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# Weak Hopf algebras with projection and weak smash bialgebra structures

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**Abstract**

In this paper we study weak Hopf algebras with projection. If $f:H \to B$, $g:B \to H$ are morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$, we prove that it is possible to find an object $B_H$, in the new category of weak Yetter–Drinfeld modules, that verifies similar conditions to the ones include in the definition of weak Hopf algebra. Finally, we define weak smash bialgebra structures and prove that, under central and cocentral conditions, $B_H$ and $H$ determine an example of them.


*Keywords:* Weak Hopf algebra; Hopf algebra; Yetter–Drinfeld modules; Weak smash bialgebra structure

## Introduction

Weak Hopf algebras are generalizations of Hopf algebras and were defined by Böhm, Nill, and Szlachányi in [4,5]. The axioms are the same as the ones for a Hopf algebra,

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except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. The main motivation for studying weak Hopf algebras comes from quantum field theory and operator algebras.

A well known result of Radford [10] gives equivalent conditions for object $A \otimes H$ equipped with smash product algebra and coalgebra to be a Hopf algebra and characterized such objects via bialgebra projection. Majid in [9] interpreted this result in the modern context of Yetter–Drinfeld modules and stated that there is a one to one correspondence between Hopf algebras in this category, denoted by ${}^H_H\mathcal{YD}$, and Hopf algebras $B$ with morphisms of Hopf algebras $f : H \to B$, $g : B \to H$ such that $g \circ f = \mathrm{id}_H$. Later, Bespalov proved the same result for braided categories with split idempotents in [2], and further pursued the development of Radford's theory in joint work with Drabant. The key point in Bespalov–Majid's theorem is to define an object $B_H$ as the equalizer of $(B \otimes g) \circ \delta_B$ and $B \otimes \eta_H$. This object is a Hopf algebra in the category ${}^H_H\mathcal{YD}$ and there exists a Hopf algebra isomorphism $\omega$ between $B$ and $B_H \bowtie H$ (the crossed product of $B_H$ and $H$). It is important to point out that in the construction of $B_H$ they use the idempotent morphism $q_H^B = \mu_B \circ (B \otimes (f \circ \lambda_H \circ g)) \circ \delta_B$ and the equality $\delta_H \circ \eta_H = \eta_H \otimes \eta_H$, that it is no possible to assume for weak Hopf algebras.

The basic motivation of the present paper is to obtain results similar to the ones related in the last paragraph, when we have morphisms of weak Hopf algebras $f : H \to B$, $g : B \to H$ in a symmetric monoidal category with split idempotents and such that $g \circ f = \mathrm{id}_H$.

In the first section of this paper, following [4,7], we give a summary of the fundamental results about weak Hopf algebras and focus our attention in the study of center and cocenter conditions for the idempotent morphisms $\Pi_H^L$, $\overline{\Pi}_H^L$, $\Pi_H^R$, and $\overline{\Pi}_H^R$ associated to a weak Hopf algebra $H$. These conditions will be used, in the last section, in order to obtain weak smash bialgebra structures. Note that the papers on weak Hopf algebras mostly consider finite weak Hopf algebras (see, for example, [4]). Here we are working with these objects without finiteness conditions.

In the next section we prove that the morphism $q_H^B$ is also idempotent when we work with weak Hopf algebras and then, if the category admits split idempotents, there exist an epimorphism $p_H^B$, a monomorphism $i_H^B$ and an object $B_H$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\;q_H^B\;} & B \\ & {\scriptstyle p_H^B}\searrow \quad \nearrow{\scriptstyle i_H^B} & \\ & B_H & \end{array} \tag{D1}$$

commutes and $p_H^B \circ i_H^B = \mathrm{id}_{B_H}$. As a consequence, we have that

$$B_H \xrightarrow{\;i_H^B\;} B \underset{(B\otimes(\Pi_H^L\circ g))\circ\delta_B}{\overset{(B\otimes g)\circ\delta_B}{\rightrightarrows}} B \otimes H \tag{D2}$$

is an equalizer diagram and

$$B\otimes H \underset{\mu_B\circ(B\otimes(f\circ\Pi_H^L))}{\overset{\mu_B\circ(B\otimes f)}{\rightrightarrows}} B \xrightarrow{p_B^H} B_H \tag{D3}$$

is a coequalizer diagram. Therefore, it is possible to find an algebra coalgebra structure for $B_H$ and morphisms $\varphi_{B_H}=p_H^B\circ\mu_B\circ(f\otimes i_H^B): H\otimes B_H\to B_H$ and $r_{B_H}=(g\otimes p_H^B)\circ\delta_B\circ i_H^B: B_H\to H\otimes B_H$ such that $(B_H,\varphi_{B_H})$ is a left $H$-module and $(B_H,r_{B_H})$ is a left $H$-comodule. Moreover, in this section we introduce the category of weak Yetter–Drinfeld modules, denoted by ${}_H^H\mathcal{WYD}$, and we show that $(B_H,\varphi_{B_H},r_{B_H})$ belongs to ${}_H^H\mathcal{WYD}$. This category is defined as follows: $M=(M,\varphi_M,r_M)$ is an object in ${}_H^H\mathcal{WYD}$ if $(M,\varphi_M)$ is a left $H$-module, $(M,r_M)$ is a left $H$-comodule and

$$\begin{aligned}\text{(a)}\quad &(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ\big((r_M\circ\varphi_M)\otimes H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)\\ &=(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ(\mu_H\otimes\varphi_M\otimes H)\circ(H\otimes c_{H,H}\otimes M\otimes H)\\ &\quad\circ\big(\delta_H\otimes r_M\otimes\Pi_H^R\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M),\\ \text{(b)}\quad &(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ\big((\delta_H\circ\eta_H)\otimes r_M\big)=r_M.\end{aligned}$$

If we have cocenter and center conditions, $\mu_H\circ c_{H,H}\circ(H\otimes\Pi_H^R)=\mu_H\circ(H\otimes\Pi_H^R)$ and $(H\otimes\Pi_H^R)\circ c_{H,H}\circ\delta_H=(H\otimes\Pi_H^R)\circ\delta_H$, the condition (a) of the last definition can be changed by

$$\begin{aligned}&(\mu_H\otimes M)\circ(H\otimes c_{M,H})\circ\big((r_M\circ\varphi_M)\otimes H\big)\circ(H\otimes c_{H,M})\circ(\delta_H\otimes M)\\ &=(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes r_M)\end{aligned}$$

and then, when we consider only finite objects, the category ${}_H^H\mathcal{WYD}$ is the category of Yetter–Drinfeld modules defined by Gabriella Böhm in [3].

Also, in the second section, using the morphism

$$t_{B_H,B_H}=(\varphi_{B_H}\otimes B_H)\circ(H\otimes c_{B_H,B_H})\circ(r_{B_H}\otimes B_H): B_H\otimes B_H\to B_H\otimes B_H$$

we obtain that $B_H$ verifies similar conditions with the ones include in the definition of weak Hopf algebra and the morphism $\omega: B_H\otimes H\to B$ defined by $\omega=\mu_B\circ(i_H^B\otimes f)$ is an isomorphism if and only if $H$ is a Hopf algebra.

Finally, in the third section, draw inspiration from the work of Caenepeel and De Groot [7], we define weak smash bialgebra structures and prove that, under central and cocentral conditions, $B_H$ and $H$ determine an example of them.

## 1. Weak Hopf algebras

In what follows, $\mathcal{C}$ denotes a symmetric monoidal category with tensor product $\otimes$, symmetry isomorphism $c$, and base object $K$. We will suppose too that $\mathcal{C}$ admits split idempotents, i.e., for every morphism $q : Y \to Y$ such that $q = q \circ q$ exists an object $Z$ and morphisms $i : Z \to Y$ and $p : Y \to Z$ such that $q = i \circ p$ and $p \circ i = \mathrm{id}_Z$.

An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where $A$ is an object in $\mathcal{C}$ and $\eta_A : K \to A$, $\mu_A : A \otimes A \to A$ are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = \mathrm{id}_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if $A$, $B$ are algebras in $\mathcal{C}$, the object $A \otimes B$ is also an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where $D$ is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$, $\delta_D : D \to D \otimes D$ are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = \mathrm{id}_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When $D$, $E$ are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

From [4] we recall the definition of weak Hopf algebra.

**Definition 1.1.** A weak Hopf algebra $H$ in $\mathcal{C}$ is by definition an algebra $(H, \eta_H, \mu_H)$ and coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$
$$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ \big(H \otimes (c_{H,H} \circ \delta_H) \otimes H\big).$$

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$$= \big(H \otimes (\mu_H \circ c_{H,H}) \otimes H\big) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H).$$

(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called antipode of $H$) verifying:

(a4-1) $\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes c_{H,H}) \circ \big((\delta_H \circ \eta_H) \otimes H\big)$;

(a4-2) $\mu_H \circ (\lambda_H \otimes H) \circ \delta_H = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ (c_{H,H} \otimes H) \circ \big(H \otimes (\delta_H \circ \eta_H)\big)$;

(a4-3) $\mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \circ \delta_H = \lambda_H$.

Observe that in the definition of Hopf algebra, (a2)–(a4) are replaced by the conditions:

(a′2) $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$.
(a′3) $\delta_H \circ \eta_H = \eta_H \otimes \eta_H$.
(a′4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ verifying:

$$\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_H.$$

Therefore, a Hopf algebra is always a weak Hopf algebra. Moreover, in [4] we can find the following equivalent conditions for a weak Hopf algebra $H$:

(1) $H$ is a Hopf algebra.
(2) $\delta_H \circ \eta_H = \eta_H \otimes \eta_H$.
(3) $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$.
(4) $\mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \varepsilon_H \otimes \eta_H$.
(5) $\mu_H \circ (\lambda_H \otimes H) \circ \delta_H = \varepsilon_H \otimes \eta_H$.

Finally, if $H$ is a weak Hopf algebra, the antipode is unique, antimultiplicative, anticomultiplicative and leaves the unit $\eta_H$ and the counit $\varepsilon_H$ invariant:

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \qquad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$
$$\lambda_H \circ \eta_H = \eta_H, \qquad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

The next proposition is a resume of Propositions (4.3)–(4.6) contained in [7].

**Proposition 1.2.** *Let $H$ be an algebra and a coalgebra such that* (a1) *holds.*

(1) *The following assertions are equivalent.*
  (1.1) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.
  (1.2) *There exists a morphism $\Pi_H^L : H \to H$ such that*

$$\big(H \otimes \Pi_H^L\big) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ \big((\delta_H \circ \eta_H) \otimes H\big).$$

  (1.3) *There exists a morphism $\Pi_H^R : H \to H$ such that*

$$\big(\Pi_H^R \otimes H\big) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ \big(H \otimes (\delta_H \circ \eta_H)\big).$$

(2) *The following assertions are equivalent.*
  (2.1) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.
  (2.2) *There exists a morphism $\overline{\Pi}_H^L : H \to H$ such that*

$$\big(\overline{\Pi}_H^L \otimes H\big) \circ \delta_H = (H \otimes \mu_H) \circ \big((\delta_H \circ \eta_H) \otimes H\big).$$

  (2.3) *There exists a morphism $\overline{\Pi}_H^R : H \to H$ such that*

$$\big(H \otimes \overline{\Pi}_H^R\big) \circ \delta_H = (\mu_H \otimes H) \circ \big(H \otimes (\delta_H \circ \eta_H)\big).$$

(3) *The following assertions are equivalent.*
  (3.1) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.
  (3.2) *There exists a morphism $\Pi_H^L : H \to H$ such that*

$$\mu_H \circ \big(H \otimes \Pi_H^L\big) = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

(3.3) *There exists a morphism* $\Pi_H^R : H \to H$ *such that*

$$\mu_H \circ \big(\Pi_H^R \otimes H\big) = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H).$$

(4) *The following assertions are equivalent.*

(4.1) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$.

