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Notes on Hilbert and Cauchy matrices ☆

Miroslav Fiedler

Academy of Sciences of the Czech Republic, Inst. of Computer Science, Pod vodáren. věží 2, 182 07 Praha 8, Czech Republic

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ABSTRACT

Inspired by examples of small Hilbert matrices, the author proves a property of symmetric totally positive Cauchy matrices, called AT-property, and consequences for the Hilbert matrix.

1. Introduction

As is well known [5], a *Cauchy matrix* (maybe even not square) is an $m \times n$ matrix assigned to $m+n$ parameters $x_1, \ldots, x_m, y_1, \ldots, y_n$ (one of them can be considered as superfluous) as follows:

$$C = \left[\frac{1}{x_i + y_j}\right], \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

For *generalized Cauchy matrices*, additional parameters $u_1, \ldots, u_m, v_1, \ldots, v_n$, have to be considered (one of which again superfluous):

$$\widehat{C} = \left[\frac{u_i v_j}{x_i + y_j}\right].$$

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E-mail address: fiedler@math.cas.cz

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If we restrict ourselves to the square case, it is well known that C is nonsingular if and only if, in addition to the general existence assumption that $x_i + y_j \neq 0$ for all i and j, the x_i's are mutually distinct as well as the y_j's are mutually distinct. In fact, there is a formula [1,4] for the determinant of $C(m = n)$

$$\det C = \frac{\prod_{i,k,i>k}(x_i - x_k)(y_i - y_k)}{\prod_{i,j=1}^{n}(x_i + y_j)}. \tag{1}$$

Clearly, such formula is easily established also for the generalized Cauchy matrix $\widehat{C}$.

Since every submatrix of a Cauchy matrix is also a Cauchy matrix, the formula (1) allows us to find the inverse matrix to C. Thus $C^{-1} = [\gamma_{ij}]$, where

$$\gamma_{ij} = (x_j + y_i)\frac{\prod_{\ell \neq i}(x_j + y_\ell)\prod_{k \neq j}(y_i + x_k)}{\prod_{\ell \neq j}(x_j - x_\ell)\prod_{k \neq i}(y_i - y_k)}. \tag{2}$$

In this note, we shall be interested in real symmetric Cauchy matrices, in particular in the positive definite and totally positive case, and in the matrix $C \circ C^{-1}$, where $\circ$ means the Hadamard entrywise product. Recall that a real matrix is totally positive if all its submatrices have positive determinant.

A simple corollary of Eq. (2) is

Theorem A. *A symmetric Cauchy matrix (for which $y_i = x_i$ for each i)*

$$C = \left[\frac{1}{x_i + x_j}\right]$$

is positive definite if and only if all the x_i's are positive and mutually distinct.

It is totally positive if and only if either $0 < x_1 < \cdots < x_n$, or $0 < x_n < \cdots < x_1$.

Corollary B. *If C is a positive definite Cauchy matrix then there exists a permutation matrix P for which PCP is totally positive.*

Let us mention that the famous Hilbert matrix (e.g. [1]) (more precisely, the finite section thereof)

$$H_n = \left[\frac{1}{i + j - 1}\right]$$

is clearly a Cauchy matrix.

In fact, the results for small Hilbert matrices were an inspiration for the author to present this note. The second inspiration was the following notion.

If A is a nonsingular matrix, then it makes sense to define the Hadamard product $A \circ (A^T)^{-1}$. We shall call it the *combined matrix of A*.

Let us recall three well known properties of the combined matrices.

