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Crack detection in elastic beams by static measurements

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Abstract

This paper deals with the identification of a single crack in a beam based on the knowledge of the damage-induced variations in the static deflection of the beam. The crack is simulated by an equivalent linear spring connecting the two adjacent segments of the beam. Sufficient conditions on static measurements which allow for the unique identification of the crack are presented and discussed. The inverse analysis provides exact closed-form expressions of position and severity of the crack as functions of deflection measurements for different boundary conditions. The theoretical results are confirmed by a comparison with static measurements on steel beams with a crack. Extension of the presented analysis to multiple cracks is briefly discussed.

Keywords: Damage assessment; Beams; Static loads

1. Introduction

In several areas of civil and mechanical engineering, at present, real challenges arising for the control, maintenance and retrofitting of existing structures and machinery concern the diagnostic identification of damages. To this purpose, nondestructive testing is of great interest under several respects, because it can provide a direct assessment of integrity of structures during service or can be employed to assess the residual resistance of a structure after the occurrence of a strong seismic event.

Within the large class of methods of nondestructive testing, static and dynamic techniques as diagnostic tools in structural mechanics have received great attention in the engineering communities in last decades. Conventional methods of nondestructive testing and evaluation such as visual inspection, radiography, thermal analysis, ultrasonic testing, are very sensitive in terms of global assessment of a structure. In fact, they usually require that the vicinity of the damage is known a priori and the portion of the structure being inspected is readily accessible. Modal analysis techniques or static methods, on the contrary, offer potential advantages for damage detection in a global scale.

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In most of the diagnostic techniques, damage identification procedures are based on dynamic data (see, for example, Adams et al., 1978; Gudmundson, 1982; Rizos et al., 1990; Hearn and Testa, 1991; Liang et al., 1992; Morassi, 1993; Narkis, 1994; Capecchi and Vestroni, 1999; Vestroni and Capecchi, 2000; Chaudhari and Maiti, 2000; Morassi, 2001; Pai and Young, 2001; Lele and Maiti, 2002; Dilena and Morassi, 2004; Gladwell, 2004, Chapter 15, for an updated review). Dynamic identification techniques, for the inverse problem of detecting a single open crack in an elastic straight beam in bending, provide explicit expressions for the position and the severity of the crack only in the case of small damages and for initially uniform beams under special boundary conditions (pinned–pinned, sliding–sliding), see, for example, Narkis (1994) and Morassi (2001). Nondestructive tests in dynamic regime provide, in general, a large number of information with respect to static tests and, furthermore, since they can be easily carried out without interrupting the operation of a system, they are profitably repeatable during service. However, in cases of simple structural system, such as straight beams, subject to damage, static tests are easily executable and provide additional information to dynamic identification without any introduction of uncertainties due to inertia distribution and damping ratios. In the specialized literature there are, in fact, studies, although less numerous, proposing identification procedures based on measurements by static tests or simultaneous use of static and dynamic data aiming at structural identification or damage detection in structures (Hajela and Soeiro, 1990; Sanayei and Onipede, 1991; Sanayei and Scampoli, 1991; Hjelmstad and Shin, 1997). An optimization procedure for damage identification in straight beams by means of bending moment measurements by static tests has been proposed by Di Paola and Bilello (2004). In this procedure the damage has been modelled as a distortion superimposed to the undamaged beam. The identification algorithm is formulated as a constrained least-squared minimization problem, where the few parameters defining the damage (position and severeness) are estimated in an iterative way.

Recently, Buda and Caddemi (2007) proposed an identification procedure of concentrated damages, like (open) cracks, for straight beams in bending under static loads. On the basis of closed form solutions obtained for an open crack modelled as linear rotational spring, Buda and Caddemi (2007) formulated the identification problem as a nonlinear optimization procedure consisting on the minimization of an error function measuring the error between the analytical model deflections and the experimental data.

Aim of this paper is to reconsider the inverse problem of detecting a single crack in an elastic straight beam in bending from static measurements in order to provide explicit expressions for the position and the severity, which represent the exact solutions of the inverse problem. Here, the attention is also focussed on finding sufficient analytical conditions which allow for a rigorous, e.g. mathematically proved, identification of the damage.

The open crack is modelled as a linear elastic rotational spring located at the cracked cross-section. The explicit expression of the damage-induced variation in the deflection of the beam, tested in the undamaged configuration and in a damaged one under the same load distribution, allows to set a procedure to solve the inverse damage identification problem. Sufficient conditions for the unique determination of both the damage location and the damage severity together with exact closed form solutions in terms of deflection measurements are obtained. The last section of the paper is devoted to some numerical applications including also evaluation of the sensitivity of the presented closed-form solutions to instrumental noise affecting the measurements.

2. Crack-induced variations in the deflection of a beam structure

The present method of crack identification is based on an explicit expression of crack-induced variations in the deflection of a beam structure. To illustrate the main idea the case of slender straight beams under transversal loads will be first considered. The approach can be straightforwardly extended to beams under longitudinal loads and to more complex beam-like structures.

Bending deflections of an elastic beam of length L are governed by the Bernoulli–Euler equation

$$-(av'')'' + p = 0 \quad \text{in } (0, L), \tag{1}$$

where $v = v(x)$ is the transversal displacement of the beam axis evaluated at the cross-section of abscissa x and $p = p(x)$ is the transversal load per unit length acting along the beam axis.

For definiteness p is assumed to be a regular load distribution, e.g. a continuous function, but this regularity request can be significantly weakened to include, in the limit case, also concentrate loads. The quantity $a = a(x)$ denotes the bending stiffness of the beam and will be assumed to be continuous and such that $a(x) \geqslant a_0$ for every $x \in [0, L]$, where a_0 is a positive constant. Also for $a(x)$, the regularity request can be weakened to include more general cases.

Let the ends of the beam be restrained by translational and rotational elastic springs. In this case the boundary conditions are the following:

$$(av'')' + h_1 v = 0 = av'' - g_1 v' \quad \text{for } x = 0, \tag{2}$$

$$(av'')' - h_2 v = 0 = av'' + g_2 v' \quad \text{for } x = L, \tag{3}$$

where h_1, h_2, $0 < h_i < \infty$, $i = 1,2$, and g_1, g_2, $0 < g_i < \infty$, $i = 1,2$, are the elastic constants of the translational and rotational springs, respectively, at the ends of the beam.