(4.2) *There exists a morphism* $\overline{\Pi}_H^L : H \to H$ *such that*

$$\mu_H \circ \big(H \otimes \overline{\Pi}_H^L\big) = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ (\delta_H \otimes H).$$

(4.3) *There exists a morphism* $\overline{\Pi}_H^R : H \to H$ *such that*

$$\mu_H \circ \big(\overline{\Pi}_H^R \otimes H\big) = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes \delta_H).$$

From Proposition 1.2, we conclude immediately the following:

$$\Pi_H^L = \mu_H \circ (H \otimes \lambda_H) \circ \delta_H = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes c_{H,H}) \circ \big((\delta_H \circ \eta_H) \otimes H\big),$$
$$\Pi_H^R = \mu_H \circ (\lambda_H \otimes H) \circ \delta_H = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ (c_{H,H} \otimes H) \circ \big(H \otimes (\delta_H \circ \eta_H)\big),$$
$$\overline{\Pi}_H^L = \big(H \otimes (\varepsilon_H \circ \mu_H)\big) \circ \big((\delta_H \circ \eta_H) \otimes H\big),$$
$$\overline{\Pi}_H^R = \big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ \big(H \otimes (\delta_H \circ \eta_H)\big).$$

The morphisms $\Pi_H^L$, $\Pi_H^R$, $\overline{\Pi}_H^L$, and $\overline{\Pi}_H^R$ are idempotent and we have (see [7]):

$$\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \qquad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \qquad \overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \qquad \Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L,$$
$$\overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \qquad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \qquad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R, \qquad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R.$$

Moreover, it is possible to prove that

$$\Pi_H^L = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^L, \qquad \Pi_H^L \circ \lambda_H = \Pi_H^L \circ \Pi_H^R = \lambda_H \circ \Pi_H^R,$$
$$\Pi_H^R = \overline{\Pi}_H^R \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R, \qquad \Pi_H^R \circ \lambda_H = \Pi_H^R \circ \Pi_H^L = \lambda_H \circ \Pi_H^L.$$

Finally, if $\lambda_H$ is bijective, in [11] we can find the equalities:

$$\overline{\Pi}_H^L = \mu_H \circ \big(H \otimes \lambda_H^{-1}\big) \circ c_{H,H} \circ \delta_H, \qquad \overline{\Pi}_H^R = \mu_H \circ \big(\lambda_H^{-1} \otimes H\big) \circ c_{H,H} \circ \delta_H.$$

**Definition 1.3.** Let $H$, $B$ be weak Hopf algebras and let $f : H \to B$ be a morphism in $\mathcal{C}$. If $f$ is an algebra and coalgebra morphism, $f$ is called a morphism of weak Hopf algebras.

**Proposition 1.4.** *Let* $H$, $B$ *be weak Hopf algebras and let* $f : H \to B$ *be a weak Hopf algebra morphism. Then* $\lambda_B \circ f = f \circ \lambda_H$.

**Proof.** First, note that the equalities $\Pi_B^L \circ f = f \circ \Pi_H^L$ and $\Pi_B^R \circ f = f \circ \Pi_H^R$ hold. Then, as a consequence, we have:

$$\begin{aligned} f \circ \lambda_H &= \mu_B \circ \big(\big(f \circ \Pi_H^R\big) \otimes (f \circ \lambda_H)\big) \circ \delta_H = \mu_B \circ \big(\big(\Pi_B^R \circ f\big) \otimes (f \circ \lambda_H)\big) \circ \delta_H \\ &= \mu_B \circ \big((\lambda_B \circ f) \otimes \big(f \circ \Pi_H^L\big)\big) \circ \delta_H = \mu_B \circ \big((\lambda_B \circ f) \otimes \big(\Pi_B^L \circ f\big)\big) \circ \delta_H \\ &= \lambda_B \circ f. \qquad \square \end{aligned}$$

**Proposition 1.5.** *Let $H$ be a weak Hopf algebra. The following assertions are equivalent*:

(1) $\mu_H \circ (\Pi_H^R \otimes H) = \mu_H \circ c_{H,H} \circ (\Pi_H^R \otimes H)$.
(2) $\mu_H \circ (\Pi_H^L \otimes H) = \mu_H \circ c_{H,H} \circ (\Pi_H^L \otimes H)$.
(3) $\mu_H \circ (\overline{\Pi}_H^R \otimes H) = \mu_H \circ c_{H,H} \circ (\overline{\Pi}_H^R \otimes H)$.
(4) $\mu_H \circ (\overline{\Pi}_H^L \otimes H) = \mu_H \circ c_{H,H} \circ (\overline{\Pi}_H^L \otimes H)$.
(5) $\Pi_H^L = \overline{\Pi}_H^R$.
(6) $\Pi_H^R = \overline{\Pi}_H^L$.

**Proof.** The assertions (1)–(4) are equivalent by [8, (1.1)].

(1) ⇒ (5). Using the equality $(\Pi_H^R \otimes H) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H$, we obtain:

$$\begin{aligned} \Pi_H^L &= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H) \circ (H \otimes \eta_H) \\ &= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ (c_{H,H} \otimes H) \circ \big(H \otimes \Pi_H^R \otimes H\big) \circ (H \otimes \delta_H) \circ (H \otimes \eta_H) \\ &= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ \big(H \otimes \Pi_H^R \otimes H\big) \circ (H \otimes \delta_H) \circ (H \otimes \eta_H) = \overline{\Pi}_H^R. \end{aligned}$$

(5) ⇒ (1). We have:

$$\begin{aligned} \mu_H \circ \big(\Pi_H^R \otimes H\big) &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \mu_H \otimes H) \\ &\quad \circ (H \otimes H \otimes c_{H,H}) \circ (H \otimes \delta_H \otimes H) \circ (H \otimes \eta_H \otimes H) \\ &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ \big(c_{H,H} \otimes \Pi_H^L\big) \circ (H \otimes \delta_H) \\ &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ \big(c_{H,H} \otimes \overline{\Pi}_H^R\big) \circ (H \otimes \delta_H) \\ &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \mu_H \otimes H) \\ &\quad \circ (H \otimes H \otimes \delta_H) \circ (H \otimes H \otimes \eta_H) \\ &= \mu_H \circ c_{H,H} \circ \big(\Pi_H^R \otimes H\big). \end{aligned}$$

(2) ⇒ (6). By the equality $(H \otimes \Pi_H^L) \circ \delta_H \circ \eta_H = \delta_H \circ \eta_H$, we obtain:

$$\begin{aligned} \Pi_H^R &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) \circ (\eta_H \otimes H) \\ &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ (H \otimes c_{H,H}) \circ \big(H \otimes \Pi_H^L \otimes H\big) \circ (\delta_H \otimes H) \circ (\eta_H \otimes H) \\ &= (H \otimes \varepsilon_H) \circ (H \otimes \mu_H) \circ \big(H \otimes \Pi_H^L \otimes H\big) \circ (\delta_H \otimes H) \circ (\eta_H \otimes H) = \overline{\Pi}_H^L. \end{aligned}$$

(6) $\Rightarrow$ (2). We have:

$$\begin{aligned}
\mu_H \circ \big(\Pi_H^L \otimes H\big) &= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (H \otimes \mu_H \otimes H) \\
&\quad \circ (\delta_H \otimes c_{H,H}) \circ (\eta_H \otimes H \otimes H) \\
&= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ \big(\overline{\Pi}_H^L \otimes c_{H,H}\big) \circ (\delta_H \otimes H) \circ c_{H,H} \\
&= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ \big(\Pi_H^R \otimes c_{H,H}\big) \circ (\delta_H \otimes H) \circ c_{H,H} \\
&= (\varepsilon_H \otimes H) \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (H \otimes \mu_H \otimes H) \\
&\quad \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes c_{H,H}) \circ (\eta_H \otimes H \otimes H) \\
&= \mu_H \circ c_{H,H} \circ \big(\Pi_H^L \otimes H\big). \qquad \square
\end{aligned}$$

**Proposition 1.6.** *Let $H$ be a weak Hopf algebra. The following assertions are equivalent*:

(1) $(\Pi_H^R \otimes H) \circ \delta_H = (\Pi_H^R \otimes H) \circ c_{H,H} \circ \delta_H$.
(2) $(\Pi_H^L \otimes H) \circ \delta_H = (\Pi_H^L \otimes H) \circ c_{H,H} \circ \delta_H$.
(3) $(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (\overline{\Pi}_H^L \otimes H) \circ c_{H,H} \circ \delta_H$.
(4) $(\overline{\Pi}_H^R \otimes H) \circ \delta_H = (\overline{\Pi}_H^R \otimes H) \circ c_{H,H} \circ \delta_H$.
(5) $\Pi_H^L = \overline{\Pi}_H^L$.
(6) $\Pi_H^R = \overline{\Pi}_H^R$.

**Proof.** It follows from Proposition 1.5 after passing to the opposite category. $\square$

## 2. The construction of $B_H$

**Proposition 2.1.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. Then the following morphism is an idempotent in $\mathcal{C}$*:

$$q_H^B = \mu_B \circ \big(B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B : B \to B.$$

**Proof.** We have:

$$\begin{aligned}
q_H^B \circ q_H^B &= \mu_B \circ \big(B \otimes (f \circ \lambda_H \circ g)\big) \circ \mu_{B\otimes B} \circ (\delta_B \otimes \delta_B) \circ \big(B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B \\
&= \mu_B \circ \big(B \otimes \big(\mu_B \circ c_{B,B} \circ \big((f \circ \lambda_H \circ g) \otimes \big(f \circ \lambda_H^2 \circ g\big)\big)\big)\big) \circ (\mu_B \otimes \delta_B) \\
&\quad \circ (B \otimes c_{B,B}) \circ \big(\delta_B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B \\
&= \mu_B \circ \big(\mu_B \otimes \big(f \circ \lambda_H \circ \Pi_H^L\big)\big) \circ (B \otimes c_{H,B}) \circ \big(B \otimes g \otimes (f \circ \lambda_H \circ g)\big) \\
&\quad \circ (\delta_B \otimes B) \circ \delta_B \\
&= \mu_B \circ \big(B \otimes \big(f \circ \lambda_H \circ g \circ \mu_B \circ \big(\Pi_B^L \otimes B\big) \circ \delta_B\big)\big) \circ \delta_B = q_H^B.
\end{aligned}$$

Note that the first equality follows from (a1), the second, the third, and the fourth ones from the associativity, the coassociativity, the naturality of $c$, the condition of morphisms of weak Hopf algebras for $f$ and $g$ and the anti(co)multiplicative nature of the antipode. Finally, in the fifth one we use the equality $\mu_B \circ (\Pi_B^L \otimes B) \circ \delta_B = \mathrm{id}_B$. $\square$

As a consequence of Proposition 2.1, we obtain that there exist an epimorphism $p_H^B$, a monomorphism $i_H^B$ and an object $B_H$ such that the diagram D1, that we can find in the introduction, commutes and $p_H^B \circ i_H^B = \mathrm{id}_{B_H}$.

**Proposition 2.2.** *Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. Let* D2 *and* D3 *be the diagrams given in the introduction. Then,* D2 *is an equalizer diagram and* D3 *is a coequalizer diagram.*

**Proof.** (1) First we will prove the equality $(B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ i_H^B = (B \otimes g) \circ \delta_B \circ i_H^B$. Composing with $p_H^B$, we obtain:

$$
\begin{aligned}
&\big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \delta_B \circ i_H^B \circ p_H^B \\
&\quad = \big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \delta_B \circ q_H^B \\
&\quad = \big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \mu_{B\otimes B} \circ (\delta_B \otimes \delta_B) \circ \big(B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B \\
&\quad = \big(B \otimes \big(\Pi_H^L \circ \mu_H\big)\big) \circ (\mu_B \otimes g \otimes H) \circ (B \otimes c_{B,B} \otimes H) \circ (B \otimes B \otimes c_{H,B}) \\
&\qquad \circ \big(B \otimes B \otimes (\lambda_H \circ g) \otimes (f \circ \lambda_H \circ g)\big) \circ (B \otimes \delta_B \otimes B) \circ (\delta_B \otimes B) \circ \delta_B \\
&\quad = (\mu_B \otimes H) \circ (B \otimes c_{H,B}) \circ \big(B \otimes \big(\Pi_H^L \circ \Pi_H^L \circ g\big) \otimes B\big) \circ \big(\delta_B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B \\
&\quad = (\mu_B \otimes H) \circ (B \otimes c_{H,B}) \circ \big(B \otimes \big(\Pi_H^L \circ g\big) \otimes B\big) \circ \big(\delta_B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B \\
&\quad = (\mu_B \otimes H) \circ (B \otimes f \otimes H) \circ (B \otimes c_{H,H}) \circ \big(B \otimes \Pi_H^L \otimes \lambda_H\big) \circ (B \otimes \delta_H) \\
&\qquad \circ (B \otimes g) \circ \delta_B \\
&\quad = (\mu_B \otimes \mu_H) \circ (B \otimes B \otimes g \otimes H) \circ (B \otimes c_{B,B} \otimes H) \circ (B \otimes B \otimes c_{H,B}) \\
&\qquad \circ \big(B \otimes B \otimes (\lambda_H \circ g) \otimes (f \circ \lambda_H \circ g)\big) \circ (B \otimes \delta_B \otimes B) \circ (\delta_B \otimes B) \circ \delta_B \\
&\quad = (\mu_B \otimes \mu_H) \circ (B \otimes B \otimes g \otimes g) \circ (B \otimes c_{B,B} \otimes B) \circ \big(B \otimes B \otimes (\delta_B \circ f \circ \lambda_H \circ g)\big) \\
&\qquad \circ (\delta_B \otimes B) \circ \delta_B \\
&\quad = (B \otimes g) \circ \delta_B \circ q_H^B = (B \otimes g) \circ \delta_B \circ i_H^B \circ p_H^B.
\end{aligned}
$$

In the last calculations we use repeatedly the associativity, the coassociativity, the naturality of $c$, the condition of morphism of weak Hopf algebras for $f$ and $g$ and the anti-(co)multiplicative nature of the antipode. Note that in the fifth one appears the idempotent character of $\Pi_H^L$.

