY - aZ^2, cYZ - aX^2, cZX - aY^2\}$ is a normalizing sequence in S, and the Koszul dual $(A')^\star$ to A' is the quotient of S by the ideal spanned by this normalizing sequence. The defining relations of $(A')^\star$ have empty zero locus in $\mathbb{P}^2 \times \mathbb{P}^2$ if $a^3 \neq c^3$. By [5, Theorem 4.2], it follows that A' is a regular graded skew Clifford algebra if $a^3 \neq c^3$. By [5, Proposition 4.5], B is a twist of a graded Clifford algebra, since S is a twist of a polynomial ring by an automorphism (since $a^3 = b^3$). $\square$

If $abc \neq 0$ and $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ and $a^3 \neq b^3 \neq c^3 \neq a^3$, then it is still open whether or not A' is directly related to a graded skew Clifford algebra.

Remark 4.4. If $\tilde{A}$ is an algebra of type A or E, then the Koszul dual of $\tilde{A}$ is the quotient of a regular graded skew Clifford algebra S (indeed, S is a skew polynomial ring). So, in this sense, such algebras are weakly related to graded skew Clifford algebras.

Question 4.5. Can the results of this article be generalized to quadratic regular algebras of global dimension four, thereby possibly enabling the classification of such algebras by using regular graded skew Clifford algebras?

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