When the elastic constants of the springs assume limit values, e.g. *0* or ∞, one has the well known *ideal* boundary conditions:

$$\text{pinned–pinned}: h_1 = \infty, \quad g_1 = 0, \quad h_2 = \infty, \quad g_2 = 0; \tag{4}$$

$$\text{clamped–free}: h_1 = \infty, \quad g_1 = \infty, \quad h_2 = 0, \quad g_2 = 0; \tag{5}$$

$$\text{pinned–clamped}: h_1 = \infty, \quad g_1 = 0, \quad h_2 = \infty, \quad g_2 = \infty; \tag{6}$$

$$\text{clamped–clamped}: h_1 = \infty, \quad g_1 = \infty, \quad h_2 = \infty, \quad g_2 = \infty. \tag{7}$$

Suppose that a crack appears at the cross-section of abscissa $s \in (0, L)$ and that the crack always remains open during the deformation of the beam.

A crack on a beam element significantly increases the flexibility due to the strain energy concentration in the vicinity of the crack tip under load. Following, for example, Freund and Herrmann (1976) and Gounaris and Dimarogonas (1988), a crack can be macroscopically modelled as an elastic link connecting the two adjacent segments of beam. In the present analysis, since only in-plane bending deflections are considered, the rotational crack compliance is assumed to be dominant in the local flexibility matrix. Therefore, an open crack is modelled by inserting an elastic rotational spring at the damaged cross-section. The values of the stiffness K of the spring are tabulated for a large number of cases, for different geometry of the cross-section and different crack shape. When a lateral crack of uniform depth δ is present in a rectangular cross-section of width b and height h, for example, the stiffness K has the expression

$$K = \frac{EI}{5.346hC\left(\frac{\delta}{h}\right)}, \tag{8}$$

where E is the Young's modulus of the beam material, I is the moment of inertia of the beam cross-section and the dimensionless local compliance $C\left(\frac{\delta}{h}\right)$ has the expression

$$\begin{aligned} C\left(\frac{\delta}{h}\right) = {} & 1.8624\left(\frac{\delta}{h}\right)^2 - 3.95\left(\frac{\delta}{h}\right)^3 + 16.375\left(\frac{\delta}{h}\right)^4 - 37.226\left(\frac{\delta}{h}\right)^5 + 76.81\left(\frac{\delta}{h}\right)^6 - 126.9\left(\frac{\delta}{h}\right)^7 \\ & + 172\left(\frac{\delta}{h}\right)^8 - 143.97\left(\frac{\delta}{h}\right)^9 + 66.56\left(\frac{\delta}{h}\right)^{10}, \end{aligned} \tag{9}$$

see, for example, Rizos et al. (1990).

Hence, the static deflection of the damaged beam, under the load distribution p, is governed by the following differential equation

$$-(a\tilde{v}'')'' + p = 0 \quad \text{in } (0, s) \cup (s, L), \tag{10}$$

where, in addition to the boundary conditions (2) and (3), one has to consider also the jump conditions

$$[\tilde{v}(s)] = [a(s)\tilde{v}''(s)] = [(a(s)\tilde{v}''(s))'] = 0, \tag{11}$$

$$a(s)\tilde{v}''(s) = K[\tilde{v}'(s)], \tag{12}$$

that are to hold at the cross-section where the crack occurs. In (10)–(12), $\widetilde{v}$ is a function belonging to $H^1(0,L)$ with square summable second derivatives, e.g. $\widetilde{v} \in H^1(0,L) \cap (H^2(0,s) \cup H^2(s,L))$, where $H^m(I)$, $m = 1,2$, are the usual Sobolev spaces on the interval $I \subset \mathbb{R}$. Moreover, $[\phi(s)] \equiv (\phi(s^+) - \phi(s^-))$ is the jump of the function ϕ at $x = s$.

Let

$$w(x) \equiv \widetilde{v}(x) - v(x) \tag{13}$$

be the function which represents the crack-induced variation in the deflection of the beam under the same load distribution p. A direct computation shows that w satisfies the following differential equation

$$(aw'')'' = 0 \quad \text{in } (0,s) \cup (s,L), \tag{14}$$

coupled with the jump conditions

$$[w(s)] = [a(s)w''(s)] = [(a(s)w''(s))'] = 0, \tag{15}$$

$$K[w'(s)] = -\widetilde{M}(s), \tag{16}$$

and a set of boundary conditions which coincide with those satisfied by the undamaged beam. In Eq. (16), the expression

$$\widetilde{M}(s) \equiv -a(s)\widetilde{v}''(s) \tag{17}$$

denotes the bending moment at the cross-section where the damage occurs, in the damaged beam, under the load distribution p.

An expressive physical interpretation of the function w can be inferred by an inspection of the governing Eq. (14) coupled with the jump conditions (15) and (16): $w(x)$ is the transversal displacement of the undamaged beam when, under the same boundary conditions, the singular angular distortion

$$\alpha = -\frac{\widetilde{M}(s)}{K} \tag{18}$$

is introduced between the two cross-sections adjacent to the damaged cross-section, at the abscissa $x = s$. The latter interpretation is in agreement with the principle of virtual distortion introduced by Di Paola (2004) for systems with uncertain parameters.

It is worth noticing that (i) no condition on the smallness of the damage has been introduced in the analysis above, that is w is defined for a crack of generic severeness; (ii) the present analysis can be easily extended to the case of multiple cracks (see at the end of this section); (iii) $w(x)$ is not identically equal to the zero function if and only if $\widetilde{M}(s) \neq 0$, or, equivalently, if the load distribution is such that the bending moment at the damaged cross-section is different from zero. This last condition $\widetilde{M}(s) \neq 0$ can be read as a sort of identificability or observability condition to be satisfied by the load distribution.

In the diagnostic *inverse* problem, one seeks to extract information on damage location and severeness from measurements of the crack-induced variation in the deflection $w(x)$ of the beam axis under a single, prescribed load distribution. Therefore, the crucial point of the inverse analysis lies on the determination of $w(x)$ and on the study of its properties.

The relative deflection w can be evaluated by solving the boundary value problem (14)–(16) with the appropriate set of boundary conditions. There is, however, a more convenient approach which is based on an extended version of the Betti–Maxwell Theorem for beam structures with a singular angular distortion. More precisely, for every $\bar{x} \in (0,L)$, one can prove that

$$w(\bar{x}) = -\alpha M_{\bar{x}}(s), \tag{19}$$

where $M_{\bar{x}}(s)$ is the bending moment at the cross-section of abscissa s, $s \in (0,L)$, of the undamaged beam when a unit transversal, positive (downwards directed) force is applied at the cross-section of abscissa $\bar{x}$.

It is not excessive to say that all the present damage identification procedure originates from Eq. (19). In fact, by recalling the definition of α given in (18), from (19) one can obtain the *fundamental identity*

$$w_s(\bar{x}) = \frac{1}{K}\widetilde{M}(s,K)M_{\bar{x}}(s), \tag{20}$$

namely, the damage-induced variation in the deflection of the beam due to a crack located at s and of severity K, under a given load distribution, is proportional to the product of two quantities. The first quantity, $\widetilde{M}(s,K)$, is the bending moment present at the cracked cross-section, of abscissa s, of the damaged beam. Note that, with the exception of statically determinate beams, $\widetilde{M}(s, K)$ depends also on K. The second quantity, $M_{\bar{x}}(s)$, is the bending moment present at the cross-section of abscissa s due to a unity force acting at the cross-section of abscissa $\bar{x}$ of the undamaged beam.