Thus, $(B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ i_H^B = (B \otimes g) \circ \delta_B \circ i_H^B$ since $p_H^B$ is an epimorphism.

Now, let $t : D \to B$ be a morphism such that $(B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ t = (B \otimes g) \circ \delta_B \circ t$. If $v = p_H^B \circ t$, since $f \circ \Pi_H^R \circ g = \Pi_B^R$ and $\mu_B \circ (B \otimes \Pi_B^R) \circ \delta_B = \mathrm{id}_B$, we have

$$\begin{aligned}
i_H^B \circ v = q_H^B \circ t &= \mu_B \circ \big(B \otimes (f \circ \lambda_H \circ g)\big) \circ \delta_B \circ t \\
&= \mu_B \circ \big(B \otimes \big(f \circ \lambda_H \circ \Pi_H^L \circ g\big)\big) \circ \delta_B \circ t \\
&= \mu_B \circ \big(B \otimes \big(f \circ \Pi_H^R \circ \Pi_H^L \circ g\big)\big) \circ \delta_B \circ t = \mu_B \circ \big(B \otimes \big(f \circ \Pi_H^R \circ g\big)\big) \circ \delta_B \circ t \\
&= \mu_B \circ \big(B \otimes \Pi_B^R\big) \circ \delta_B \circ t = t.
\end{aligned}$$

Trivially, the morphism $v$ is unique and therefore, the diagram is an equalizer diagram.

(2) The proof of this assertion is analogous and we leave the calculus to the reader. □

**Remark 2.3.** One can replace in D2 the morphism $\Pi_H^L$ by $\overline{\Pi}_H^R$. Then, using the equality $\overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L$, it is easy to show that

$$B_H \xrightarrow{i_H^B} B \underset{(B\otimes(\overline{\Pi}_H^L\circ g))\circ\delta_B}{\overset{(B\otimes g)\circ\delta_B}{\rightrightarrows}} B \otimes H$$

is an equalizer diagram in $\mathcal{C}$. Analogously, if in D3, we change $\Pi_H^L$ by $\overline{\Pi}_H^L$, the diagram

$$B \otimes H \underset{\mu_B\circ(B\otimes(f\circ\overline{\Pi}_H^L))}{\overset{\mu_B\circ(B\otimes f)}{\rightrightarrows}} B \xrightarrow{p_B^H} B$$

is a coequalizer diagram in $\mathcal{C}$ since $\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L$.

**Proposition 2.4.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. Then*

(1) $(B_H, \eta_{B_H} = p_H^B \circ \eta_B, \mu_{B_H} = p_H^B \circ \mu_B \circ (i_H^B \otimes i_H^B))$ *is an algebra in* $\mathcal{C}$.
(2) $(B_H, \varepsilon_{B_H} = \varepsilon_B \circ i_H^B, \delta_{B_H} = (p_H^B \otimes p_H^B) \circ \delta_B \circ i_H^B)$ *is a coalgebra in* $\mathcal{C}$.

**Proof.** We will verify (1), and leave the assertion (2) to the reader.

Note that the morphisms $\eta_{B_H}$ and $\mu_{B_H}$ are the factorizations, through the equalizer $i_H^B$, of the morphisms $\eta_B$ and $\mu_B \circ (i_H^B \otimes i_H^B)$. It is an easy exercise to show that $(B_H, \eta_{B_H}, \mu_{B_H})$ is an algebra in $\mathcal{C}$. □

**Proposition 2.5.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. Then*

(1) $(B_H, \varphi_{B_H} = p_H^B \circ \mu_B \circ (f \otimes i_H^B))$ *is a left $H$-module in $\mathcal{C}$ (i.e. $\varphi_{B_H} \circ (\eta_H \otimes B_H) = \mathrm{id}_{B_H}$ and $\varphi_{B_H} \circ (\varphi_{B_H} \otimes B_H) = \varphi_{B_H} \circ (\mu_H \otimes B_H)$).*

(2) $(B_H, r_{B_H} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B)$ *is a left* $H$*-comodule in* $\mathcal{C}$ (*i.e.* $(\varepsilon_{B_H} \otimes B_H) \circ r_{B_H} = \mathrm{id}_{B_H}$ *and* $(H \otimes r_{B_H}) \circ r_{B_H} = (\delta_H \otimes B_H) \circ r_{B_H}$).

**Proof.** (1) Let $y_B : H \otimes B_H \to B$ be the morphism given by

$$y_B = \mu_B \circ \big(B \otimes (\mu_B \circ c_{B,B})\big) \circ \big(f \otimes (f \circ \lambda_H) \otimes B\big) \circ \big(\delta_H \otimes i_H^B\big).$$

This morphism verifies the equality $(B \otimes g) \circ \delta_B \circ y_B = (B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ y_B$. Indeed, using $(B \otimes (\Pi_H^L \circ g)) \circ \delta_B \circ i_H^B = (B \otimes g) \circ \delta_B \circ i_H^B$ and (3.3) of Proposition 1.2, we obtain:

$$\begin{aligned}
&(B \otimes g) \circ \delta_B \circ y_B \\
&\quad = \mu_{B\otimes H} \circ \big(\big((B \otimes g) \circ \delta_B\big) \otimes \mu_{B\otimes H}\big) \circ \big(B \otimes \big((B \otimes g) \circ \delta_B\big) \otimes \big((B \otimes g) \circ \delta_B\big)\big) \\
&\qquad \circ (B \otimes c_{B,B}) \circ \big(f \otimes (\lambda_B \circ f) \otimes B\big) \circ \big(\delta_H \otimes i_H^B\big) \\
&\quad = (\mu_B \otimes H) \circ (B \otimes \mu_{B\otimes H}) \circ (B \otimes B \otimes \mu_H \otimes B \otimes H) \circ (B \otimes c_{H,B} \otimes H \otimes B \otimes H) \\
&\qquad \circ \big(\big((f \otimes H) \circ \delta_H\big) \otimes \big(\big((B \otimes g) \circ \delta_B\big) \circ i_H^B\big) \otimes \big((f \otimes H) \circ \delta_H\big)\big) \circ (H \otimes c_{H,B_H}) \\
&\qquad \circ (H \otimes \lambda_H \otimes B_H) \circ (\delta_H \otimes B_H) \\
&\quad = (\mu_B \otimes H) \circ (B \otimes \mu_{B\otimes H}) \circ \big(B \otimes B \otimes \big(\mu_H \circ \big(H \otimes \Pi_H^L\big)\big) \otimes B \otimes H\big) \\
&\qquad \circ (B \otimes c_{H,B} \otimes H \otimes B \otimes H) \circ \big(\big((f \otimes H) \circ \delta_H\big) \otimes \big(\big((B \otimes g) \circ \delta_B\big) \circ i_H^B\big) \\
&\qquad \otimes \big((f \otimes H) \circ \delta_H\big)\big) \circ (H \otimes c_{H,B_H}) \circ (H \otimes \lambda_H \otimes B_H) \circ (\delta_H \otimes B_H) \\
&\quad = (\mu_B \otimes H) \circ (B \otimes \mu_{B\otimes H}) \circ \big(B \otimes B \otimes \big(\big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes c_{H,H}) \\
&\qquad \circ (\delta_H \otimes H)\big) \otimes \big((f \otimes H) \circ \delta_H \circ \lambda_H\big)\big) \circ (B \otimes c_{H,B} \otimes c_{H,H}) \\
&\qquad \circ \big(\big((f \otimes H) \circ \delta_H\big) \otimes c_{H,B} \otimes H\big) \circ \big(\delta_H \otimes \big((B \otimes g) \circ \delta_B \circ i_H^B\big)\big) \\
&\quad = (\mu_B \otimes H) \circ (B \otimes \mu_B \otimes H) \circ \big(B \otimes B \otimes \big(c_{H,B} \circ \big(\Pi_H^L \otimes (f \circ \lambda_H)\big) \circ \delta_H\big)\big) \\
&\qquad \circ \big(B \otimes B \otimes \big(\big((\varepsilon_H \circ \mu_H) \otimes H\big) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)\big)\big) \circ (B \otimes c_{H,B} \otimes H) \\
&\qquad \circ \big(\big((f \otimes H) \circ \delta_H\big) \otimes \big((B \otimes g) \circ \delta_B \circ i_H^B\big)\big) \\
&\quad = (\mu_B \otimes H) \circ (B \otimes \mu_B \otimes H) \circ \big(B \otimes B \otimes \big(c_{H,B} \circ \big(\Pi_H^L \otimes (f \circ \lambda_H)\big) \circ \delta_H \circ \mu_H\big)\big) \\
&\qquad \circ (B \otimes c_{H,B} \otimes H) \circ \big(\big((f \otimes H) \circ \delta_H\big) \otimes \big(\big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \delta_B \circ i_H^B\big)\big) \\
&\quad = (\mu_B \otimes H) \circ \big(B \otimes \big(c_{H,B} \circ \big(\Pi_H^L \otimes (f \circ \lambda_H)\big) \circ \delta_H\big)\big) \circ (B \otimes g) \circ \delta_B \circ \mu_B \circ \big(f \otimes i_H^B\big).
\end{aligned}$$

On the other hand, since $\Pi_H^L$ is an idempotent morphism, we have

$$\begin{aligned}
&\big(B \otimes \big(\Pi_H^L \circ g\big)\big) \circ \delta_B \circ y_B \\
&\quad = \big(B \otimes \Pi_H^L\big) \circ (\mu_B \otimes H) \circ \big(B \otimes \big(c_{H,B} \circ \big(\Pi_H^L \otimes (f \circ \lambda_H)\big) \circ \delta_H\big)\big) \circ (B \otimes g) \circ \delta_B \\
&\qquad \circ \mu_B \circ \big(f \otimes i_H^B\big)
\end{aligned}$$

$$= (\mu_B \otimes H) \circ \big(B \otimes \big(c_{H,B} \circ \big(\Pi_H^L \otimes (f \circ \lambda_H)\big) \circ \delta_H\big)\big) \circ (B \otimes g) \circ \delta_B \circ \mu_B \circ \big(f \otimes i_H^B\big).$$

Therefore, there exists an unique morphism $\varphi_{B_H} : H \otimes B_H \to B_H$ verifying the equality $i_H^B \circ \varphi_{B_H} = y_B$ and, as a consequence,

$$\varphi_{B_H} = p_H^B \circ y_B = p_H^B \circ \mu_B \circ \big(f \otimes i_H^B\big).$$

Finally, it is easy to show that $(B_H, \varphi_{B_H})$ is a left $H$-module.

(2) The proof of this assertion is similar to the one developed in (1) and we leave it to the reader. □

**Definition 2.6.** Let $H$ be a weak Hopf algebra. We shall denote by ${}_H^H\mathcal{WYD}$ the category of left weak Yetter–Drinfeld modules over $H$. That is, $M = (M, \varphi_M, r_M)$ is an object in ${}_H^H\mathcal{WYD}$ if $(M, \varphi_M)$ is a left $H$-module, $(M, r_M)$ is a left $H$-comodule and

(a) $$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ \big((r_M \circ \varphi_M) \otimes H\big) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$
$$= (\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\mu_H \otimes \varphi_M \otimes H) \circ (H \otimes c_{H,H} \otimes M \otimes H)$$
$$\circ \big(\delta_H \otimes r_M \otimes \Pi_H^R\big) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M).$$

(b) $$(\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ \big((\delta_H \circ \eta_H) \otimes r_M\big) = r_M.$$

Let $M$, $N$ in ${}_H^H\mathcal{WYD}$. The morphism $f : M \to N$ is a morphism in the category ${}_H^H\mathcal{WYD}$ if $f \circ \varphi_M = \varphi_N \circ (H \otimes f)$ and $(H \otimes f) \circ r_M = r_N \circ f$.

