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Note

Best packing of rods into boxes

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Abstract

We determine the maximum number of $1 \times 1 \times \cdots \times 1 \times n$ rods which will fit inside an arbitrary d-dimensional rectangular box, on condition that they be packed parallel to the edges of the box.

1. Introduction

By an $a_1 \times a_2 \times \cdots \times a_d$ brick, or box, we mean a rectangular parallelpiped in d-dimensional Euclidean space with edges of lengths $a_1, a_2, \ldots, a_d$. We may think of it as the set of points $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ with $0 \leqslant x_i < a_i$ for $i = 1, 2, \ldots, d$. Much has been written on the general problem of how much of a given box can be filled with non-overlapping translates of bricks of specified sizes, usually under the following two assumptions:

(1) Bricks may only be packed with their edges parallel to those of the box.

(2) All edge lengths of bricks and boxes are integral.

These are genuine restrictions, as it has been shown [7] that more bricks may sometimes be packed by placing them at skew angles, and also [2] that irrational tiles (e.g. $1 \times \sqrt{2}$) will pack a large area more efficiently than rational ones, the uncovered area for a large square of side x being at most $O(\log x)$ in one case, as against $O(x)$ in the other.

If the bricks have integral edge lengths, then subject to (1) we may as well assume that the box does too, for it is easy to see that the number of bricks which will fit into an $a_1 \times a_2 \times \cdots \times a_d$ box is not reduced when each a_i is replaced by its integer part. (It suffices to show that replacing each translate $u + A$ by $[u] + A$, where A is a brick

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with integral edges, and $[u]$ denotes $([u_1],[u_2],\ldots,[u_d])$ for $u=(u_1,u_2,\ldots,u_d)\in\mathbb{R}^d$, causes no overlapping of the bricks. If $u+A$, $v+B$ are translates of integral bricks A,B in the original packing, then each of $[u]+A$, $[v]+B$ is a union of disjoint translates of the unit cube $C=\{(\theta_1,\theta_2,\ldots,\theta_d)\in\mathbb{R}^d\,|\,0\leqslant\theta_i<1\}$, so they can only overlap if they contain a common translate $x+C$ where $x\in\mathbb{Z}^d$. But then since $u-[u],v-[v]\in C$ we have

$$x+u-[u]+v-[v]\in(u-[u]+x+C)\cap(v-[v]+x+C)\subseteq(u+A)\cap(v+B)$$

contradicting the disjointness of the original packing.)

Here we shall assume (1) and (2). A brick is called *harmonic* if after a suitable permutation of edge lengths we have $a_i\,|\,a_{i+1}$ for $1\leqslant i\leqslant d-1$, and is called a *rod* if all but one of the a_i is equal to 1. If the remaining a_i is equal to n we call it a rod of length n, or an *n-rod*. The remarks in the preceding paragraph show that there is no loss of generality in 2 dimensions in assuming that a brick has relatively prime edge lengths, and in particular that a harmonic brick is a rod since, for example, packing 4×6 bricks into a 9×11 box is equivalent by change of scale to packing 2×3 bricks into a $4\frac{1}{2}\times5\frac{1}{2}$ and hence into a 4×5 box.

The best packing of rods (and hence of harmonic bricks) in 2 dimensions appears to have been discovered independently by several people, but the first in-depth treatment was probably that of Brualdi and Foregger [6] who stated their result in the form of a max/min equality. They define a *representing set* to be a set of unit cells inside a given box which meets every brick no matter where it is placed inside the box. If one can exhibit a packing of k bricks into the box, and at the same time a representing set of k cells, it is then clear that both the packing and the representing set are optimal, in the sense that they use the maximum number of bricks and the minimum number of cells, respectively. Brualdi and Foregger showed that this is always possible for harmonic bricks in 2 dimensions, and also in 3 dimensions subject to certain restrictions. In particular the max/min equality holds for rods in 3 dimensions. They also showed that equality does not hold for any nonharmonic brick, so that the optimal packing and the optimal representing set are then more difficult to determine.

Barnes and Shearer [5] studied representing sets for rectangles in the infinite plane and found that only in the harmonic case does the optimal density of a representing set equal the packing density of the brick. They determined the optimal representing set for all rectangles except those having length/width ratio greater than 2, with fractional part between $\frac{1}{3}$ and $\frac{1}{2}$. This gap still remains to be filled. Little is known in higher dimensions.