It is worth noticing that $\widetilde{M}(s, K)$ depends on the assigned load distribution, whereas $M_{\bar{x}}(s)$ only depends on the properties of the undamaged beam (boundary conditions and stiffness coefficient a) and on the force location $\bar{x}$. Moreover, no matter how the boundary conditions and the coefficient a are, the function $M_{\bar{x}}(s)$ is *a continuous, piecewise linear function of the variable s*.

Finally, it is worth noticing that Eq. (19) allows for a great simplification on the calculations needed to obtain w, because it only requires the evaluation of the bending moment on the undamaged beam under a unit transversal force, and this can be done by well established techniques.

The extension of the fundamental identity (20) to multiple open cracks is immediate. With the usual notation and by modelling each crack with an elastic rotational spring of stiffness K_i at the damaged cross-section of abscissa s_i, $i = 1,\ldots,n$, one has

$$w(\bar{x}) = \frac{1}{K_1}\widetilde{M}(s_1)M_{\bar{x}}(s_1) + \cdots + \frac{1}{K_n}\widetilde{M}(s_n)M_{\bar{x}}(s_n). \tag{21}$$

3. Identification of a crack in an elastic beam from static tests

In this section the inverse problem of identifying a single open crack in a uniform elastic beam by using static measurements will be closely investigated. An extension of the proposed diagnostic technique to beams with varying bending stiffness will be presented in the last part of this section.

The main goal of this paper is to find a minimal set of sufficient conditions on static measurements which allows for the unique identification of the crack. The key point of the diagnostic procedure will be presented by considering separately the cases corresponding to the boundary conditions (4) and (7). Clamped–free (5) and pinned–clamped (6) boundary conditions can be discussed similarly.

Without loss of generality, let $\alpha \neq 0$, that is the given load distribution satisfies the observability condition $\widetilde{M}(s) \neq 0$, see Eqs. (16)–(18).

3.1. Pinned–pinned beam

An easy computation shows that the solution w of the boundary value problem (14)–(16), coupled with the boundary conditions (4), is given by

$$w(x) = -\alpha \cdot \begin{cases} x(1-\frac{s}{L}), & 0 \leqslant x \leqslant s, \\ s(1-\frac{x}{L}), & s \leqslant x \leqslant L, \end{cases} \tag{22}$$

where $\alpha = -\widetilde{M}(s,K)/K$. Note that $\mathrm{sign}(w(x)) = -\mathrm{sign}(\alpha)$ in $(0,L)$, that is $w(x) \neq 0$ in $(0,L)$. Assume that two measurements of the relative transversal displacement w are taken at two points of the beam axis of abscissa η_1, η_2, where

$$0 < \eta_1 < s < \eta_2 < L. \tag{23}$$

From Eq. (22), by dividing side by side the expressions of $w(\eta_1)$ and $w(\eta_2)$ the unknown α disappears and the following single linear equation on the damage location s is obtained:

$$s\left(\left(1-\frac{\eta_2}{L}\right)r_{12} + \frac{\eta_1}{L}\right) = \eta_1, \tag{24}$$

where $r_{12} \equiv w(\eta_1)/w(\eta_2) > 0$. Since $((1 - \eta_2/L)r_{12} + \eta_1/L) > 0$, one has

$$s = \frac{\eta_1}{(1 - \frac{\eta_2}{L}) r_{12} + \frac{\eta_1}{L}}. \tag{25}$$

Eq. (25) says that if the relative displacement w is measured at two points η_1, η_2 which are at the left and at the right of the damage, respectively, e.g. $\eta_1 < s < \eta_2$, then the position of the crack can be uniquely determined. Moreover, Eq. (25) is a closed form expression for the damage location s.

It has to be noted that, by dividing side by side the relative displacement $w(x)$ at two points both lying on the same side of the damage, the unknowns α and s disappear. As a result, the damage location s cannot be identified.

Once the damage localization problem is solved by means of Eq. (25), the damage severity K can be easily determined. By measuring the relative displacement $w(x)$ at $x = \eta_2$ and by Eq. (20) the following linear equation on K is obtained:

$$w(\eta_2) = \frac{1}{K} \widetilde{M}(s) s \left(1 - \frac{\eta_2}{L}\right). \tag{26}$$

Since, as for any statically determinate system, the bending moment $\widetilde{M}$ doesn't depend on K and $w(\eta_2) \neq 0$, this equation can be solved to obtain the closed form expression for the damage severity:

$$K = \frac{\widetilde{M}(s) s (1 - \frac{\eta_2}{L})}{w(\eta_2)}. \tag{27}$$

Hence it can be stated that, for the case of a pinned–pinned beam, the condition on the displacement measurements $\eta_1 < s < \eta_2$ can be recognized as the sufficient condition for damage localization.

In a real damage identification procedure, since the damage location is not a priori known, one cannot be sure whether the sufficient condition $\eta_1 < s < \eta_2$ is satisfied by two arbitrarily chosen abscissae η_1, η_2. In order to address this point, let us suppose that two measurements lie at the left of the crack, e.g. $0 < \eta_1 < \eta_2 < s$, and the closed form expression (25) is erroneously adopted for damage localization. In this case displacement measurements $w(\eta_1)$, $w(\eta_2)$ are both expressed by equation $(22)_1$ as follows

$$w(\eta_1) = -\alpha \eta_1 \left(1 - \frac{s}{L}\right), \quad w(\eta_2) = -\alpha \eta_2 \left(1 - \frac{s}{L}\right). \tag{28}$$

Substitution of Eqs. (28) into Eq. (25) leads to

$$s_{\text{ident}} = \eta_2 \quad \text{for } 0 < \eta_1 < \eta_2 < s, \tag{29}$$

hence, in this case, the identified damage position s_{ident} coincides with the measurement position η_2.

Analogously, by taking both measurements at the right of the damage, the following condition can be proved

$$s_{\text{ident}} = \eta_1 \quad \text{for } 0 < s < \eta_1 < \eta_2. \tag{30}$$

Equations (29) and (30) allow to recognize in practice those pairs of measurements which do not satisfy the sufficient condition $\eta_1 < s < \eta_2$, even though the damage position is not a priori known.