**Remark 2.7.** Note that the last definition is not exactly the same as one of Gabriella Böhm's in [3] even in finite dimensions and even after passing to the opposite algebra. The essential difference appears in (a) since this equality involves the idempotent morphism $\Pi_H^R$. The origin of this new condition come from the properties that verifies $B_H$ (see 2.8).

On the other hand, if we have $(H \otimes \Pi_H^R) \circ c_{H,H} \circ \delta_H = (H \otimes \Pi_H^R) \circ \delta_H$ and $\mu_H \circ c_{H,H} \circ (H \otimes \Pi_H^R) = \mu_H \circ (H \otimes \Pi_H^R)$, the condition (a) of the last definition can be changed by

$$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ \big((r_M \circ \varphi_M) \otimes H\big) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$
$$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes r_M).$$

Indeed, if $(H \otimes \Pi_H^R) \circ c_{H,H} \circ \delta_H = (H \otimes \Pi_H^R) \circ \delta_H$ and $\mu_H \circ c_{H,H} \circ (H \otimes \Pi_H^R) = \mu_H \circ (H \otimes \Pi_H^R)$, by Propositions 1.5 and 1.6, we obtain the equalities $\Pi_H^L = \overline{\Pi}_H^L = \overline{\Pi}_H^R = \Pi_H^R$. Therefore,

$$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\mu_H \otimes \varphi_M \otimes H) \circ (H \otimes c_{H,H} \otimes M \otimes H)$$
$$\circ \big(\delta_H \otimes r_M \otimes \Pi_H^R\big) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$$
$$= \big((\mu_H \circ c_{H,H}) \otimes M\big) \circ (H \otimes c_{M,H}) \circ (\mu_H \otimes \varphi_M \otimes H) \circ (H \otimes c_{H,H} \otimes M \otimes H)$$

$$\circ\big(\delta_H\otimes r_M\otimes \Pi_H^R\big)\circ(H\otimes c_{H,M})\circ\big((\delta_H\circ c_{H,H})\otimes M\big)$$
$$=(\mu_H\otimes\varphi_M)\circ\big(\big(\mu_H\circ\big(\Pi_H^R\otimes H\big)\circ\delta_H\big)\otimes c_{H,H}\otimes M\big)\circ(\delta_H\otimes r_M)$$
$$=(\mu_H\otimes\varphi_M)\circ(H\otimes c_{H,H}\otimes M)\circ(\delta_H\otimes r_M).$$

Then, under central and cocentral conditions, when we consider only finite objects ($M$ in $\mathcal{C}$ is said to be finite if there exists $M^*$ in $\mathcal{C}$ such that $(M\otimes -, M^*\otimes -, \alpha_M, \beta_M)$ is an adjoint pair), the category ${}_H^H\mathcal{WYD}$ is the category of Yetter–Drinfeld modules defined by Gabriella Böhm in [3].

**Proposition 2.8.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g:B\to H$ and $f:H\to B$ be morphisms of weak Hopf algebras such that $g\circ f=\mathrm{id}_H$. Then $(B_H,\varphi_{B_H},r_{B_H})$ belongs to ${}_H^H\mathcal{WYD}$.*

**Proof.** Composing with the monomorphism $H\otimes i_H^B$ and using the (co)associativity, the naturality of $c$, the condition of weak Hopf algebra morphism for $f$ and $g$, the anti(co)-multiplicative nature of the antipode, and $(B\otimes q_H^B)\circ\delta_B\circ i_H^B=\delta_B\circ i_H^B$, we obtain:

$$\big(\mu_H\otimes i_H^B\big)\circ(H\otimes c_{B_H,H})\circ\big((r_{B_H}\circ\varphi_{B_H})\otimes H\big)\circ(H\otimes c_{H,B_H})\circ(\delta_H\otimes B_H)$$
$$=(\mu_H\otimes B)\circ(g\otimes c_{B,H})\circ\big(\big(\delta_B\circ\mu_B\circ\big(B\otimes(f\circ\lambda_H\circ g)\big)\big)\otimes H\big)$$
$$\circ\big(\big(\mu_{B\otimes B}\circ\big((\delta_B\circ f)\otimes\big(\delta_B\circ i_H^B\big)\big)\big)\otimes H\big)\circ(H\otimes c_{H,B_H})\circ(\delta_H\otimes B_H)$$
$$=(\mu_H\otimes B)\circ(g\otimes c_{B,H})\circ(\mu_{B\otimes B}\otimes H)\circ(B\otimes B\otimes\mu_{B\otimes B}\otimes H)$$
$$\circ(\delta_B\otimes\delta_B\otimes\delta_B\otimes H)\circ(f\otimes c_{B,B}\otimes H)\circ\big(H\otimes(\lambda_B\circ f)\otimes i_H^B\otimes H\big)$$
$$\circ(\delta_H\otimes c_{H,B_H})\circ(\delta_H\otimes B_H)$$
$$=(\mu_H\otimes B)\circ(H\otimes\mu_H\otimes\mu_B)\circ(H\otimes H\otimes\mu_{H\otimes B}\otimes B)$$
$$\circ(H\otimes H\otimes c_{B,H}\otimes c_{B,H}\otimes B)\circ(H\otimes c_{B,H}\otimes c_{B,H}\otimes c_{B,H})$$
$$\circ(H\otimes B\otimes H\otimes B\otimes c_{B,H}\otimes H)$$
$$\circ\big(\big((H\otimes f)\circ\delta_H\big)\otimes\big((g\otimes B)\circ\delta_B\big)\otimes\big(\big(\big(\big((f\circ\lambda_H)\otimes\lambda_H\big)\circ\delta_H\big)\otimes H\big)\circ\delta_H\big)\big)$$
$$\circ(H\otimes c_{H,B})\circ\big(\delta_H\otimes i_H^B\big)$$
$$=(\mu_H\otimes B)\circ(H\otimes\mu_H\otimes B)\circ\big(H\otimes H\otimes\big(c_{B,H}\circ(\mu_B\otimes H)\circ(\mu_B\otimes B\otimes H)\big)\big)$$
$$\circ\big(H\otimes c_{B,H}\otimes c_{B,B}\otimes\Pi_H^R\big)\circ(H\otimes B\otimes c_{B,H}\otimes c_{H,B})$$
$$\circ\big(H\otimes\big((B\otimes\lambda_B)\circ(f\otimes f)\circ\delta_H\big)\otimes c_{H,H}\otimes B\big)\circ\big(\delta_H\otimes H\otimes\big((g\otimes B)\circ\delta_B\big)\big)$$
$$\circ\big(\delta_H\otimes i_H^B\big)$$
$$=(\mu_H\otimes B)\circ(\mu_H\otimes c_{B,H})$$
$$\circ\big(H\otimes H\otimes\big(\mu_B\circ(\mu_B\otimes B)\circ(B\otimes c_{B,B})\circ\big(\big(f\otimes(\lambda_B\circ f)\big)\circ\delta_H\big)\otimes B\big)\otimes H\big)$$
$$\circ\big(H\otimes c_{H,H}\otimes B\otimes\Pi_H^R\big)\circ\big(H\otimes H\otimes\big(\big(g\otimes q_H^B\big)\circ\delta_B\circ i_H^B\big)\otimes H\big)$$

$$\circ (\delta_H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$$
$$= \big(\mu_H \otimes i_H^B\big) \circ (H \otimes c_{B_H,H}) \circ (\mu_H \otimes \varphi_{B_H} \otimes H) \circ (H \otimes c_{H,H} \otimes B_H \otimes H)$$
$$\circ \big(\delta_H \otimes r_{B_H} \otimes \Pi_H^R\big) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H).$$

Therefore,

$$(\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ \big((r_{B_H} \circ \varphi_{B_H}) \otimes H\big) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$$
$$= (\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ (\mu_H \otimes \varphi_{B_H} \otimes H) \circ (H \otimes c_{H,H} \otimes B_H \otimes H)$$
$$\circ \big(\delta_H \otimes r_{B_H} \otimes \Pi_H^R\big) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H).$$

Finally, by similar arguments, it is easy to show the equality

$$(\mu_H \otimes \varphi_{B_H}) \circ (H \otimes c_{H,H} \otimes B_H) \circ \big((\delta_H \circ \eta_H) \otimes r_{B_H}\big) = r_{B_H}. \qquad \square$$

**Proposition 2.9.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. Then, if*

$$t_{B_H,B_H} = (\varphi_{B_H} \otimes B_H) \circ (H \otimes c_{B_H,B_H}) \circ (r_{B_H} \otimes B_H),$$

*we have the following*:

(1) $\delta_{B_H} \circ \mu_{B_H} = (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes t_{B_H,B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H})$.

(2) $\varepsilon_{B_H} \circ \mu_{B_H} \circ (\mu_{B_H} \otimes B_H)$
$= (\varepsilon_{B_H} \otimes \varepsilon_{B_H}) \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ (B_H \otimes \delta_{B_H} \otimes B_H)$
$= (\varepsilon_{B_H} \otimes \varepsilon_{B_H}) \circ (\mu_{B_H} \otimes \mu_{B_H}) \circ \big(B_H \otimes (t_{B_H,B_H} \circ \delta_{B_H}) \otimes B_H\big)$.

(3) $(\delta_{B_H} \otimes B_H) \circ \delta_{B_H} \circ \eta_{B_H}$
$= (B_H \otimes \mu_{B_H} \otimes B_H) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ (\eta_{B_H} \otimes \eta_{B_H})$
$= \big(B_H \otimes (\mu_{B_H} \circ t_{B_H,B_H}) \otimes B_H\big) \circ (\delta_{B_H} \otimes \delta_{B_H}) \circ (\eta_{B_H} \otimes \eta_{B_H})$.

(4) *There exists an unique morphism $\lambda_{B_H} : B_H \to B_H$ in $\mathcal{C}$ such that*

$$i_H^B \circ \lambda_{B_H} = \mu_B \circ \big((f \circ g) \otimes \lambda_B\big) \circ \delta_B \circ i_H^B$$

*and verifying:*

(4-1) $\mu_{B_H} \circ (B_H \otimes \lambda_{B_H}) \circ \delta_{B_H}$
$= \big((\varepsilon_{B_H} \circ \mu_{B_H}) \otimes B_H\big) \circ (B_H \otimes t_{B_H,B_H}) \circ \big((\delta_{B_H} \circ \eta_{B_H}) \otimes B_H\big)$.

(4-2) $\mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H}$
$= \big(B_H \otimes (\varepsilon_{B_H} \circ \mu_{B_H})\big) \circ (t_{B_H,B_H} \otimes B_H) \circ \big(B_H \otimes (\delta_{B_H} \circ \eta_{B_H})\big)$.

(4-3) $\mu_{B_H} \circ (\mu_{B_H} \otimes B_H) \circ (\lambda_{B_H} \otimes B_H \otimes \lambda_{B_H}) \circ (\delta_{B_H} \otimes B_H) \circ \delta_{B_H} = \lambda_{B_H}$.