Theorem C. *All row sums of every combined matrix are equal to one. The combined matrices of a nonsingular matrix A and $(A^T)^{-1}$ coincide. Multiplication of a nonsingular matrix by nonsingular diagonal matrices from any side does not change the combined matrix.*

A less known property was presented in [2]:

Theorem D. *Let $A = [a_{ij}]$ be a real symmetric positive definite matrix, let $A^{-1} = [\alpha_{ij}]$. Then the combined matrix $M = A \circ A^{-1}$ of A with entries m_{ij} has the following properties:*

1. *$M - I$ is positive semidefinite, $Me = e$; here, I is the identity matrix and e is the vector of all ones.*
2. $$2\max_i\left(\sqrt{m_{ii}} - 1\right) \leqslant \sum_i\left(\sqrt{m_{ii}} - 1\right). \tag{3}$$

Remark 1. It seems still an open problem to characterize the set of all combined matrices of $n \times n$ positive definite matrices. For $n \leqslant 3$, 1. and 2. give a complete characterization [4].

Remark 2. It is easy to see that the combined matrix of an M-matrix as well as of an inverse M-matrix is an M-matrix. The possible diagonal entries of such matrices were characterized in [3].

The characterization is similar to that in 2, above: For each i, $m_{ii} \geqslant 1$, and

$$2 \max_i (m_{ii} - 1) \leqslant \sum_i (m_{ii} - 1).$$

In the sequel, we shall use the following two identities:

Lemma E. *Let for $n \geqslant 2, x_1, \dots, x_n$ be indeterminates. Then the following holds:*

1. *If n is even, then*

$$\sum_{k=1}^{n} \prod_{j \neq k} \frac{x_k + x_j}{x_k - x_j} = 0 \tag{4}$$

identically.

2. *If n is odd, then*

$$\sum_{k=1}^{n} \prod_{j \neq k} \frac{x_k + x_j}{x_k - x_j} = 1 \tag{5}$$

identically.

Proof. One can use the Lagrange identities, but we shall apply a direct proof.

Multiply the left-hand side of (4) by $\prod_{i>j}(x_i - x_j)$. We obtain a homogeneous polynomial of degree $\binom{n}{2}$. It is easily seen that this polynomial is divisible by each $x_i - x_j$, $i \neq j$, thus by $\prod_{i>j}(x_i - x_j)$, of degree $\binom{n}{2}$ again. The left-hand side of (4) is thus an integral constant. To determine it, consider the term $x_n^{n-1} x_{n-1}^{n-2} \cdots x_2$ of the highest weight of indices. For n even, we get zero, for n odd, we get one. $\square$

2. Results

We first introduce a new notion which seems to be rather artificial.

Let $G = [g_{ik}]$ be an $n \times n$ square matrix with nonnegative diagonal entries. We say that G has the *alternate trace property*, shortly AT-property, if

$$\sum_{k=1}^{n} (-1)^{k-1} \sqrt{g_{kk}} = \begin{cases} 1 & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}. \end{cases}$$

Observe that the identity matrix as well as any combined matrix of a nonsingular diagonal matrix have the AT-property. Our main task will be the following result:

Theorem 2.1. *The combined matrix of every symmetric totally positive Cauchy matrix has the AT-property.*

Proof. Suppose first that the Cauchy matrix C corresponds to the n-tuple x_k satisfying $0 < x_1 < \cdots < x_n$ according to Theorem A. Then the formulae (2) yield for the diagonal entries m_{ii} of $C \circ C^{-1}$, due to positivity,

$$\sqrt{m_{ii}} = (-1)^{n-i} \prod_{k \neq i} \frac{x_i + x_k}{x_i - x_k}.$$

The AT-property then follows immediately from Lemma E.

If C corresponds to positive x_i's in reverse order than in Theorem A, the result follows from the fact that the matrix JCJ, where J is the skew identity matrix, has the same AT-property as C. $\square$

Remark 3. By Corollary B, the assumption that C is totally positive can be removed; of course, the corresponding property would be more complicated.

Remark 4. By Theorem C, the same assertion as in Theorem 2.1 holds for totally positive generalized Cauchy matrices.