On the basis of Eqs. (29) and (30) it is possible to set the following identification procedure, which is not based on an a priori knowledge of the damage position, by means of repeated application of Eq. (25). Choose η_1 next to the left support and vary η_2 from η_1 towards the right support, then substitution of pairs η_1, η_2 in Eq. (25) leads to identified values of s increasing with η_2 (see Eq. (29)); when η_2 reaches the exact damage position s the identified damage position remains constant at its exact value, since the sufficient condition $\eta_1 < s < \eta_2$ is satisfied.

A similar behavior is encountered by choosing η_2 next to the right support and varying η_1 from η_2 backwards towards the left support.

3.2. Clamped–clamped beam

The relative displacement w for the clamped–clamped beam is given by

$$w(x) = -\alpha \cdot \begin{cases} \frac{2x^2(L-x)^2}{L^3} - \left(1 - \frac{(L+2x)(L-x)^2}{L^3}\right)(s-x), & 0 \leqslant x \leqslant s, \\ -\frac{x(L-x)^2}{L^2} + \frac{(L+2x)(L-x)^2}{L^3}s, & s \leqslant x \leqslant L. \end{cases} \tag{31}$$

By measuring w at the points η_1, η_2, with $0 < \eta_1 < s < \eta_2 < L$, one has

$$w(\eta_1) = -\alpha(A_1(\eta_1) + B_1(\eta_1)(s - \eta_1)), \tag{32}$$

$$w(\eta_2) = -\alpha(A_2(\eta_2) + B_2(\eta_2)s), \tag{33}$$

where $A_1(\eta_1) > 0$, $A_2(\eta_2) < 0$, $B_2(\eta_2) > 0$ and $B_1(\eta_1) = -\eta_1^2(3L - 2\eta_1)/L^3 < 0$ are given constants.

Here, the damage location problem is more complicated than the previous case, since $w(\eta_1)$, or $w(\eta_2)$, might now vanish for some choices of s, η_1 and η_2. One can distinguish two situations. If $w(\eta_2) = 0$, then Eq. (33) gives directly

$$s = -\frac{A_2(\eta_2)}{B_2(\eta_2)} = \frac{\eta_2}{1 + 2\frac{\eta_2}{L}}. \tag{34}$$

Otherwise, if $w(\eta_2) \neq 0$, by dividing $w(\eta_1)$ by $w(\eta_2)$, Eqs. (32) and (33) yield to the following linear equation on the unknown variable s:

$$s(r_{12}B_2(\eta_2) - B_1(\eta_1)) = A_1(\eta_1) - B_1(\eta_1)\eta_1 - r_{12}A_2(\eta_2), \tag{35}$$

where $r_{12} \equiv w(\eta_1)/w(\eta_2)$. A direct calculation shows that the coefficient of s does not vanish, so that Eq. (35) has a unique solution. In fact, by taking into account the expressions of A_i and B_i, $i = 1, 2$, one has:

$$r_{12}B_2(\eta_2) - B_1(\eta_1) = -\frac{\alpha\eta_1^2(L - \eta_2)^2}{w(\eta_2)L^4}(2L + \eta_2 - \eta_1), \tag{36}$$

which is different from zero.

Analogous considerations hold if, from the beginning, it is assumed that $w(\eta_1) \neq 0$.

Concerning the determination of the damage severity, the expression of α can be used to obtain the following equation for K:

$$w(\eta_2)K = \widetilde{M}(s, K)(A_2(\eta_2) + B_2(\eta_2)s). \tag{37}$$

Differently from the previous case, the bending moment $\widetilde{M}(s, K)$ now depends, usually in non-linear way, on K, so that numerical methods have to be used to solve (37) in terms of K.

For the case under study there are two other choices for couples of measurements of $w(x)$.

By taking two measurements of $w(x)$ at the points η_1, η_2 such that $0 < \eta_1 < \eta_2 < s$ one has

$$r_{12} = \frac{w(\eta_1)}{w(\eta_2)} = \frac{A_1(\eta_1) + B_1(\eta_1)(s - \eta_1)}{A_1(\eta_2) + B_1(\eta_2)(s - \eta_2)}, \tag{38}$$

under the condition $w(\eta_2) \neq 0$. After simple algebra, Eq. (38) leads to the following equation for s

$$s(r_{12}B_1(\eta_2) - B_1(\eta_1)) = A_1(\eta_1) - B_1(\eta_1)\eta_1 + r_{12}(B_1(\eta_2)\eta_2 - A_1(\eta_2)). \tag{39}$$

The coefficient of s is different from zero, e.g.

$$r_{12}B_1(\eta_2) - B_1(\eta_1) = -\frac{\alpha\eta_1^2\eta_2^2}{L^4}(\eta_2 - \eta_1), \tag{40}$$

and therefore Eq. (39) gives uniquely the damage location s.

If $w(\eta_2) = 0$, then, from Eq. (32) with η_1 replaced by η_2, one has directly

$$s = -\frac{A_1(\eta_2)}{B_1(\eta_2)} + \eta_2 = \frac{2 - \frac{\eta_2}{L}}{3 - \frac{\eta_2}{L}}L. \tag{41}$$

Finally, similar considerations hold true when the two measurements $w(\eta_1)$, $w(\eta_2)$ are taken at the right of the damaged cross-section, that is $0 < s < \eta_1 < \eta_2 < L$. In brief, if $w(\eta_2) \neq 0$, the following expression for s is obtained

$$s = \frac{A_2(\eta_1) - r_{12}A_2(\eta_2)}{r_{12}B_2(\eta_2) - B_2(\eta_1)}, \tag{42}$$

where the denominator $(r_{12}B_2(\eta_2) - B_2(\eta_1))$ always is different from zero. Otherwise, if $w(\eta_2) = 0$, then the expression for s is given by Eq. (34).

It can be concluded that, for the case of clamped–clamped beams, any pair of measurements can be adopted for the exact damage localization, provided that the correct explicit expression is used as follows:

(a) Eq. (35) for $0 < \eta_1 < s < \eta_2 < L$,
(b) Eq. (39) for $0 < \eta_1 < \eta_2 < s$,
(c) Eq. (42) for $s < \eta_1 < \eta_2 < L$.

However, since the damage position is not a priori known, one cannot be sure of the explicit expression to be adopted. By arguing as in the last part of the previous section, the following relevant properties, holding when Eq. (35) is erroneously adopted, can be derived:

(i) Eq. (35) provides $\eta_2 < s_{\text{ident}} < s$ for $0 < \eta_1 < \eta_2 < s$,
(ii) Eq. (35) provides $s < s_{\text{ident}} < \eta_1$ for $s < \eta_1 < \eta_2 < L$.

Properties (i) and (ii) are employed for an identification procedure presented in the numerical application.

3.3. Non-uniform beams

In this part it will be shown how the previous arguments can be adapted to investigate the more general case of non-uniform beams, e.g., beams with continuous, nonconstant bending stiffness $a(x)$ satisfying the condition $a(x) \geqslant a_0$ for every $x \in [0, L]$, where a_0 is a positive constant. In particular, the attention will be focussed on the damage localization problem.