**Proof.** (1) This assertion follows from the equalities $(B\otimes q_H^B)\circ\delta_B\circ i_H^B=\delta_B\circ i_H^B$, $q_H^B\circ\mu_B\circ(f\otimes i_H^B)=\mu_B\circ(B\otimes\mu_B)\circ(B\otimes c_{B,B})\circ(f\otimes(\lambda_B\circ f)\otimes B)\circ(\delta_H\otimes i_H^B)$, and $\mu_B\circ(B\otimes(f\circ\Pi_H^R\circ g))\circ\delta_B=\mathrm{id}_B$. Indeed,

$$
\begin{aligned}
&(\mu_{B_H}\otimes\mu_{B_H})\circ(B_H\otimes t_{B_H,B_H}\otimes B_H)\circ(\delta_{B_H}\otimes\delta_{B_H})\\
&\quad=\big((p_H^B\circ\mu_B)\otimes(p_H^B\circ\mu_B)\big)\circ\big(B\otimes(q_H^B\circ\mu_B\circ(f\otimes B))\otimes B\otimes B\big)\\
&\qquad\circ\big(q_H^B\otimes g\otimes c_{B,B}\otimes B\big)\circ\big(B\otimes\delta_B\otimes q_H^B\otimes B\big)\circ\big((\delta_B\circ i_H^B)\otimes(\delta_B\circ i_H^B)\big)\\
&\quad=\big((p_H^B\circ\mu_B)\otimes(p_H^B\circ\mu_B)\big)\\
&\qquad\circ\big(\big(\mu_B\circ\big(B\otimes\big(f\circ\Pi_H^R\circ g\big)\big)\circ\delta_B\big)\otimes(\mu_B\circ c_{B,B})\otimes B\otimes B\big)\\
&\qquad\circ\big(B\otimes(f\circ\lambda_H\circ g)\otimes c_{B,B}\otimes B\big)\circ\big(\delta_B\otimes B\otimes q_H^B\otimes B\big)\circ\big((\delta_B\circ i_H^B)\otimes(\delta_B\circ i_H^B)\big)\\
&\quad=\big((p_H^B\circ\mu_B)\otimes(p_H^B\circ\mu_B)\big)\circ(\mu_B\otimes B\otimes B\otimes B)\circ(B\otimes c_{B,B}\otimes B\otimes B)\\
&\qquad\circ\big(\big(\big(B\otimes(f\circ\lambda_H\circ g)\big)\circ\delta_B\big)\otimes c_{B,B}\otimes B\big)\circ\big(B\otimes B\otimes q_H^B\otimes B\big)\\
&\qquad\circ\big((\delta_B\circ i_H^B)\otimes(\delta_B\circ i_H^B)\big)\\
&\quad=\big((p_H^B\circ\mu_B)\otimes(p_H^B\circ\mu_B)\big)\circ\big(B\otimes(f\circ\lambda_H\circ g)\otimes B\otimes B\big)\circ(\mu_{B\otimes B}\otimes B\otimes B)\\
&\qquad\circ(\delta_B\otimes\delta_B\otimes B\otimes B)\circ\delta_{B\otimes B}\circ\big(i_H^B\otimes i_H^B\big)\\
&\quad=\big(p_H^B\otimes p_H^B\big)\circ\delta_B\circ\mu_B\circ\big(i_H^B\otimes i_H^B\big)\\
&\quad=\delta_{B_H}\circ\mu_{B_H}.
\end{aligned}
$$

(2) Using the equalities $p_H^B\circ\mu_B\circ(B\otimes q_H^B)=p_H^B\circ\mu_B$, $(B\otimes q_H^B)\circ\delta_B\circ i_H^B=\delta_B\circ i_H^B$, and $\varepsilon_B\circ\mu_B\circ(B\otimes q_H^B)=\varepsilon_B\circ\mu_B$, we obtain:

$$
\begin{aligned}
&\varepsilon_{B_H}\circ\mu_{B_H}\circ(\mu_{B_H}\otimes B_H)\\
&\quad=\varepsilon_B\circ q_H^B\circ\mu_B\circ\big(B\otimes q_H^B\big)\circ(B\otimes\mu_B)\circ\big(i_H^B\otimes i_H^B\otimes i_H^B\big)\\
&\quad=\varepsilon_B\circ\mu_B\circ(B\otimes\mu_B)\circ\big(i_H^B\otimes i_H^B\otimes i_H^B\big)\\
&\quad=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ(B\otimes\delta_B\otimes B)\circ\big(i_H^B\otimes i_H^B\otimes i_H^B\big)\\
&\quad=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ\big(B\otimes\big((q_H^B\otimes q_H^B)\circ\delta_B\big)\otimes B\big)\circ\big(i_H^B\otimes i_H^B\otimes i_H^B\big)\\
&\quad=(\varepsilon_{B_H}\otimes\varepsilon_{B_H})\circ(\mu_{B_H}\otimes\mu_{B_H})\circ(B_H\otimes\delta_{B_H}\otimes B_H).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&(\varepsilon_{B_H}\otimes\varepsilon_{B_H})\circ(\mu_{B_H}\otimes\mu_{B_H})\circ\big(B_H\otimes(t_{B_H,B_H}\circ\delta_{B_H})\otimes B_H\big)\\
&\quad=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ\big(B\otimes\big(\mu_B\circ\big((f\circ g)\otimes B\big)\big)\otimes B\otimes B\big)\\
&\qquad\circ(B\otimes B\otimes c_{B,B}\otimes B)\circ\big(B\otimes\big(\delta_B\circ q_H^B\big)\otimes B\otimes B\big)\circ(B\otimes\delta_B\otimes B)
\end{aligned}
$$

$$\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ\Big(B\otimes\big(\mu_B\circ\big((f\circ g)\otimes B\big)\big)\otimes B\otimes B\Big)$$

$$\circ(B\otimes B\otimes c_{B,B}\otimes B)\circ\Big(B\otimes\Big((\mu_B\otimes B)\circ\Big(B\otimes\Big(c_{B,B}\circ\big(q_H^B\otimes(f\circ\lambda_H\circ g)\big)\Big)\Big)$$

$$\circ(\delta_B\otimes B)\circ\delta_B\Big)\otimes B\otimes B\Big)\circ(B\otimes\delta_B\otimes B)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_H\otimes\varepsilon_H)\circ(\mu_H\otimes\mu_H)\circ(H\otimes\mu_H\otimes H\otimes H)$$

$$\circ\Big(H\otimes H\otimes\Big(c_{H,H}\circ\Big(\big(g\circ q_H^B\big)\otimes\big(\mu_H\circ\big((\lambda_H\circ g)\otimes g\big)\big)\Big)\Big)\otimes H\Big)$$

$$\circ(H\otimes\delta_B\otimes B\otimes H)\circ(g\otimes\delta_B\otimes g)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_H\otimes\varepsilon_H)\circ(\mu_H\otimes\mu_H)\circ(H\otimes\mu_H\otimes H\otimes H)$$

$$\circ\Big(H\otimes H\otimes\Big(c_{H,H}\circ\Big(\big(g\circ\Pi_B^L\big)\otimes\big(\Pi_H^R\circ g\big)\Big)\Big)\otimes H\Big)$$

$$\circ(g\otimes g\otimes B\otimes B\otimes g)\circ(B\otimes\delta_B\otimes B\otimes B)\circ(B\otimes\delta_B\otimes B)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ(B\otimes\mu_B\otimes B\otimes B)\circ\Big(B\otimes B\otimes\Big(c_{B,B}\circ\big(\Pi_B^L\otimes\lambda_B\big)\Big)\otimes B\Big)$$

$$\circ(B\otimes\delta_B\otimes B\otimes B)\circ(B\otimes\delta_B\otimes B)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ\Big(B\otimes\big(\delta_B\circ\Pi_B^L\big)\otimes B\Big)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_H\otimes\varepsilon_H)\circ(\mu_H\otimes\mu_H)\circ\big(g\otimes(\delta_H\circ g)\otimes g\big)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_B\otimes\varepsilon_B)\circ(\mu_B\otimes\mu_B)\circ(B\otimes\delta_B\otimes B)\circ\left(i_H^B\otimes i_H^B\otimes i_H^B\right)$$

$$=(\varepsilon_{B_H}\otimes\varepsilon_{B_H})\circ(\mu_{B_H}\otimes\mu_{B_H})\circ(B_H\otimes\delta_{B_H}\otimes B_H).$$

In the previous equalities, the first one follows from $\varepsilon_B\circ\mu_B\circ(B\otimes q_H^B)=\varepsilon_B\circ\mu_B$ and $(B\otimes q_H^B)\circ\delta_B\circ i_H^B=\delta_B\circ i_H^B$. In the second one, we apply

$$\delta_B\circ q_H^B=(\mu_B\otimes B)\circ\Big(B\otimes\Big(c_{B,B}\circ\big(q_H^B\otimes(f\circ\lambda_H\circ g)\big)\Big)\Big)\circ(\delta_B\otimes B)\circ\delta_B.$$

The third one follows from the fact that $f$ and $g$ are morphisms of weak Hopf algebras. In the fourth and the fifth ones, we use the equalities $g\circ q_H^B=\Pi_H^L\circ g$, $(B\otimes(\Pi_H^L\circ g))\circ\delta_B\circ i_H^B=(B\otimes g)\circ\delta_B\circ i_H^B$, $\Pi_H^L\circ g=g\circ\Pi_B^L$, and $\Pi_H^R\circ\Pi_H^L=\lambda_H\circ\Pi_H^L$. The sixth one follows from

$$\delta_B\circ\Pi_B^L=(\mu_B\otimes B)\circ\Big(B\otimes\Big(c_{B,B}\circ\big(\Pi_B^L\otimes\lambda_B\big)\Big)\Big)\circ(\delta_B\otimes B)\circ\delta_B.$$

The seventh and the eighth ones follow from the fact that $f$ and $g$ are morphisms of weak Hopf algebras and by $g\circ\Pi_B^L\circ i_H^B=g\circ i_H^B$. Finally, the last one it is easy to proof.

(3) The arguments are dual to the ones used in the proof of (2), thus it is that we leave the details to the reader.

(4) Let $\Theta_H^B$ be the morphism $\Theta_H^B=\mu_B\circ((f\circ g)\otimes\lambda_B)\circ\delta_B\circ i_H^B: B_H\to B$. This morphism verifies that $(B\otimes g)\circ\delta_B\circ\Theta_H^B=(B\otimes(\Pi_H^L\circ g))\circ\delta_B\circ\Theta_H^B$ and, as a con-

sequence, there exists an unique morphism $\lambda_{B_H}: B_H \to B_H$ such that $i_H^B \circ \lambda_{B_H} = \Theta_H^B$. Therefore,

$$\lambda_{B_H} = p_H^B \circ \Theta_H^B.$$

Using the equalities $(B \otimes q_H^B) \circ \delta_B \circ i_H^B = \delta_B \circ i_H^B$, $p_H^B \circ \mu_B \circ (B \otimes q_H^B) = p_H^B \circ \mu_B$, $\mu_B \circ (B \otimes (f \circ \Pi_H^R \circ g)) \circ \delta_B = \mathrm{id}_B$ and $i_H^B \circ \lambda_{B_H} \circ p_H^B = \mu_B \circ ((f \circ g) \otimes \lambda_B) \circ \delta_B$, we prove (4.1)–(4.3). Indeed,

$$\begin{aligned}
&\mu_{B_H} \circ (B_H \otimes \lambda_{B_H}) \circ \delta_{B_H} \\
&\quad = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ \big(B \otimes \big(\mu_B \circ \big((f \circ \lambda_H \circ g) \otimes (f \circ g)\big)\big) \otimes \lambda_B\big) \\
&\qquad \circ (\delta_B \otimes \delta_B) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \big(\big(\mu_B \circ \big(B \otimes \big(f \circ \Pi_H^R \circ g\big)\big) \circ \delta_B\big) \otimes \lambda_B\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ (B \otimes \lambda_B) \circ \delta_B \circ i_H^B \\
&\quad = \big((\varepsilon_B \circ \mu_B) \otimes p_H^B\big) \circ (B \otimes c_{B,B}) \circ \big((\delta_B \circ \eta_B) \otimes i_H^B\big) \\
&\quad = \big((\varepsilon_B \circ \mu_B) \otimes p_H^B\big) \circ \big(\big(\mu_B \circ \big(q_H^B \otimes (f \circ g)\big) \circ \delta_B\big) \otimes c_{B,B}\big) \circ \big((\delta_B \circ \eta_B) \otimes i_H^B\big) \\
&\quad = \big((\varepsilon_{B_H} \circ \mu_{B_H}) \otimes B_H\big) \circ (B_H \otimes t_{B_H,B_H}) \circ \big((\delta_{B_H} \circ \eta_{B_H}) \otimes B_H\big),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\mu_{B_H} \circ (\lambda_{B_H} \otimes B_H) \circ \delta_{B_H} \\
&\quad = p_H^B \circ \mu_B \circ \big(\big(i_H^B \circ \lambda_{B_H} \circ p_H^B\big) \otimes B\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \big(\big(\mu_B \circ \big((f \circ g) \otimes \lambda_B\big) \circ \delta_B\big) \otimes B\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \Pi_B^R\big) \circ \delta_B \circ i_H^B \\
&\quad = \big(\big(p_H^B \circ \mu_B\big) \otimes (\varepsilon_B \circ \mu_B)\big) \circ \big((f \circ g) \otimes c_{B,B} \otimes B\big) \circ (\delta_B \otimes \delta_B) \circ \big(i_H^B \otimes \eta_B\big) \\
&\quad = \big(B_H \otimes (\varepsilon_{B_H} \circ \mu_{B_H})\big) \circ (t_{B_H,B_H} \otimes B_H) \circ \big(B_H \otimes (\delta_{B_H} \circ \eta_{B_H})\big),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\mu_{B_H} \circ (\mu_{B_H} \otimes B_H) \circ (\lambda_{B_H} \otimes B_H \otimes \lambda_{B_H}) \circ (\delta_{B_H} \otimes B_H) \circ \delta_{B_H} \\
&\quad = p_H^B \circ \mu_B \circ \big(\big(\mu_B \circ \big((f \circ g) \otimes \lambda_B\big) \circ \delta_B\big) \otimes \big(\mu_B \circ (B \otimes \lambda_B) \circ \delta_B\big)\big) \circ \delta_B \circ i_H^B \\
&\quad = p_H^B \circ \mu_B \circ \big((f \circ g) \otimes \big(\mu_B \circ (\mu_B \otimes B) \circ (\lambda_B \otimes B \otimes \lambda_B) \circ (\delta_B \otimes B) \circ \delta_B\big)\big) \\
&\qquad \circ \delta_B \circ i_H^B = \lambda_{B_H}. \qquad \square
\end{aligned} \tag{4.3}$$