Barnes [1] gives the best packing of arbitrary $m\times n$ integral bricks in sufficiently large boxes, showing that the obvious necessary conditions derived from simultaneous packing with m-rods and with n-rods are also sufficient when the box is large. There is reason to believe that this principle holds in higher dimensions for a brick whose edge lengths a_i are pairwise coprime — hence the fundamental importance of knowing the best packing with rods. The reason for the coprime restriction is to ensure that an integer defined in [3], called the *variety dimension*, turns out to be zero.

This number gives an asymptotic measure of the rate of growth of the wasted volume in the best packing of a large box, and is zero when this wasted volume is bounded independently of the size of the box.

In general, the best packing of a box with an arbitrary brick A is likely to be determined by the best packing with each harmonic brick which packs A, rather than by each rod which packs A. A key role is played by *flakes*, by which we mean bricks of type $1 \times 1 \times \cdots \times 1 \times n \times n \times \cdots \times n$, a concept intermediate between that of a rod and a harmonic brick. For example it can be shown that an arbitrary set of cells in $\mathbb{R}^d$ can be packed with positive and negative[1] copies of a brick A if and only if it can be so packed with copies of each flake which packs A. The only bricks with variety dimension greater than zero for which the best packing is known for all boxes are the $1 \times 2 \times 2$ and $1 \times 2 \times 4$ bricks in 3 dimensions. The answer, recently determined in [4], suggests that the generalisation from rods to harmonic bricks is likely to be the most difficult step in a theory of best packing with arbitrary bricks. In this paper, we shall complete the first step of such a theory by showing that Brualdi and Foregger's max/min equality holds for rods in d dimensions.

2. Main results

If an $a_1 \times a_2 \times \cdots \times a_d$ box is to be packed with n-rods, we may assume that $a_i \geqslant n$ for $1 \leqslant i \leqslant d$, for if say $a_1 < n$ then the only way to pack the box is to pack each $1 \times a_2 \times \cdots \times a_d$ layer separately. Thus the problem merely reduces to one in $d - 1$ dimensions.

Let $r_1, r_2, \ldots, r_d$ be the remainders when $a_1, a_2, \ldots, a_d$ are divided by n, and let $s_i = n - r_i$. When $d = 1$, the optimal packing of a length a_1 box with n-rods obviously leaves a hole of length r_1. For $d = 2$, we may pack an $a_1 \times a_2$ box leaving a hole of size either $r_1 \times r_2$ or $s_1 \times s_2$ (see Fig. 1).

Brualdi and Foregger call these packings Type I and Type II, respectively, and show that one of them is the optimal packing. Type I is optimal if $r_1 + r_2 \leqslant n$ and Type II is optimal if $r_1 + r_2 \geqslant n$. (If $r_1 + r_2 = n$, they both have the same size hole.)

Let S be a subset of $\{1, 2, \ldots, d\}$ of even cardinality, partitioned into pairs in an arbitrary way. Define $t_1, t_2, \ldots, t_d$ by

$$t_i = \begin{cases} r_i & \text{if } i \notin S, \\ s_i & \text{if } i \in S. \end{cases}$$

For each $i \notin S$ we pack a length a_i box leaving a length r_i hole, and for each pair (i, j) in the partition of S we pack an $a_i \times a_j$ box leaving an $s_i \times s_j$ hole. The cartesian product

[1] A generalised form of packing using negative bricks is introduced in [3], e.g. superimposing a negative 2×6 and positive 3×6 rectangle gives a packing of a 1×6 box with three positive and two negative 2×3 bricks.

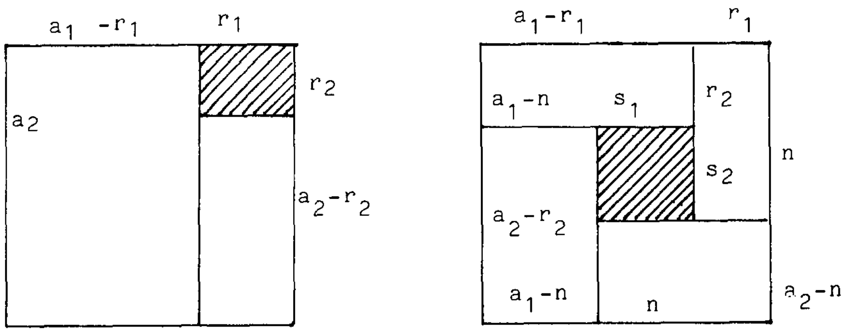

Fig. 1.

of all these packings gives a packing of an $a_1 \times a_2 \times \cdots \times a_d$ box leaving a hole of size $t_1 \times t_2 \times \cdots \times t_d$. For a suitable choice of the set S, we will show that this packing is optimal by constructing a representing set which meets each rod in exactly one cell, but does not meet the hole.