Again, the starting point is the special form of the expression (19) for the variation w of the transversal displacement caused by the crack, under a given load distribution, and, in particular, the linear dependence of $w(x)$ on the bending moment $M_x(s)$. It is recalled that $M_x(s)$ is the bending moment present at the cross-section of the undamaged beam of abscissa s induced by a unit transversal force acting downwards at the point of the beam axis of abscissa x.

For statically determinate beams, such as the pinned–pinned case, the bending moment function $M_x(s)$ doesn't depend on the bending stiffness $a(x)$. Therefore, the results proved for the uniform case can be directly extended also to non-uniform beams.

For the remaining cases, pinned–clamped and clamped–clamped, one can proceed as follows. For the sake of completeness, the pinned–clamped beam will be considered in detail, the other case being analogous.

By expression (19) it turns out that the variation w of the transversal displacement caused by the crack has the following expression:

$$w(x) = -\alpha \cdot \begin{cases} T_x(0)s - (s - x), & 0 \leqslant x \leqslant s, \\ T_x(0)s, & s \leqslant x \leqslant L. \end{cases} \tag{43}$$

Here, $T_x(0)$ is the shear force present at the left end of the beam due to a unit transversal force acting downwards at the point of the beam axis of abscissa x. Note that $T_x(0)$ is assumed to be positive if the shear force is directed upwards. A direct calculation shows that

$$T_x(0) = \frac{\int_x^L \frac{\xi - x}{a(\xi)} \xi \, d\xi}{\int_0^L \left(\int_s^L \frac{\xi}{a(\xi)} \, d\xi \right) ds}, \quad x \in [0, L], \tag{44}$$

that is, $T_x(0)$ is a regular function of C^1 class for $x \in [0, L]$ such that

$$0 < T_x(0) < 1 \quad \text{for } x \in (0, L), \quad T_0(0) = 1, \ \ T_L(0) = 0. \tag{45}$$

As it will be apparent in the sequel, property (45) plays an important role in solving the damage localization problem. From the mechanical point of view, property (45) has a clear meaning: it says that the vertical reaction at the pinned left end is directed upwards and that its magnitude cannot exceed the magnitude of the applied force.

Now, by measuring w, for example, at the points η_1, η_2, with $0 < \eta_1 < s < \eta_2 < L$, one has

$$w(\eta_1) = -\alpha(T_{\eta_1}(0)s - (s - \eta_1)), \tag{46}$$

$$w(\eta_2) = -\alpha T_{\eta_2}(0)s. \tag{47}$$

By (43) and (45) the relative displacement $w(\eta_2)$ is always different from zero. Therefore, one can divide (46) by (47) obtaining, after a reordering of the terms, the following linear equation on s:

$$s(r_{12}T_{\eta_2}(0) - T_{\eta_1}(0) + 1) = \eta_1, \tag{48}$$

where $r_{12} \equiv w(\eta_1)/w(\eta_2)$. The coefficient of the damage location s is given by

$$r_{12}T_{\eta_2}(0) - T_{\eta_1}(0) + 1 = -\frac{\alpha\eta_1}{w(\eta_2)}T_{\eta_2}(0), \tag{49}$$

and, by (45) again, it is different from zero. It follows that Eq. (48) admits the unique solution

$$s = \frac{\eta_1}{r_{12}T_{\eta_2}(0) - T_{\eta_1}(0) + 1}. \tag{50}$$

The case in which the measurements $w(\eta_1)$ and $w(\eta_2)$ are both taken at the left of the damage can be discussed analogously. Let $0 < \eta_1 < \eta_2 < s < L$. If $w(\eta_2) = 0$, then by (46) it follows that

$$s = \frac{\eta_1}{1 - T_{\eta_2}(0)}, \tag{51}$$

where, by (45), the denominator is positive. Otherwise, if $w(\eta_2) \neq 0$, then the following equation in s can be formed

$$s(r_{12}(T_{\eta_2}(0) - 1) - (T_{\eta_1}(0) - 1)) = \eta_1 - r_{12}\eta_2. \tag{52}$$

A direct calculation shows that the coefficient of s is equal to

$$r_{12}(T_{\eta_2}(0) - 1) - (T_{\eta_1}(0) - 1) = -\frac{\alpha}{w(\eta_2)}(\eta_2(1 - T_{\eta_1}(0)) + \eta_1(1 - T_{\eta_2}(0))), \tag{53}$$

which is different from zero because of condition (45). Therefore, Eq. (52) can be uniquely solved with respect to the damage location s.

Finally, the case of two measurements at the points η_1, η_2 such that $0 < s < \eta_1 < \eta_2 < L$, dividing side by side the expressions of $w(\eta_1)$ and $w(\eta_2)$, leads to the disappearance of both unknowns α and s. As a result, the damage location s cannot be identified.

Up till now, the bending stiffness $a(x)$ has been assumed to be a continuous function in $[0, L]$. The considerations above show that this regularity request can be weakened to include also a bounded function $a(x)$ with a finite number of jump discontinuities. This fact, for example, allows one to extend the above results to *stepped beams*, that is beams with bending stiffness of the form $a(x) = c_i$, $x \in (x_{i-1}, x_i)$, for some subdivision of the interval $[0, L]$ and some set of positive constants c_i. Some applications of the proposed diagnostic technique to stepped beams have been worked out in (Ret, 2004).

3.4. Remark on multiple crack identification

A full treatment of the identification of multiple cracks by static measurements is beyond the goals of this paper. However, for the sake of completeness, a succinct account of the main findings will be presented in the sequel for a pinned–pinned beam with n cracks located at the cross-sections of abscissae $0 < s_1 < s_2 < \cdots < s_n < L$. The bending stiffness $a = a(x)$ of the beam is supposed to be a continuous function satisfying the condition $a(x) \geqslant a_0$ for every $x \in [0, L]$, where a_0 is a positive constant.

A direct calculation based on expression (21) shows that the crack-induced variation in the transversal deflection of the beam under the same load distribution is given by

$$w(x) = \begin{cases} \frac{x}{L}\sum_{i=1}^{n}(-L+s_i)\alpha_i, & 0 \leqslant x \leqslant s_1, \\ \frac{x}{L}\sum_{i=k}^{n}(-L+s_i)\alpha_i + (-1+\frac{x}{L})\sum_{i=1}^{k-1} s_i\alpha_i, & s_{k-1} \leqslant x \leqslant s_k,\ \ k=2,\ldots,n, \\ (-1+\frac{x}{L})\sum_{i=1}^{n} s_i\alpha_i, & s_n \leqslant x \leqslant L, \end{cases} \tag{54}$$

where, according to (18), the singular distortion α_i is defined as $\alpha_i = -\frac{\widetilde{M}(s_i)}{K_i}$.