Theorem 2.2. *The combined matrix of every principal submatrix of the Hilbert matrix has the AT-property.*

In addition, if a square submatrix of the Hilbert matrix has consecutive rows and consecutive columns, then its inverse as well as its combined matrix have integral entries. The diagonal entries of the combined matrix are squares of integers.

Proof. The first part is a corollary to Theorem 2.1 since any principal submatrix of the Hilbert matrix is a totally positive Cauchy matrix. To prove the second part, observe that by (2), it suffices to show that in the case of consecutive rows and columns and the substitution of the corresponding integers, the ratio

$$\frac{\prod_{\ell\neq i}(x_j+y_\ell)\prod_{k\neq j}(y_i+x_k)}{\prod_{\ell\neq j}(x_j-x_\ell)\prod_{k\neq i}(y_i-y_k)}$$

is an integer.

Change in the denominator the summation index ℓ to k and k to ℓ. Then the whole ratio can be written as the product of four ratios

$$\prod_{k<j}\frac{y_i+x_k}{x_j-x_k}\prod_{k>j}\frac{y_i+x_k}{x_j-x_k}\prod_{\ell<i}\frac{x_j+y_\ell}{y_i-y_\ell}\prod_{\ell>i}\frac{x_j+y_\ell}{y_i-y_\ell}.$$

Each of the ratios is an integer since the numerators are consecutive integers. In the case that $i=j$ and $x_k=y_k$ for each k, two and two of the above ratios coincide. $\square$

Let us add an alternative proof of Theorem 2.1 which makes the AT-property more understandable.

Observation 1. Let C be a nonsingular Cauchy matrix. Then there exist diagonal nonsingular matrices D_1 and D_2, such that

$$C^{-1}=D_1C^TD_2. \tag{6}$$

Proof. In the notation of (2), we can rewrite (2) in the form

$$\begin{aligned}\gamma_{ij}&=(x_j+y_i)\frac{\prod_{\ell\neq i}(x_j+y_\ell)\prod_{k\neq j}(y_i+x_k)}{\prod_{\ell\neq j}(x_j-x_\ell)\prod_{k\neq i}(y_i-y_k)}\\&=\frac{1}{x_j+y_i}U_jV_i,\end{aligned}$$

where

$$\begin{aligned}U_j&=(x_j+y_j)\prod_{k\neq j}\frac{x_j+y_k}{x_j-x_k},\\V_i&=(x_i+y_i)\prod_{k\neq i}\frac{y_i+x_k}{y_i-y_k}.\end{aligned}$$

The nonsingular matrices $D_1=\operatorname{diag}(V_1,\dots,V_n)$, $D_2=\operatorname{diag}(U_1,\dots,U_n)$ fulfil then (6). $\square$

Observation 2. A matrix Q satisfying diag $Q = \text{diag}Q^{-1}$ has the AT-property if and only if the matrix QS has the trace property, i.e.

$$\text{tr}QS = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even,} \end{cases}$$

where $S = \text{diag}(1, -1, 1, \ldots, (-1)^{n-1})$.

Proof. Indeed, if $q_{11}, q_{22}, \ldots, q_{nn}$ are the common diagonal entries of Q and Q^{-1}, then $q_{11}, -q_{22}, \ldots, (-1)^{n-1}q_{nn}$ are the diagonal entries of QS.

We complete the proof that C has the AT-property. Let C be a symmetric totally positive Cauchy matrix. By Observation 1 and the symmetry of C, there exists a nonsingular diagonal matrix D_0 such that $C^{-1} = D_0CD_0$. It is well known that the inverse of a nonsingular totally positive matrix has the checkerboard sign-pattern. Therefore, there exists a diagonal matrix D with positive diagonal entries, such that