**Proposition 2.10.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g: B \to H$ and $f: H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. Let $\omega: B_H \otimes H \to B$ be the morphism defined by $\omega = \mu_B \circ (i_H^B \otimes f)$. If we define $\omega': B \to B_H \otimes H$ by $\omega' = (p_H^B \otimes g) \circ \delta_B$, we have that $\omega \circ \omega' = \mathrm{id}_B$. Moreover, $\omega$ is an isomorphism if and only if $H$ is a Hopf algebra.*

**Proof.** We have

$$\omega \circ \omega' = \mu_B \circ \big(q_H^B \otimes (f \circ g)\big) \circ \delta_B = \mu_B \circ \big(B \otimes \big(f \circ \Pi_H^R \circ g\big)\big) \circ \delta_B = \mathrm{id}_B .$$

If $\omega$ is an isomorphism we have $\omega^{-1} = \omega'$. Then, $\omega' \circ \omega = \mathrm{id}_{B_H \otimes H}$ and therefore $\eta_{B_H} \otimes \varepsilon_H = p_H^B \circ f$. Thus, $q_H^B \circ f = \eta_B \otimes \varepsilon_H$ and, as a consequence, we obtain $f \circ \Pi_H^L = \eta_B \otimes \varepsilon_H$. This equality implies that $\Pi_H^L = \eta_H \otimes \varepsilon_H$ or, equivalently, $H$ is a Hopf algebra.

Conversely, it is well know that if $H$ is a Hopf algebra $\omega$ is an isomorphism with inverse $\omega'$ (see [1]). □

## 3. Weak smash bialgebra structures

**Definition 3.1.** An algebra without unity in $\mathcal{C}$ is a pair $A = (A, \mu_A)$ where $A$ is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$ is a morphism in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$.

A coalgebra without counity in $\mathcal{C}$ is a pair $C = (C, \delta_C)$ where $C$ is an object in $\mathcal{C}$ and $\delta_C : C \to C \otimes C$ is a morphism in $\mathcal{C}$ such that $(\delta_C \otimes C) \circ \delta_C = (C \otimes \delta_C) \circ \delta_C$.

**Definition 3.2.** Let $A$ be an algebra without unity in $\mathcal{C}$. We say that $e : K \to A$ is a preunit if

$$\mu_A \circ (e \otimes A) = \mu_A \circ (A \otimes e) = \mu_A \circ (A \otimes \mu_A) \circ (A \otimes e \otimes e).$$

**Definition 3.3.** Let $A$ and $B$ be algebras and let $R : B \otimes A \to A \otimes B$ be a morphism. We say that $(A, B, R)$ is a weak smash product structure (see, e.g., [6]) if $A \#_R B = (A \otimes B, \mu_{A \#_R B} = (\mu_A \otimes \mu_B) \circ (A \otimes R \otimes B))$ is an algebra without unity and with preunit $\eta_A \otimes \eta_B$.

**Proposition 3.4.** *Let $A$ and $B$ be algebras and let $R : B \otimes A \to A \otimes B$ be a morphism. Then $(A, B, R)$ is a weak smash product structure if and only if*

(1) $R \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (B \otimes R)$;
(2) $R \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (R \otimes A)$;
(3) $R \circ (\eta_B \otimes A) = (\mu_A \otimes B) \circ (A \otimes R) \circ (A \otimes \eta_B \otimes \eta_A)$;
(4) $R \circ (B \otimes \eta_A) = (A \otimes \mu_B) \circ (R \otimes B) \circ (\eta_B \otimes \eta_A \otimes B)$.

**Proof.** See [7, (3.2)]. □

In a similar way, it is possible to define a precounit and a weak smash coproduct structure $(C, D, S)$, being $C$ and $D$ coalgebras and $S : C \otimes D \to D \otimes C$ a morphism.

**Definition 3.5.** Let $D$ be a coalgebra without counit. A precounit on $D$ is a morphism $\epsilon : D \to K$ satisfying

$$(\epsilon \otimes D) \circ \delta_D = (D \otimes \epsilon) \circ \delta_D = (\epsilon \otimes \epsilon \otimes D) \circ (\delta_D \otimes D) \circ \delta_D.$$

**Definition 3.6.** Let $C$ and $D$ be coalgebras and let $S : C \otimes D \to D \otimes C$ be a morphism. We say that $(C, D, S)$ is a weak smash coproduct structure (see, e.g., [6]) if $C \ltimes_S D = (C \otimes D, \delta_{C \ltimes_S D} = (C \otimes S \otimes D) \circ (\delta_C \otimes \delta_D))$ is a coalgebra without counit and with precounit $\varepsilon_C \otimes \varepsilon_D$.

**Proposition 3.7.** *Let $C$ and $D$ be coalgebras and let $S : C \otimes D \to D \otimes C$ be a morphism. Then $(C, D, S)$ is a weak smash coproduct structure if and only if*

(1) $(\delta_D \otimes C) \circ S = (D \otimes S) \circ (S \otimes D) \circ (C \otimes \delta_D)$;
(2) $(D \otimes \delta_C) \circ S = (S \otimes C) \circ (C \otimes S) \circ (\delta_C \otimes D)$;
(3) $(\varepsilon_D \otimes C) \circ S = (C \otimes \varepsilon_D \otimes \varepsilon_C) \circ (C \otimes S) \circ (\delta_C \otimes D)$;
(4) $(D \otimes \varepsilon_C) \circ S = (\varepsilon_D \otimes \varepsilon_C \otimes D) \circ (S \otimes D) \circ (C \otimes \delta_D)$.

**Proof.** See [7, (3.8)]. $\square$

**Proposition 3.8.** *Let $H$ be a weak Hopf algebra, and $(A, \varphi_A)$ an algebra, which is also a left $H$-module, such that $\varphi_A \circ (H \otimes \mu_A) = \mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$. The object $(A, \varphi_A)$ is called a left $H$-module algebra if the following equivalent conditions hold*:

(1) $\varphi_A \circ (\mu_H \otimes \eta_A) = (\varphi_A \otimes \varepsilon_H) \circ (H \otimes \eta_A \otimes \mu_H) \circ (\delta_H \otimes H)$.
(2) $\varphi_A \circ (\mu_H \otimes \eta_A) = (\varepsilon_H \otimes \varphi_A) \circ (\mu_H \otimes H \otimes \eta_A) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$.
(3) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) = \mu_A \circ c_{A,A} \circ (\varphi_A \otimes A) \circ (H \otimes \eta_A \otimes A)$.
(4) $\varphi_A \circ (\Pi_H^L \otimes A) = \mu_A \circ (\varphi_A \otimes A) \circ (H \otimes \eta_A \otimes A)$.
(5) $\varphi_A \circ (\overline{\Pi}_H^L \otimes A) \circ (H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.
(6) $\varphi_A \circ (\Pi_H^L \otimes A) \circ (H \otimes \eta_A) = \varphi_A \circ (H \otimes \eta_A)$.

**Proof.** This proposition is the left version of [7, 4.15]. $\square$

**Proposition 3.9.** *Let $H$ be a weak Hopf algebra, and $(B, r_B)$ an algebra, which is also a left $H$-comodule, such that $\mu_{B\otimes H} \circ (r_B \otimes r_B) = r_B \circ \mu_B$. The object $(B, r_B)$ is called a left $H$-comodule algebra if the following equivalent conditions hold*:

(1) $(H \otimes r_B) \circ r_B \circ \eta_B = (H \otimes (\mu_H \circ c_{H,H}) \otimes B) \circ (\delta_H \otimes r_B) \circ (\eta_H \otimes \eta_B)$.
(2) $(H \otimes r_B) \circ r_B \circ \eta_B = (H \otimes \mu_H \otimes B) \circ (\delta_H \otimes r_B) \circ (\eta_H \otimes \eta_B)$.
(3) $(\Pi_H^R \otimes B) \circ r_B = (H \otimes (\mu_B \circ c_{B,B})) \circ (r_B \otimes B) \circ (\eta_B \otimes B)$.
(4) $(\overline{\Pi}_H^L \otimes B) \circ r_B = (H \otimes \mu_B) \circ (r_B \otimes B) \circ (\eta_B \otimes B)$.
(5) $(\Pi_H^R \otimes B) \circ r_B \circ \eta_B = r_B \circ \eta_B$.
(6) $(\overline{\Pi}_H^L \otimes B) \circ r_B \circ \eta_B = r_B \circ \eta_B$.

**Proof.** See [7, 4.11]. $\square$

**Proposition 3.10.** *Let $H$ be a weak Hopf algebra. Let $A$ be a left $H$-comodule algebra and $B$ a left $H$-module algebra. If $R := (\varphi_B \otimes A) \circ (H \otimes c_{A,B}) \circ (r_A \otimes B) : A \otimes B \to B \otimes A$, then $(B, A, R)$ is a weak smash product structure.*

**Proof.** Similar to the proof of [7, 4.16]. □

**Proposition 3.11.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g: B \to H$ and $f: H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. If $\Pi_B^R$ satisfies the equality $\mu_B \circ (\Pi_B^R \otimes B) = \mu_B \circ c_{B,B} \circ (\Pi_B^R \otimes B)$ then $(B_H, H, R)$ is a weak smash product structure, being $R = (\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$. Moreover, the morphism $\omega = \mu_B \circ (i_H^B \otimes f): B_H \otimes H \to B$ is multiplicative and verifies that $\omega \circ (\eta_{B_H} \otimes \eta_H) = \eta_B$.*

**Proof.** Trivially, $(H, \delta_H)$ is a left $H$-comodule algebra. Moreover, by Proposition 2.4, $B_H$ is an algebra and by Proposition 2.5 $(B_H, \varphi_{B_H})$ is a left $H$-module.

On the other hand, $(B_H, \varphi_{B_H})$ satisfies the equality

$$\mu_{B_H} \circ (\varphi_{B_H} \otimes \varphi_{B_H}) \circ (H \otimes c_{H,B_H} \otimes B_H) \circ (\delta_H \otimes B_H \otimes B_H) = \varphi_{B_H} \circ (H \otimes \mu_{B_H}).$$

Indeed:

$$\begin{aligned}
& i_H^B \circ \mu_{B_H} \circ (\varphi_{B_H} \otimes \varphi_{B_H}) \circ (H \otimes c_{H,B_H} \otimes B_H) \circ (\delta_H \otimes B_H \otimes B_H) \\
&\quad = \mu_B \circ (\mu_B \otimes \mu_B) \circ \big(B \otimes (\mu_B \circ c_{B,B}) \otimes B \otimes (\mu_B \circ c_{B,B})\big) \\
&\qquad \circ \big(f \otimes (f \circ \lambda_H) \otimes B \otimes f \otimes (f \circ \lambda_H) \otimes B\big) \circ (\delta_H \otimes B \otimes \delta_H \otimes B) \\
&\qquad \circ (H \otimes c_{H,B} \otimes B) \circ \big(\delta_H \otimes i_H^B \otimes i_H^B\big) \\
&\quad = \mu_B \circ (B \otimes \mu_B) \circ \big(\mu_B \otimes \big(\mu_B \circ \big(\Pi_B^R \otimes B\big)\big) \otimes B\big) \circ \big(f \otimes B \otimes f \otimes B \otimes (f \circ \lambda_H)\big) \\
&\qquad \circ (H \otimes B \otimes H \otimes c_{H,B}) \circ (H \otimes B \otimes \delta_H \otimes B) \circ (H \otimes c_{H,B} \otimes B) \circ \big(\delta_H \otimes i_H^B \otimes i_H^B\big) \\
&\quad = \mu_B \circ (B \otimes \mu_B) \circ \big(\mu_B \otimes \big(\mu_B \circ c_{B,B} \circ \big(\Pi_B^R \otimes B\big)\big) \otimes B\big) \\
&\qquad \circ \big(f \otimes B \otimes f \otimes B \otimes (f \circ \lambda_H)\big) \circ (H \otimes B \otimes H \otimes c_{H,B}) \circ (H \otimes B \otimes \delta_H \otimes B) \\
&\qquad \circ (H \otimes c_{H,B} \otimes B) \circ \big(\delta_H \otimes i_H^B \otimes i_H^B\big) \\
&\quad = \mu_B \circ \big(\mu_B \otimes \big(\mu_B \circ (B \otimes \mu_B) \circ (\lambda_B \otimes B \otimes \lambda_B) \circ (\delta_B \otimes B) \circ \delta_B\big)\big) \circ (\mu_B \otimes B \otimes f) \\
&\qquad \circ (f \otimes B \otimes c_{H,B}) \circ (H \otimes c_{H,B} \otimes B) \circ \big(\delta_H \otimes i_H^B \otimes i_H^B\big) \\
&\quad = i_H^B \circ \varphi_{B_H} \circ (H \otimes \mu_{B_H}).
\end{aligned}$$

In the last calculations, the first equality follows by definition of $\varphi_{B_H}$, the second one by the condition of morphism of weak Hopf algebras for $f$, the third one by $\mu_B \circ (\Pi_B^R \otimes B) = \mu_B \circ c_{B,B} \circ (\Pi_B^R \otimes B)$ and finally, in the fourth one, we use the weak Hopf algebra structure of $H$.