Let the relative displacement w be measured at the $2n$ points η_1, η_i, η'_i, η_{n+1}, $i = 2,\ldots,n$, satisfying the conditions

$$0 < \eta_1 < s_1 < \eta_2 < \eta'_2 < \cdots < s_{k-1} < \eta_k < \eta'_k < s_k < \cdots < s_n < \eta_{n+1} < L. \tag{55}$$

Conditions (55) mean that w is measured at the $2n$ points η_k and η'_k between each pair of consecutive cracks located at s_{k-1} and s_k and, moreover, it is measured at a single point in the two segments of beam which are delimited by the left support and the first crack (η_1), and the right support and the nth crack (η_{n+1}).

It can be shown that, if $\alpha_i \neq 0$ for every $i = 1,\ldots,n$, then this set of measurements allows for the unique determination of the position and severity of the n cracks. The proof of this result involves the solution of a system of $2n$ non-linear equations in the $2n$ unknowns s_i, K_i, $i = 1,\ldots,n$, and can be found in a paper by Caddemi and Morassi (2006).

Moreover, under the same assumptions, closed form expressions of the damage parameters in terms of the static data are available. For example, in the case of a pinned–pinned beam with two cracks, one obtains the following expressions for the damage locations s_1, s_2, and for the angular distortions α_1, α_2 (related to the equivalent rotational spring stiffnesses K_1, K_2) in terms of the measured deflections:

$$s_1 = \frac{(\eta_2 w(\eta'_2) - \eta'_2 w(\eta_2))\eta_1}{-w(\eta_1)(\eta'_2 - \eta_2) + (w(\eta'_2) - w(\eta_2))\eta_1}, \tag{56}$$

$$s_2 = \frac{Lw(\eta_3)(\eta'_2 - \eta_2) - (\eta_2 w(\eta'_2) - \eta'_2 w(\eta_2))(\eta_3 - L)}{w(\eta_3)(\eta'_2 - \eta_2) - (w(\eta'_2) - w(\eta_2))(\eta_3 - L)}, \tag{57}$$

$$\alpha_1 = -\frac{w(\eta_1)}{\eta_1} - \frac{w(\eta_2) - w(\eta'_2)}{\eta'_2 - \eta_2}, \tag{58}$$

$$\alpha_2 = \frac{w(\eta_2) - w(\eta'_2)}{\eta'_2 - \eta_2} + \frac{w(\eta_3)}{\eta_3 - L}. \tag{59}$$

Finally, as in the identification of a single crack, it is possible to set an identification procedure which is not based on an a priori knowledge of the relative position of the cracks and of the measurement points, see Caddemi and Morassi (2006) for a detailed treatment.

4. Numerical applications

In the preceding sections it has been shown how to employ the measurement of static deflections of a cracked beam so as to assess the location as well as the severity of the damage. Aiming to account for the practical use of the results above within the analysis of real cases, the present section is devoted to outlining some applications of numerical character.

Among several numerical tests performed, some results of the damage identification for pinned–pinned and clamped–clamped cases are presented and discussed in detail in the sequel. They are representative of the results obtained in the investigation and of the main features of the proposed method of crack detection.

In particular, the diagnostic technique is applied to a steel beam of the series IPE 200, of length $L = 6$ m, having bending stiffness of the cross section equal to $a = 4080$ MNm^2, with a single open crack located at the cross section of abscissa $s = 2.8$ m and whose severity is equivalent to a rotational spring stiffness $K = 10$ GNm/rad. In order to reproduce a commonly used experimental set-up, the load distribution acting on the beam is chosen as a concentrated load $P = 10$ KN applied at the mid point of the beam span.

Whatever case of boundary conditions is treated, as it was shown previously, measurements of the relative transversal displacement at two points η_1 and η_2 are needed to localize a single crack.

4.1. Pinned–pinned beam

In order to apply the closed form expression provided by equation (25) for damage localization in a pinned–pinned beam, two relative displacement measurements $w(\eta_1)$, $w(\eta_2)$ have to be taken at η_1, η_2 at the left and at the right of the crack, respectively, e.g. $0 < \eta_1 < s < \eta_2 < L$ (sufficient condition for identification). However, since in the inverse identification problem the damage location is not a priori known, the procedure proposed in the section concerning the pinned–pinned beam is here adopted. In particular, by taking the measurement $w(\eta_1)$ next to the left support at $\eta_1 = 0.1$ m, and the second measurement $w(\eta_2)$ at η_2 increasing along a grid of points with step 0.5 m and using Eq. (25), the results plotted in Fig. 1a have been obtained. Analysis of Fig. 1a shows that the estimated damage position s_{ident} increases with η_2. However, once η_2 reaches the actual damage position 2.8 m, the identified damage position will remain constant at its actual value.

On the other hand, in Fig. 1b the results provided by equation (25), concerning $\eta_2 = 5.9$ m fixed next to the right support and η_1 increasing along a grid of points with step 0.5 m, are plotted. The identified damage position s_{ident} keeps its actual value 2.8 m for $\eta_1 \leqslant 2.8$ m; when η_1 becomes greater than 2.8 m then s_{ident} increases with η_1.

It can be concluded that the deflection of the beam has to be measured in a grid of points $\{x_i\}_{i=1}^{N}$, $0 < x_1 < \cdots < x_N < L$, along the beam axis. The identified damage positions, obtained by replacing the experimental measurements into equation (25), lying on an horizontal line provide the actual damage position. Both, Fig. 1a and b, have been obtained by employing twelve measurements, however, even six, equally spaced, measurements along the beam span would provide the same information and lead to the actual damage position. Once s is identified, the actual damage severity K is provided by Eq. (27).

4.2. Clamped–clamped beam

In this case closed form solutions of damage localization are represented by Eqs. (35), (39), (42) for measurement positions $0 < \eta_1 < s < \eta_2 < L$, $0 < \eta_1 < \eta_2 < s$, $s < \eta_1 < \eta_2 < L$, respectively.