Theorem 1. *Let S be a subset of $\{1,2,\ldots,d\}$ of even cardinality for which $\sum_{i=1}^{d} t_i$ is minimal. Then the packing described above is optimal — hence an $a_1 \times a_2 \times \cdots \times a_d$ box can accommodate precisely*

$$\frac{1}{n}\left[\prod_{i=1}^{d} a_i - \prod_{i=1}^{d} t_i\right]$$

n-rods.

Remark. In practice $t_1, t_2, \ldots, t_d$ (and hence S) may be computed from $r_1, r_2, \ldots, r_d$ by arranging them in nonincreasing order and then complementing (i.e. subtracting from n) those pairs (r_1, r_2), (r_3, r_4), etc., whose sum exceeds n.

Proof of Theorem 1. We define a *perfect representing set* (PRS) to be a set T of unit cells in $\mathbb{R}^d$ such that every n-rod contains exactly one cell of T. For example, if the cells are assigned integer coordinates in the obvious way, then those cells $(x_1, x_2, \ldots, x_d)$ with $\sum_{i=1}^{d} x_i \equiv 0 \pmod{n}$ form a PRS. It is clear that any PRS must necessarily be periodic of period n in each direction, and hence may be specified by giving a set of n^{d-1} cells in the $n \times n \times \cdots \times n$ torus which meets every n-rod. Also, applying an arbitrary translation to a PRS yields another PRS. Hence it will serve our purpose to find a PRS inside the $n \times n \times \cdots \times n$ torus, together with any $t_1 \times t_2 \times \cdots \times t_d$ subbox which does not meet this PRS.

When $\sum_{i=1}^{d} t_i$ is minimal we have $t_i + t_j \leqslant n$ for all $i \neq j$ since otherwise the pair (t_i, t_j) could be replaced by $(n - t_i, n - t_j)$, and S replaced by its symmetric difference with $\{i, j\}$, thereby reducing the sum. In particular if $a \leqslant b$ are the two largest t_i, we have $a + b \leqslant n$ and hence $a \leqslant \frac{1}{2}n$, from which it follows that a $t_1 \times t_2 \times \cdots \times t_d$ box will fit inside an $a \times a \times \cdots \times a \times (n-a)$ box. We construct a PRS which avoids this latter box.

Label the cells of the $n \times n \times \cdots \times n$ torus with coordinates $(x_1, x_2, \ldots, x_d)$ where $1 \leqslant x_i \leqslant n$. Now choose any $a \times a$ latin square based on the symbols $1, 2, \ldots, a$. By a standard result (cf. Theorem 2.2 of [8]) any $r \times s$ latin rectangle containing symbols $1, 2, \ldots, n$ can be extended to an $n \times n$ latin square provided each symbol appears at least $r + s - n$ times in the rectangle. In particular, since $a + a - n \leqslant 0$ we may extend our $a \times a$ latin square to an $n \times n$ latin square based on the symbols $1, 2, \ldots, n$ so that only the symbols $1, 2, \ldots, a$ appear in its initial $a \times a$ subsquare. We now regard the large square as a multiplication table, making $\{1, 2, \ldots, n\}$ into a (nonassociative!) semigroup, by defining the product $x * y$ to be the entry in row x, column y. The set $\{1, 2, \ldots, a\}$ is closed under this binary operation, i.e. forms a subsemigroup, and the product has the property that if one of x, y is fixed while the other runs through the values $1, 2, \ldots, n$, then the product $x * y$ runs through these same values in some order.

We define an extended product $x_1 * x_2 * \cdots * x_k$ by performing the operations from left to right — e.g. $x * y * z$ means $(x * y) * z$. An easy induction on k shows that if all but one of the x_i are held fixed while the remaining one runs through the values $1, 2, \ldots, n$ then the product $x_1 * x_2 * \cdots * x_k$ also runs through $1, 2, \ldots, n$ in some order.

Let T be the set of cells $(x_1, x_2, \ldots, x_d)$ in the $n \times n \times \cdots \times n$ torus for which $x_1 * x_2 * \cdots * x_{d-1} = x_d$. The previous remark shows that T is a PRS and the fact that $\{1, 2, \ldots, a\}$ is closed under the operation shows that T avoids the $a \times a \times \cdots \times a \times (n-a)$ box consisting of those cells with

$$1 \leqslant x_i \leqslant a, \quad i = 1, 2, \ldots, d-1, \qquad a + 1 \leqslant x_d \leqslant n.$$

This completes the proof. □

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