We have too $i_H^B \circ \varphi_{B_H} \circ (\Pi_H^L \otimes \eta_{B_H}) = q_H^B \circ f \circ \Pi_H^L = f \circ \Pi_H^L \circ \Pi_H^L = f \circ \Pi_H^L = q_H^B \circ f = i_H^B \circ \varphi_{B_H} \circ (H \otimes \eta_{B_H})$. Therefore, $\varphi_{B_H} \circ (\Pi_H^L \otimes \eta_{B_H}) = \varphi_{B_H} \circ (H \otimes \eta_{B_H})$ and $(B_H, \varphi_{B_H})$ is a left $H$-module algebra. Moreover, by Proposition 3.10, $(B_H, H, R)$ is a weak smash product structure, being $R = (\varphi_{B_H} \otimes H) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H)$.

Finally, since $\mu_B \circ (\Pi_B^R \otimes B) = \mu_B \circ c_{B,B} \circ (\Pi_B^R \otimes B)$, we obtain that $\omega \circ \eta_{B_H \otimes H} = \eta_B$ and $w \circ \mu_{B_H \#_R H} = \mu_B \circ (\omega \otimes \omega)$. Indeed,

$$\begin{aligned}
\omega\circ\mu_{B_H\#_R H} &= \mu_B\circ(\omega\otimes\mu_B)\circ\big(B_H\otimes H\otimes\big(\mu_B\circ\big(B\otimes\Pi_B^R\big)\big)\otimes B\big)\\
&\quad\circ\big(B_H\otimes H\otimes i_H^B\otimes f\otimes f\big)\circ(B_H\otimes H\otimes c_{H,B_H}\otimes H)\\
&\quad\circ(B_H\otimes\delta_H\otimes B_H\otimes H)\\
&= \mu_B\circ(\omega\otimes\mu_B)\circ\big(B_H\otimes H\otimes\big(\mu_B\circ c_{B,B}\circ\big(B\otimes\Pi_B^R\big)\big)\otimes B\big)\\
&\quad\circ\big(B_H\otimes H\otimes i_H^B\otimes f\otimes f\big)\circ(B_H\otimes H\otimes c_{H,B_H}\otimes H)\\
&\quad\circ(B_H\otimes\delta_H\otimes B_H\otimes H)\\
&= \mu_B\circ(\mu_B\otimes\mu_B)\circ\big(B\otimes\big(\mu_B\circ\big(B\otimes\Pi_B^R\big)\circ\delta_B\big)\otimes B\otimes B\big)\\
&\quad\circ\big(i_H^B\otimes f\otimes i_H^B\otimes f\big)=\mu_B\circ(\omega\otimes\omega). \qquad\square
\end{aligned}$$

The proofs of the following two propositions are similar to the ones of Propositions 3.8 and 3.9.

**Proposition 3.12.** *Let $H$ be a weak Hopf algebra. Let $(C, r_C)$ be a coalgebra, which is also a left $H$-comodule, such that $(H\otimes\delta_C)\circ r_C=(\mu_H\otimes C\otimes C)\circ(H\otimes c_{C,H}\otimes C)\circ(r_C\otimes r_C)\circ\delta_C$. The object $(C, r_C)$ is called a left $H$-comodule coalgebra if the following equivalent conditions hold*:

(1) $(\delta_H\otimes\varepsilon_C)\circ r_C=(\mu_H\otimes H)\circ(H\otimes\varepsilon_C\otimes\delta_H)\circ(r_C\otimes\eta_H)$.
(2) $(\delta_H\otimes\varepsilon_C)\circ r_C=(\mu_H\otimes H\otimes\varepsilon_C)\circ(H\otimes c_{H,H}\otimes C)\circ(\delta_H\otimes r_C)\circ(\eta_H\otimes C)$.
(3) $(\overline{\Pi}_H^R\otimes C)\circ r_C=(H\otimes\varepsilon_C\otimes C)\circ(r_C\otimes C)\circ c_{C,C}\circ\delta_C$.
(4) $(\Pi_H^L\otimes C)\circ r_C=(H\otimes\varepsilon_C\otimes C)\circ(r_C\otimes C)\circ\delta_C$.
(5) $(\overline{\Pi}_H^R\otimes\varepsilon_C)\circ r_C=(H\otimes\varepsilon_C)\circ r_C$.
(6) $(\Pi_H^L\otimes\varepsilon_C)\circ r_C=(H\otimes\varepsilon_C)\circ r_C$.

**Proposition 3.13.** *Let $H$ be a weak Hopf algebra. Let $(D,\varphi_D)$ be a coalgebra, which is also a left $H$-module, such that $\delta_D\circ\varphi_D=(\varphi_D\otimes\varphi_D)\circ\delta_{H\otimes D}$. The object $(D,\varphi_D)$ is called a left $H$-module coalgebra if the following equivalent conditions hold*:

(1) $\varepsilon_D\circ\varphi_D\circ(\mu_H\otimes D)=(\varepsilon_H\otimes\varepsilon_D)\circ(\mu_H\otimes\varphi_D)\circ(H\otimes(c_{H,H}\circ\delta_H)\otimes D)$.
(2) $\varepsilon_D\circ\varphi_D\circ(\mu_H\otimes D)=(\varepsilon_H\otimes\varepsilon_D)\circ(\mu_H\otimes\varphi_D)\circ(H\otimes\delta_H\otimes D)$.
(3) $\varphi_D\circ(\Pi_H^R\otimes D)=(\varepsilon_D\otimes D)\circ(\varphi_D\otimes D)\circ(H\otimes(c_{D,D}\circ\delta_D))$.
(4) $\varphi_D\circ(\overline{\Pi}_H^R\otimes D)=(\varepsilon_D\otimes D)\circ(\varphi_D\otimes D)\circ(H\otimes\delta_D)$.
(5) $\varepsilon_D\circ\varphi_D\circ(\Pi_H^R\otimes D)=\varepsilon_D\circ\varphi_D$.
(6) $\varepsilon_D\circ\varphi_D\circ(\overline{\Pi}_H^R\otimes D)=\varepsilon_D\circ\varphi_D$.

**Proposition 3.14.** *Let $C$ be a left $H$-comodule coalgebra, $D$ a left $H$-comodule algebra, and $S:=(\varphi_D\otimes C)\circ(H\otimes c_{C,D})\circ(r_C\otimes D): C\otimes D\to D\otimes C$. Then, $(C, D, S)$ is a weak smash coproduct structure.*

**Proof.** Dual to Proposition 3.10. □

**Proposition 3.15.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. If $\Pi_B^R$ satisfies the equality $(\Pi_B^R \otimes B) \circ \delta_B = (\Pi_B^R \otimes B) \circ c_{B,B} \circ \delta_B$ then $(B_H, H, S)$ is a weak smash coproduct structure, being $S = (\mu_H \otimes B) \circ (H \otimes c_{B,H}) \circ (r_B \otimes H)$. Moreover, the morphism $\omega' = (p_H^B \otimes g) \circ \delta_B : B \to B_H \otimes H$ is comultiplicative and verifies that $(\varepsilon_{B_H} \otimes \varepsilon_H) \circ \omega' = \varepsilon_B$.*

**Proof.** The calculations are similar to the ones developed in the proof of Proposition 3.11 and we leave the details to the reader. □

**Definition 3.16.** Let $H$ be a weak bialgebra, $A$, $B$ algebras coalgebras, and $R : B \otimes A \to A \otimes B$, $S : A \otimes B \to B \otimes A$ two morphisms. We say that $(A, B, R, S)$ is a weak smash bialgebra structure if

(1) $(A, B, R)$ is a weak smash product structure;
(2) $(A, B, S)$ is a weak smash coproduct structure;
(3) $\delta_{A\ltimes_S B} \circ \mu_{A\#_R B}$
$= (\mu_{A\#_R B} \otimes \mu_{A\#_R B}) \circ (A \otimes B \otimes c_{A\otimes B, A\otimes B} \otimes A \otimes B) \circ (\delta_{A\ltimes_S B} \otimes \delta_{A\ltimes_S B})$;
(4) $\varepsilon_{A\ltimes_S B} \circ \mu_{A\#_R B} \circ (\mu_{A\#_R B} \otimes A \otimes B)$
$= (\varepsilon_{A\ltimes_S B} \otimes \varepsilon_{A\ltimes_S B}) \circ (\mu_{A\#_R B} \otimes \mu_{A\#_R B}) \circ (A \otimes B \otimes \delta_{A\ltimes_S B} \otimes A \otimes B)$
$= (\varepsilon_{A\ltimes_S B} \otimes \varepsilon_{A\ltimes_S B}) \circ (\mu_{A\#_R B} \otimes \mu_{A\#_R B}) \circ (A \otimes B \otimes (c_{A\otimes B, A\otimes B} \circ \delta_{A\ltimes_S B}) \otimes A \otimes B)$;
(5) $(\delta_{A\ltimes_S B} \otimes A \otimes B) \circ \delta_{A\ltimes_S B} \circ \eta_{A\#_R B}$
$= (A \otimes B \otimes \mu_{A\#_R B} \otimes A \otimes B) \circ (\delta_{A\ltimes_S B} \otimes \delta_{A\ltimes_S B}) \circ (\eta_{A\#_R B} \otimes \eta_{A\#_R B})$
$= (A \otimes B \otimes (\mu_{A\#_R B} \circ c_{A\otimes B, A\otimes B}) \otimes A \otimes B) \circ (\delta_{A\ltimes_S B} \otimes \delta_{A\ltimes_S B}) \circ (\eta_{A\#_R B} \otimes \eta_{A\#_R B})$.