In Fig. 2a and b results obtained by the adoption of Eq. (35) only are reported. In particular, in Fig. 2a, with η_1 fixed at 0.1 m, the identified damage position is plotted for η_2 increasing along the beam span. In Fig. 2b, for η_2 fixed at 5.9 m, the identified damage position is plotted for η_1 increasing along the beam span. Both Fig. 2a and b show that the exact damage localization is provided by the horizontal lines obtained by

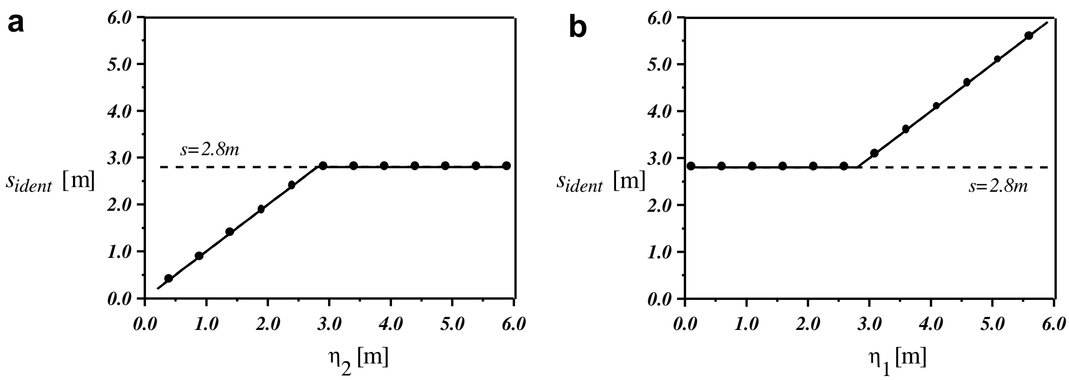

Fig. 1. Pinned–pinned beam: Identified damage position s_{ident}: (a) versus measurement position η_2, for fixed measurement position $\eta_1 = 0.1$ m; (b) versus measurement position η_1, for fixed measurement position $\eta_2 = 5.9$ m.

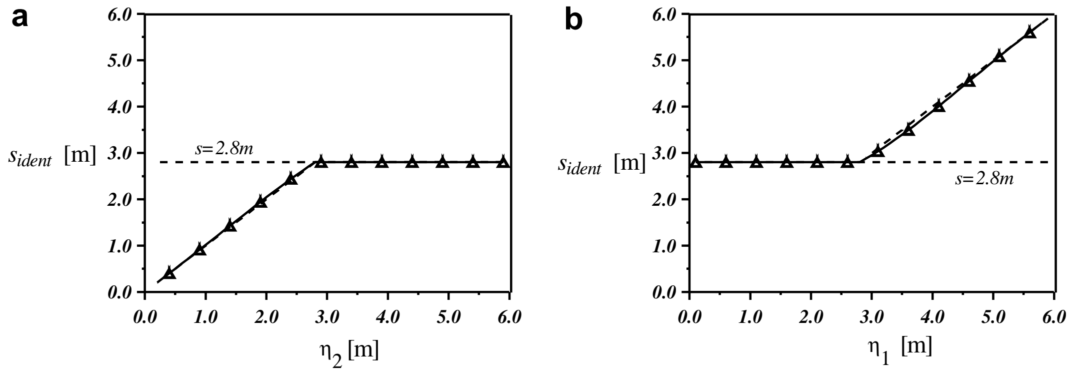

Fig. 2. Clamped–clamped beam: Identified damage position s_{ident}: (a) versus measurement position η_2, for fixed measurement position $\eta_1 = 0.1$ m; (b) versus measurement position η_1, for fixed measurement position $\eta_2 = 5.9$ m.

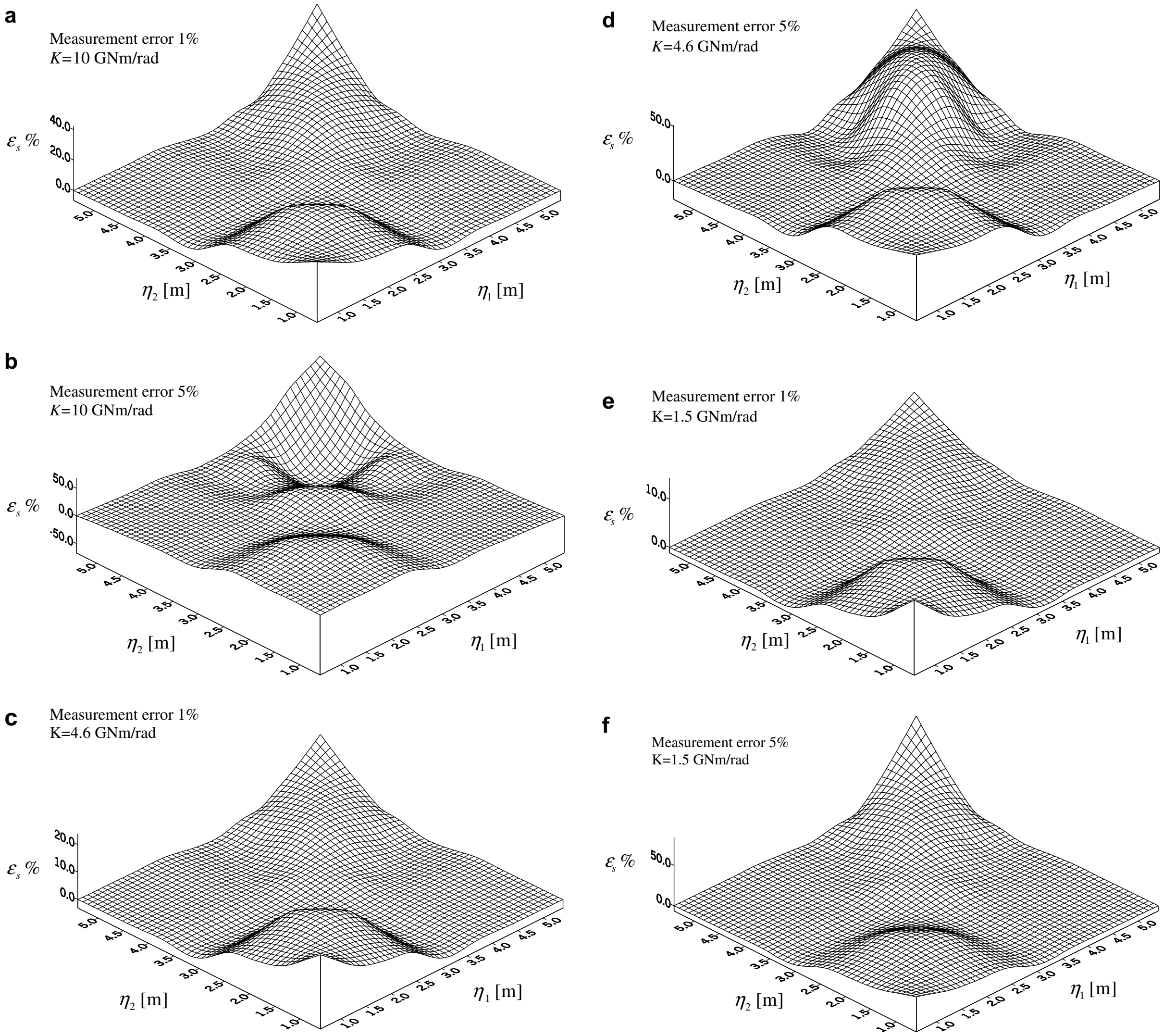

Fig. 3. **a**–**f** Clamped–clamped beam: percentage error $\varepsilon_s = 100(s_{\text{ident}} - s)/s$ on the damage location with η_1,η_2 varying along the beam span, for error levels 1% and 5% on the measurement data, and for different values of equivalent stiffness rotational spring: $K = 10$ GNm/rad (small damage), $K = 4.6$ GNm/rad (moderate damage), $K = 1.5$ GNm/rad (severe damage).

making use of Eq. (35). For those pairs of measurements both lying on the same side of damage, Eq. (35) provides values s_{ident} slightly higher than η_2 or lower than η_1 (broken lines in Fig. 2a and b, respectively).