$$C^{-1} = DSCSD.$$

The matrix $Q = D^{\frac{1}{2}}CD^{\frac{1}{2}}$ has thus the property that

$$(QS)^{-1} = QS,$$

i.e., QS is involutory. Therefore, the eigenvalues of QS are 1 and -1 only. On the other hand, Q is positive definite, so that the matrix $Q^{\frac{1}{2}}SQ^{\frac{1}{2}}$ which has the same eigenvalues as QS is congruent to S. Thus the eigenvalues of QS are 1 and -1 with the same multiplicity if n is even, the multiplicity of 1 being greater by one if n is odd. Also, the diagonal entries of Q and its inverse coincide. By Observation 2, Q has the AT-property. The fact that $Q \circ Q^{-1} = C \circ C^{-1}$ now completes the proof. It also shows the relationship with the involutory property of QS. $\square$

Example. The submatrix

$$G = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

of the Hilbert matrix has the inverse

$$\begin{bmatrix} 300 & -900 & 630 \\ -900 & 2880 & -2100 \\ 630 & -2100 & 1575 \end{bmatrix}.$$

Thus

$$G \circ G^{-1} = \begin{bmatrix} 100 & -225 & 126 \\ -225 & 576 & -350 \\ 126 & -350 & 225 \end{bmatrix}$$

is an integral matrix. It clearly has the AT-property. The condition (6) also holds with $U_1 = V_1 = 30$, $U_2 = V_2 = -120$, $U_3 = V_3 = 105$. The involutory matrix QS is then diag($\sqrt{30}, \sqrt{120}, \sqrt{105}$$G$diag$\left(\sqrt{30}, -\sqrt{120}, \sqrt{105}\right)$, i.e.

$$\begin{bmatrix} 10 & -15 & 3\sqrt{14} \\ 15 & -24 & 5\sqrt{14} \\ 3\sqrt{14} & -5\sqrt{14} & 15 \end{bmatrix}.$$

The trace condition is fulfilled. Observe that the Hadamard power of QS is the modulus of $G \circ G^{-1}$.

Remark 5. It seems of interest that the real positive definite matrices A for which equality in (3) is attained [2, Theorem 3.3] have the property that (up to multiplication by a nonsingular diagonal matrix from both sides and simultaneous permutation of rows and columns) AS is involutory, S being a diagonal matrix $\mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)$, satisfying $\mathrm{tr}S = n - 2$. Indeed, it was proved in [2] that such matrix has the form (up to multiplication by a nonsingular diagonal matrix from both sides and simultaneous permutation of rows and columns)

$$A = \begin{bmatrix} B & b \\ b^T & \beta \end{bmatrix},$$

where $\beta \geqslant 1$ is a number, B an $(n-1) \times (n-1)$ matrix of the form $I + (\beta - 1)uu^T$, u a unit real vector, and b the $\sqrt{\beta^2 - 1}$-multiple of u. If S is the diagonal matrix $\mathrm{diag}(1, \ldots, 1, -1)$, then

$$AS = \begin{bmatrix} B & -b \\ b^T & -\beta \end{bmatrix}$$

is involutory since

$$A^{-1} = \begin{bmatrix} B & -b \\ -b^T & \beta \end{bmatrix}.$$

Addendum. An amusing corollary of Lemma E is the following property of the tableaux of the numbers $t_{ij} = \frac{i+j}{|i-j|}$ for $i \neq j$, $t_{ii} = 1$, $i, j = 0, 1, \ldots$:

$$\begin{bmatrix} 1 & \frac{1}{1} & \frac{2}{2} & \frac{3}{3} & \frac{4}{4} & \cdots \\ \frac{1}{1} & 1 & \frac{3}{1} & \frac{4}{2} & \frac{5}{3} & \cdots \\ \frac{2}{2} & \frac{3}{1} & 1 & \frac{5}{1} & \frac{6}{2} & \cdots \\ \cdots & & & & & \end{bmatrix}.$$

Choose any “principal minor" of even order; then the sum of all the products in odd rows is equal to the sum of all the products in even rows. If the minor has odd order, the first sum exceeds the second by one.

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