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Super-efficiency in DEA by effectiveness of each unit in society

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ABSTRACT

One of the most important topics in management science is determining the efficiency of Decision Making Units (DMUs). The Data Envelopment Analysis (DEA) technique is employed for this purpose. In many DEA models, the best performance of a DMU is indicated by an efficiency score of one. There is often more than one DMU with this efficiency score. To rank and compare efficient units, many methods have been introduced under the name of super-efficiency methods. Among these methods, one can mention Andersen and Petersen's (1993) [1] super-efficiency model, and the slack-based measure introduced by Tone (2002) [4]. Each of the methods proposed for ranking efficient DMUs has its own advantages and shortcomings. In this paper, we present a super-efficiency method by which units that are more effective and useful in society have better ranks. In fact, in order to determine super-efficiency by this method, the effectiveness of each unit in society is considered rather than the cross-comparison of the units. To do so, we divide the inputs and outputs into two groups, desirable and undesirable, at the discretion of the manager, and assign weights to each input and output. Then we determine the rank of each DMU according to the weights and the desirability of inputs and outputs.

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1. Introduction

It is often necessary in real performance assessment practice to rank a group of decision making units (DMUs) in terms of their efficiencies. Several authors have proposed methods for ranking the best performers. See [1–5] among others. In this paper, we propose a ranking methodology for DMUs based on the SBM. This paper is organized as follows. Section 2, briefly introduces the SBM model. In Section 3 we present a new super efficiency model by using the SBM model and management weight. Numerical examples are provided in Section 4 and the paper concludes in Section 5.

2. Slacks-based measure of efficiency

We will deal with n DMUs with the input and output matrices $X = (x_{ij}) \in R^{m\times n}$ and $Y = (y_{rj}) \in R^{s\times n}$, respectively. We assume that the data set is nonnegative, i.e. $X \geq 0$ and $Y \geq 0$ the production possibility set p is defined as:

$$p = \{(x, y) | x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0\} \tag{1}$$

where λ is a non-negative vector in R^n. We consider an expression for describing a certain $\text{DMU}_o = (x_o, y_o)$ as:

$$x_o = X\lambda + s^-, \tag{2}$$

$$y_o = Y\lambda - s^+. \tag{3}$$

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With $\lambda \geq 0, s^- \geq 0$ and $s^+ \geq 0$. The vector $s^- \in R^m$ and $s^+ \in R^s$ indicate the input excess and output shortfall of this expression, respectively, and are called slacks. From the condition $X \geq 0$ and $\lambda \geq 0$, it holds that

$$x_o \geq s^-. \tag{4}$$

Using s^- and s^+, we define an index ρ as follows:

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{ro}}. \tag{5}$$

It can be verified that ρ satisfies the properties (i) units invariant and (ii) monotone decreasing in input/output slacks. Furthermore, from (4), it holds that

$$0 < \rho \leq 1. \tag{6}$$

In an effort to estimate the efficiency of (x_o, y_o), we formulate the following fractional program [SBM] in λ, s^- and s^+.

[SBM]

$$\begin{aligned} &\text{Min} \quad \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{ro}} \\ &\text{subject to} \\ &\quad x_o = X\lambda + s^- \\ &\quad y_o = Y\lambda + s^+ \\ &\quad \lambda \geq 0,\ s^- \geq 0,\ s^+ \geq 0. \end{aligned} \tag{7}$$

[SBM] can be transformed into a linear program using the Charnes–Cooper transformation in a similar way to the CCR model (see [6]). Let an optimal solution for [SBM] be $(\rho^*, \lambda^*, s^{-*}, s^{+*})$. Based on this optimal solution, we define a DMU as being SBM-efficient as follows.

Definition 1 (*SBM-Efficient*)**.** A DMU (x_o, y_o) is SBM-efficient, if $\rho^* = 1$.

This condition is equivalent to $s^{-*} = 0$ and $s^{+*} = 0$, i.e., no input excesses and no output shortfalls in any optimal solution.

3. Proposed method

After specifying SBM-efficient DMUs by using Model (7), we consider the effectiveness of these units in society in order to rank them. From the management point of view, the larger a group, the more difficult it is to manage it; furthermore, larger groups are more effective in society. In the DEA technique, the less the input consumption and the more the output production of a DMU compared to those of other DMUs, the better its performance and, hence, the higher its efficiency score. To rank efficient units, however, we had better consider the effectiveness of the units in society. Consider, for instance, a unit that is efficient and employs 1000 people, compared to another one that is also efficient but employs 50 people. Although both units are DEA-efficient, it stands to reason that the former has a more positive effect in society by providing more job opportunities. Setting priorities for outputs can be performed similarly. For example, one can consider an output that provides a basic or strategic need of an area or society. One such output in a petroleum refinery is gasoline, which plays a more important role in the life and economy of society than the other products. We can use such priorities for determining super-efficiency. To this end, we divide the inputs and outputs into two groups, desirable and undesirable, as follows:

$D_i^+ = \{$inputs that are more useful to society when used in larger amounts$\}$.
$D_i^- = \{$inputs that are more useful to society when used in smaller amounts$\}$.
$D_o^+ = \{$outputs that are more useful to society when produced in larger amounts$\}$.
$D_o^- = \{$outputs that are more useful to society when produced in smaller amounts$\}$.

It must be noted that $D_i^+ \cup D_i^- \subseteq \{1, 2, \ldots, m\}$ and $D_o^+ \cup D_o^- \subseteq \{1, 2, \ldots, s\}$. In other words, there might be inputs and outputs that do not belong to either set. Such inputs and outputs are not usually important and effective in society, and their consumption or production only leads to increasing the profit or decreasing the costs of the DMU. We propose the following method to determine priority in efficient DMUs. The underlying idea is similar to the Sequential Multi objective problem solving (SEMop) method which is indeed a method for solving multi-objective problems [7]. We use following assumptions to utilize this method:

Table 1
Data.