As far as the severity of the damage is concerned, a direct calculation shows that Eq. (37) gives the following closed form expression for the stiffness K of the rotational spring which is used to model the crack (for $w(\eta_2) \neq 0$):

$$K = \begin{cases} \frac{P}{2}\left(s - \frac{L}{4}\right)\frac{(A_2(\eta_2)+B_2(\eta_2)s)}{w(\eta_2)} - \frac{4a}{L^3}(3s^2 - 3sL + L^2) & \text{for} \quad 0 < s < \frac{L}{2}, \\ -\frac{P}{2}\left(s - \frac{3L}{4}\right)\frac{(A_2(\eta_2)+B_2(\eta_2)s)}{w(\eta_2)} - \frac{4a}{L^3}(3s^2 - 3sL + L^2) & \text{for} \ \frac{L}{2} < s < L. \end{cases} \tag{60}$$

4.3. Measurement errors

The damage analysis has been developed in absence of error so far, but, as it is well known, the results of most identification techniques strictly depend on possible measurement errors and on the severity of the damage to be identified. To take the effect of errors in the experimental data into account and to evaluate the sensitivity of the proposed diagnostic method, different pairs of measurements at η_1, η_2, in a clamped–clamped beam, corrupted by random errors, have been studied. In particular, random errors of magnitude equal to 1% and 5% of the displacement measured on the damaged beam have been included in the analysis. Moreover, to consider damage configurations of different severity, values of the stiffness K of the rotational spring used to simulate the crack have been chosen such that the transversal displacement $\tilde{v}(\frac{L}{2})$ in the damaged beam is the 5% ("small" damage, $K = 10$ GNm/rad), 10% ("moderate" damage, $K = 4.6$ GNm/rad), 25% ("severe" damage, $K = 1.5$ GNm/rad), bigger than the corresponding value $v(\frac{L}{2})$ for the undamaged beam.

The results of identification have been obtained by making use, for each pair of measurements, of the correspondent formulas (35), (39) and (42) so as to assess the sensitivity to measurement errors of all of the proposed closed form expressions.

In Fig. 3a–f the percentage error $\epsilon_s = \frac{s_{\text{ident}} - s}{s} \times 100$ on the damage location, is plotted for η_1, η_2 varying along a grid of points, for random errors 1% and 5% and for increasing levels of damage.

In particular, three different zones have to be distinguished in Fig. 3:

zone 1: $(\eta_1, \eta_2) \leqslant s = 2.8$ m, i.e. both measurements at the left of the damage;
zone 2: $(\eta_1, \eta_2) \geqslant s = 2.8$ m, i.e. both measurements at the right of the damage;
zone 3: $(\eta_1 < s = 2.8 \text{ m} < \eta_2) \cup (\eta_2 < s = 2.8\ m < \eta_1)$, i.e. one measurement at the left and one measurement at the right of the damage.

Fig. 3a–f shows that the maximum error is always reached for measurements belonging to zones 1 and 2 (closed form solutions provided by Eqs. (39) and (42), respectively). More precisely, for small damage ($K = 10$ GNm/rad), the percentage error on damage localization reaches values up to 40% for 1% measurement error (Fig. 3a), and 50% for 5% measurement error (Fig. 3b) in zones 1 and 2. For moderate damage ($K = 4.6$ GNm/rad), the percentage error reaches values up to 20% for 1% measurement error (Fig. 3c) and 50% for 5% measurement error (Fig. 3d). For severe damage ($K = 1.5$ GNm/rad), the percentage error reaches values up to 10% for 1% measurement error (Fig. 3e) and 50% for 5% measurement error (Fig. 3f).

It can be observed that the measurement errors in the damage identification procedure are largely magnified in those cases where measurements are taken both at the right or both at the left of the damage. Moreover, it has to be remarked that the error propagation through the identification procedure is not proportional to the measurement error.

Inspection of Fig. 3a–f show also that the sensitivity to measurement errors is considerably lower for those measurement positions lying on different sides of the damage, i.e. zone 3, hence by adopting in the proposed damage identification procedure the closed form expression provided by Eq. (35).

More precisely, for small damage (Fig. 3a and b) the level of the measurement errors of 1% and 5% could be amplified in the damage position identification since the maximum values 2% and 20% are reached, respectively; for moderate damage (Fig. 3c and d) the errors of 1% and 5% are not magnified in the damage position identification; on the contrary, for severe damage (Fig. 3e and f) the maximum error is 0.4% for 1% measurement error and 2% for 5% measurement error.

It can be concluded that, for a given level of noise, the accuracy of the crack localization increases for increasing levels of damage severity. In fact, when the damage is small, the crack-induced variations on the beam deformation are masked by the errors on the measured data and, therefore, the estimate of the crack location becomes worse.

5. Conclusions

This paper was concerned with the identification of a single crack in an elastic straight beam in bending from the knowledge of static measurements. It was shown how an appropriate choice of pairs of measurements of the damage-induced variations in the transversal displacements with respect to the crack position may be useful for the unique identification of the damage. Closed form expressions for identification of crack position and severeness were provided for different measurements positions. However, in practice, since the measurement positions with respect to the crack are not known a priori, a procedure for damage localization, based on detection of deflection measurements in a grid of points along the beam axis and successive applications of different closed form solutions, was proposed. Numerical results are in good agreement with the theory.

The proposed identification procedure can be extended to cases of beams in presence of multiple cracks. The case of a pinned–pinned beam with two cracks has been briefly discussed. However a consistent and more general approach is currently under study and will be object of a forthcoming paper.

The effect of errors due to the presence of noise in the acquisition of experimental data was also explored. In particular, the sensitivity to instrumental noise of the damage identification procedure was shown to be acceptable when errors are small with respect to deflections induced by damage and the measurement points are chosen in suitable regions of the beam axis. However, on the basis of the closed form solutions provided throughout the paper, a full probabilistic analysis of the identified parameters by modelling the noise as stochastic variables superimposed to the measured data will be object of a future study.

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