**Proposition 3.17.** *Let $H$, $B$ be weak Hopf algebras in $\mathcal{C}$. Let $g : B \to H$ and $f : H \to B$ be morphisms of weak Hopf algebras such that $g \circ f = \mathrm{id}_H$. If $\mu_B \circ (\Pi_B^R \otimes B) = \mu_B \circ c_{B,B} \circ (\Pi_B^R \otimes B)$, $(\Pi_B^R \otimes B) \circ \delta_B = (\Pi_B^R \otimes B) \circ c_{B,B} \circ \delta_B$ and $R$, $S$ are the morphisms defined in Propositions* 3.11 *and* 3.15*, respectively, we have that $(B_H, H, R, S)$ is a weak smash bialgebra structure.*

**Proof.** By Propositions 3.11 and 3.15, $(B_H, H, R)$ is a weak smash product structure and $(B_H, H, S)$ is a weak smash coproduct structure. Now, we are going to show (3). Put

$$\Upsilon = (\mu_H \otimes B_H) \circ (H \otimes c_{B_H,H}) \circ \big((r_{B_H} \circ \varphi_{B_H}) \otimes H\big) \circ (H \otimes c_{H,B_H}) \circ (\delta_H \otimes B_H),$$
$$\Omega = (\mu_H \otimes \varphi_{B_H}) \circ (H \otimes c_{H,H} \otimes B_H) \circ (\delta_H \otimes r_{B_H});$$

then, if

$$\mu_B \circ \big(\Pi_B^R \otimes B\big) = \mu_B \circ c_{B,B} \circ \big(\Pi_B^R \otimes B\big), \qquad \big(\Pi_B^R \otimes B\big) \circ \delta_B = \big(\Pi_B^R \otimes B\big) \circ c_{B,B} \circ \delta_B,$$

by Propositions 1.5, 1.6, and Remark 2.7 we have that $\Upsilon = \Omega$. Therefore,

$$\delta_{B_H \ltimes S H} \circ \mu_{B_H \#_R H}$$
$$= (\mu_{B_H} \otimes \mu_H \otimes B_H \otimes H) \circ (B_H \otimes B_H \otimes H \otimes c_{B_H,H} \otimes H)$$

$$
\begin{aligned}
&\circ \big(B_H \otimes \varphi_{B_H} \otimes (r_{B_H} \circ \mu_{B_H}) \otimes \mu_H \otimes \mu_H\big)\\
&\circ (B_H \otimes H \otimes c_{B_H,B_H} \otimes B_H \otimes \delta_{H\otimes H}) \circ \big(B_H \otimes r_{B_H} \otimes (\delta_{B_H} \circ \varphi_{B_H}) \otimes H \otimes H\big)\\
&\circ (\delta_{B_H} \otimes H \otimes c_{H,B_H} \otimes H) \circ (B_H \otimes \delta_H \otimes B_H \otimes H)\\
={}& (\mu_{B_H} \otimes \mu_H \otimes B_H \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes H \otimes c_{B_H,H} \otimes H \otimes H)\\
&\circ (B_H \otimes H \otimes B_H \otimes \mu_{H\otimes B_H} \otimes c_{H,H} \otimes H)\\
&\circ (B_H \otimes \mu_H \otimes B_H \otimes r_{B_H} \otimes \Upsilon \otimes H \otimes \delta_H)\\
&\circ (B_H \otimes H \otimes H \otimes c_{B_H,B_H} \otimes H \otimes B_H \otimes H \otimes H)\\
&\circ (B_H \otimes H \otimes c_{B_H,H} \otimes c_{H,B_H} \otimes B_H \otimes H \otimes H)\\
&\circ (B_H \otimes r_{B_H} \otimes \delta_H \otimes \delta_{B_H} \otimes H \otimes H) \circ (\delta_{B_H} \otimes H \otimes c_{H,B_H} \otimes H)\\
&\circ (B_H \otimes \delta_H \otimes B_H \otimes H)\\
={}& (\mu_{B_H} \otimes \mu_H \otimes B_H \otimes \mu_H) \circ (B_H \otimes \varphi_{B_H} \otimes H \otimes c_{B_H,H} \otimes H \otimes H)\\
&\circ (B_H \otimes H \otimes B_H \otimes \mu_{H\otimes B_H} \otimes c_{H,H} \otimes H)\\
&\circ (B_H \otimes \mu_H \otimes B_H \otimes r_{B_H} \otimes \Omega \otimes H \otimes \delta_H)\\
&\circ (B_H \otimes H \otimes H \otimes c_{B_H,B_H} \otimes H \otimes B_H \otimes H \otimes H)\\
&\circ (B_H \otimes H \otimes c_{B_H,H} \otimes c_{H,B_H} \otimes B_H \otimes H \otimes H)\\
&\circ (B_H \otimes r_{B_H} \otimes \delta_H \otimes \delta_{B_H} \otimes H \otimes H) \circ (\delta_{B_H} \otimes H \otimes c_{H,B_H} \otimes H)\\
&\circ (B_H \otimes \delta_H \otimes B_H \otimes H)\\
={}& (B_H \otimes H \otimes \mu_{B_H} \otimes H) \circ (\mu_{B_H} \otimes H \otimes B_H \otimes \varphi_{B_H} \otimes \mu_H)\\
&\circ (B_H \otimes \varphi_{B_H} \otimes \mu_H \otimes B_H \otimes H \otimes c_{H,B_H} \otimes H)\\
&\circ (B_H \otimes H \otimes c_{H,B_H} \otimes c_{B_H,H} \otimes \delta_H \otimes B_H \otimes H)\\
&\circ (B_H \otimes \mu_H \otimes \mu_H \otimes c_{B_H,B_H} \otimes c_{H,H} \otimes B_H \otimes H)\\
&\circ (B_H \otimes \delta_{H\otimes H} \otimes B_H \otimes c_{H,B_H} \otimes \mu_H \otimes B_H \otimes H)\\
&\circ (B_H \otimes H \otimes c_{B_H,H} \otimes H \otimes B_H \otimes H \otimes c_{B_H,H} \otimes H)\\
&\circ (B_H \otimes r_{B_H} \otimes \delta_H \otimes B_H \otimes r_{B_H} \otimes \delta_H) \circ (\delta_{B_H} \otimes H \otimes \delta_{B_H} \otimes H)\\
={}& (\mu_{B_H \#_R H} \otimes \mu_{B_H \#_R H}) \circ (B_H \otimes H \otimes c_{B_H\otimes H, B_H\otimes H} \otimes B_H \otimes H)\\
&\circ (\delta_{B_H \ltimes SH} \otimes \delta_{B_H \ltimes SH}).
\end{aligned}
$$

Finally we will prove (4). The assertion (5) is analogous and we leave the calculations for the reader. Firstly, note that

$$
\begin{aligned}
\mu_H \circ \big((g \circ i_H^B) \otimes H\big) &= \mu_H \circ (g \otimes H) \circ \big(\big((\varepsilon_B \otimes B) \circ \delta_B\big) \otimes H\big) \circ \big(i_H^B \otimes H\big)\\
&= \mu_H \circ \big(\big(\Pi_H^L \circ g\big) \otimes H\big) \circ \big(\big((\varepsilon_B \otimes B) \circ \delta_B\big) \otimes H\big) \circ \big(i_H^B \otimes H\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mu_H \circ c_{H,H} \circ \big(\big(\Pi_H^L \circ g\big) \otimes H\big) \circ \big(\big((\varepsilon_B \otimes B) \circ \delta_B\big) \otimes H\big) \\
&\quad \circ \big(i_H^B \otimes H\big) \\
&= \mu_H \circ c_{H,H} \circ \big(\big(g \circ i_H^B\big) \otimes H\big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\varepsilon_{B_H \ltimes SH} \circ \mu_{B_H \#_R H} \circ (B_H \otimes H \otimes \mu_{B_H \#_R H}) \\
&\quad = \big((\varepsilon_B \circ \mu_B) \otimes (\varepsilon_H \circ \mu_H)\big) \circ (\mu_B \otimes \mu_B \otimes H \otimes H) \circ \big(\mu_B \otimes i_H^B \otimes f \otimes i_H^B \otimes H \otimes H\big) \\
&\quad\quad \circ \big(i_H^B \otimes f \otimes B_H \otimes H \otimes c_{H,B_H} \otimes H\big) \circ (B_H \otimes H \otimes B_H \otimes H \otimes \mu_H \otimes B_H \otimes H) \\
&\quad\quad \circ (B_H \otimes H \otimes B_H \otimes c_{H,H} \otimes H \otimes B_H \otimes H) \circ (B_H \otimes H \otimes c_{H,B_H} \otimes \delta_H \otimes B_H \otimes H) \\
&\quad\quad \circ (B_H \otimes \delta_H \otimes B_H \otimes H \otimes B_H \otimes H) \\
&\quad = (\varepsilon_H \otimes \varepsilon_H) \circ \mu_{H \otimes H} \circ (\mu_{H \otimes H} \otimes H \otimes H) \circ (\mu_H \otimes H \otimes \mu_H \otimes H \otimes H \otimes H) \\
&\quad\quad \circ \big(\big(g \circ i_H^B\big) \otimes \delta_H \otimes \big(g \circ i_H^B\big) \otimes \delta_H \otimes \big(g \circ i_H^B\big) \otimes H\big) \\
&\quad = \varepsilon_H \circ \mu_H \circ \big(\big(g \circ i_H^B\big) \otimes H\big) \circ (\mu_{B_H} \otimes \mu_H) \circ (B_H \otimes \mu_{B_H \otimes H} \otimes H) \\
&\quad\quad \circ (B_H \otimes c_{H,B_H} \otimes c_{H,B_H} \otimes H) \\
&\quad = \big((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)\big) \circ (\mu_H \otimes c_{H,H} \otimes H) \circ (\mu_H \otimes \mu_H \otimes H \otimes H) \\
&\quad\quad \circ \big(\big(g \circ i_H^B\big) \otimes H \otimes \big(g \circ i_H^B\big) \otimes \delta_H \otimes \big(g \circ i_H^B\big) \otimes H\big) \\
&\quad = \big((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)\big) \circ \big(\mu_H \otimes (\delta_H \circ \mu_H) \otimes c_{H,H} \otimes H\big) \\
&\quad\quad \circ \big(\big(g \circ i_H^B\big) \otimes H \otimes \big(g \circ i_H^B\big) \otimes \delta_H \otimes \big(g \circ i_H^B\big) \otimes H\big) \\
&\quad = \big((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)\big) \\
&\quad\quad \circ \big(\big(g \circ i_H^B\big) \otimes (\delta_H \circ \mu_H) \otimes H \otimes \mu_H \otimes c_{H,H} \otimes H\big) \\
&\quad\quad \circ \big(B_H \otimes H \otimes H \otimes c_{H,H} \otimes \delta_H \otimes \big(g \circ i_H^B\big) \otimes H\big) \\
&\quad\quad \circ \big(B_H \otimes H \otimes \big(\delta_H \circ g \circ i_H^B\big) \otimes \delta_H \otimes B_H \otimes H\big) \\
&\quad = \big((\varepsilon_B \circ \mu_B) \otimes (\varepsilon_H \circ \mu_H) \otimes (\varepsilon_B \circ \mu_B) \otimes (\varepsilon_H \circ \mu_H)\big) \\
&\quad\quad \circ (\mu_B \otimes B \otimes \mu_H \otimes c_{B,H} \otimes \mu_B \otimes H \otimes H) \\
&\quad\quad \circ \big(i_H^B \otimes f \otimes c_{H,B} \otimes g \otimes B \otimes H \otimes f \otimes c_{H,B} \otimes H\big) \\
&\quad\quad \circ \big(B_H \otimes \delta_H \otimes B \otimes \delta_B \otimes H \otimes \delta_H \otimes i_H^B \otimes H\big) \\
&\quad\quad \circ \big(B_H \otimes H \otimes \big(\delta_B \circ i_H^B\big) \otimes \delta_H \otimes B_H \otimes H\big) \\
&\quad = (\varepsilon_{B_H \ltimes SH} \otimes \varepsilon_{B_H \ltimes SH}) \circ (\mu_{B_H \#_R H} \otimes \mu_{B_H \#_R H}) \circ (B_H \otimes H \otimes \delta_{B_H \ltimes SH} \otimes B_H \otimes H).
\end{aligned}
$$

In the last calculations, the first and the eighth equalities follows from $\varepsilon_B \circ \mu_B \circ (B \otimes q_H^B) = \varepsilon_B \circ \mu_B$ and $(B \otimes q_H^B) \circ \delta_B \circ i_H^B = \delta_B \circ i_H^B$. In the second, the fifth, the sixth, and the seventh ones, we use that $H$ is a weak Hopf algebra and $f$ and $g$ are morphisms of weak Hopf algebras. Finally, the third and the fourth ones follows from the equality $\mu_H \circ ((g \circ i_H^B) \otimes H) = \mu_H \circ c_{H,H} \circ ((g \circ i_H^B) \otimes H)$.

In a similar way, it is not difficult to see that

$$\begin{aligned} &\varepsilon_{B_H \ltimes S H} \circ \mu_{B_H \#_R H} \circ (B_H \otimes H \otimes \mu_{B_H \#_R H}) \\ &\quad = (\varepsilon_{B_H \ltimes_S H} \otimes \varepsilon_{B_H \ltimes_S H}) \circ (\mu_{B_H \#_R H} \otimes \mu_{B_H \#_R H}) \\ &\qquad \circ \big(B_H \otimes H \otimes (c_{B_H \otimes H, B_H \otimes H} \circ \delta_{B_H \ltimes_S H}) \otimes B_H \otimes H\big). \qquad \square \end{aligned}$$

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