DMU	Data					
	x_1	x_2	x_3	x_4	y_1	y_2
D1	80	600	54	8	90	5
D2	65	200	97	1	58	1
D3	83	400	72	4	60	7
D4	40	1000	75	7	80	10
D5	52	600	20	3	72	8
D6	94	700	36	5	96	6

1. We choose upper and lower limits for each input and output among efficient DMUs as follows:

$$E = \{j \mid \rho_j^* = 1\} \tag{8}$$

$$x_i^{*u} = \underset{j\in E}{\text{Max}} |x_{ij}| \quad i = 1, \dots, m$$

$$x_i^{*l} = \underset{j\in E}{\text{Min}} |x_{ij}| \quad i = 1, \dots, m$$

$$y_r^{*u} = \underset{j\in E}{\text{Max}} |y_{rj}| \quad r = 1, \dots, s$$

$$y_r^{*l} = \underset{j\in E}{\text{Min}} |y_{rj}| \quad r = 1, \dots, s.$$

2. We indicate aspiration levels for each input and output. The utility inputs and outputs – regarding to definition of sets $D_i^-, D_i^+, D_o^-, D_o^+$ – are as follows:

$$\bar{x} = x_i^{*l} \quad \forall i \ (i \in D_i^-), \qquad \bar{x} = x_i^{*u} \quad \forall i \ (i \in D_i^+) \tag{9}$$

$$\bar{y} = y_r^{*l} \quad \forall r \ (r \in D_o^-), \qquad \bar{y} = y_r^{*u} \quad \forall r \ (r \in D_o^+). \tag{10}$$

3. In this step, we define (d_i, d_r) for each (DMU_j s.t. $j \in E$) as follows:

$$\forall i \in D_i^+ \quad d_{ij} = \frac{x_{ij}}{\bar{x}_i + \xi}, \qquad \forall i \in D_i^- \quad d_{ij} = \frac{\bar{x}_i}{x_{ij} + \xi} \tag{11}$$

$$\forall r \in D_r^+ \quad d_{rj} = \frac{y_{rj}}{\bar{y}_r + \xi}, \qquad \forall r \in D_r^- \quad d_{rj} = \frac{\bar{y}_r}{y_{rj} + \xi}. \tag{12}$$

This makes both inputs and outputs dimensionless. Notice, ξ is representative of a small and non-zero number which is utilized to prevent a division by zero.

Each $d_{ij} \geq 1$, $d_{rj} \geq 1$ shows that ith input and rth output in DMU_j provide purposed aspiration levels. So we define D_j, as follows:

$$D_j = \sum_{i\in I} d_{ij} + \sum_{r\in R} d_{rj} \quad I = D_i^+ \cup D_i^-, \ R = D_r^+ \cup D_r^-.$$

Notice, the larger the D_j, the more successful DMU_j, in providing purposed objectives for each input and output, and it is possible to rank efficient DMUs with higher D_j.

In the next section, we apply the proposed method to an example to determine super-efficiency or rank efficient units.

4. Numerical example

This example consists of six efficient DMUs (power plant locations) with four inputs and two outputs as listed below:

x_1 = manpower required
x_2 = construction costs in millions of dollars
x_3 = annual maintenance costs in millions of dollars
x_4 = number of villages to be evacuated
y_1 = power generated in megawatts
y_2 = safety level.

Table 1 shows the data.

Since all DMUs are efficient, in order to select the best alternative among them we employ super-efficiency by the proposed method. In the above example, the first input, manpower, has a positive effect in society by creating job opportunities. The fourth input is the number of villages to be evacuated to install the power plants. It is well understood that evacuating a village has an undesirable social, cultural, and psychological effect in society, especially among the dwellers of

Table 2
Results from Eq. (11).

DMU	D1	D2	D3	D4	D5	D6
D_j	2.4135	2.3957	2.4580	2.4016	2.4362	2.8000
Rank	4	6	2	5	3	1

that village. The amounts of the second and third inputs are important only to the company, but not to society. Therefore, we define sets D_i^+, D_i^- as follows:

As for the outputs, higher levels of power generation and safety are $D_i^+ = \{x_1\}$, $D_i^- = \{x_4\}$ which are more desirable and have a positive effect on people's satisfaction with the DMU. So, we define sets D_o^+, D_0^- as follows:

$$D_o^+ = \{y_1, y_2\}, \qquad D_o^- = \{\}.$$

As all DMUs in the above example are efficient, by relations (8)–(10), we have:

$$E = \{\text{DMU}_1, \text{DMU}_2, \text{DMU}_3, \text{DMU}_4, \text{DMU}_5, \text{DMU}_6\}$$
$$\bar{x}_1 = x_1^{*u} = 94, \qquad \bar{x}_4 = x_4^{*l} = 1, \qquad \bar{y}_1 = y_1^{*u} = 96, \qquad \bar{y}_2 = y_2^{*u} = 10.$$

Table 2 contains the results obtained by relations (11) and (12), by which the rank of each DMU has been determined.

5. Conclusion

In this paper, we presented a new methodology for measuring super-efficiency, or ranking efficient DMUs. Since most of the existing methods for ranking efficient DMUs lack any economic or managerial justification and are merely mathematical models with theoretical assumptions, it was attempted in the proposed method to perform ranking based on criteria that are more effective and more acceptable in society. In fact, in measuring efficiency *working well* is assessed, while in measuring super-efficiency *doing good work* or *effectiveness* of DMUs is assessed.

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