On Henstock–Kurzweil and McShane integrals of Banach space-valued functions

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Abstract

This paper deals with the relation between the McShane integral and the Henstock–Kurzweil integral for the functions mapping a compact interval $I_0 \subset \mathbb{R}^m$ into a Banach space $X$ and some other questions in connection with the McShane integral and the Henstock–Kurzweil integral of Banach space-valued functions. We prove that if a Banach space-valued function $f$ is Henstock–Kurzweil integrable on $I_0$ and satisfies Property (P), then $I_0$ can be written as a countable union of closed sets $E_n$ such that $f$ is McShane integrable on each $E_n$ when $X$ contains no copy of $c_0$. We further give an answer to the Karták’s question.

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1. Introduction

It is known that the McShane integral and the Henstock–Kurzweil integral are two kinds of the Riemann-type integral. For real-valued functions the McShane integral is equivalent to the Lebesgue integral and the Henstock–Kurzweil integral is equivalent to the Perron integral. R.A. Gordon [7] generalized the definition of the McShane integral for real-valued functions to functions from intervals in $\mathbb{R}$ to Banach spaces and discussed some of its properties. S.S. Cao in [10] defined the Henstock–Kurzweil integral for Banach space-valued functions. It is easy
to see from the corresponding definitions that for Banach space-valued functions the McShane integrability implies Henstock–Kurzweil integrability.

We are looking into the following problem: Is it true that if a Banach space-valued function $f$ is Henstock–Kurzweil integrable on an interval $I_0$ then $I_0$ can be written as a countable union of closed sets $E_n$ such that $f$ is McShane integrable on each $E_n$?

This is an interesting and unanswered question. In this paper we give under suitable condition over the function $f$ (see Property (P)) an affirmative answer to it when Banach space $X$ contains no copy of $c_0$.

Furthermore, in his memoir [1] K.M. Ostaszewski mentioned that “the question posed by Kartáč in [5]—whether, for a Perron-integrable function, one can find a nondegenerate interval on which it is Lebesgue-integrable—remains unanswered.” For the one-dimensional real-function’s case this is a known result of [14] and for the Banach space’s case. Because the Henstock lemma does not hold in an infinite-dimensional Banach space (see [10]). In this paper, we use the other way to give an answer to the question posed by Kartáč in [5] for the $m$-dimensional case of the integrals of Banach-space-valued functions. This is to prove that for a Henstock–Kurzweil integrable function, one can find a non-degenerate interval on which it is McShane integrable when Banach space $X$ contains no copy of $c_0$. Some other questions in connection with the McShane integral and the Henstock–Kurzweil integral are also studied.

2. Preliminaries

Let $I_0$ be a compact interval in $\mathbb{R}^m$ (or $\mathbb{R}^1$) and $E \subset \mathbb{R}^m$ (or $\mathbb{R}^1$) a measurable subset of $I_0$. $\mu(E)$ stands for the Lebesgue measure. The Lebesgue integral of a function $f$ over a set $E$ will be denoted by $(L)\int_E f$. $X$ is a real Banach space with the norm $\| \cdot \|$ and $X^*$ its dual. $B(X^*) = \{x^* \in X^*; \|x^*\| \leq 1\}$ is the closed unit ball in $X^*$.

We say that the intervals $I$ and $J$ are nonoverlapping if $\text{int}(I) \cap \text{int}(J) = \emptyset$. By $\text{int} J$ the interior of $J$ is denoted.

A partial $M$-partition $D$ in $I_0$ is a finite collection of interval-point pairs $(I, \xi)$ with nonoverlapping intervals $I \subset I_0$, $\xi \in I_0$ being the associated point of $I$. Requiring $\xi \in I$ for the associated point of $I$ we get the concept of a partial $K$-partition $D$ in $I_0$. We write $D = \{(I, \xi)\}$.

A partial $M$-partition $D = \{(I, \xi)\}$ in $I_0$ is an $M$-partition of $I_0$ if the union of all the intervals $I$ equals $I_0$ and similarly for a $K$-partition.

Let $\delta$ be a positive function defined on the interval $I_0$. A partial $M$-partition ($K$-partition) $D = \{(I, \xi)\}$ is said to be $\delta$-fine if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$ where $B(\xi, \delta(\xi)) = \{t \in \mathbb{R}^m; \text{dist}(\xi, t) < \delta(\xi)\}$ and dist is the metric in $\mathbb{R}^m$.

**Definition 1.** An $X$-valued function $f$ is said to be McShane integrable on $I_0$ if there exists $S_f \in X$ such that for every $\varepsilon > 0$, there exists $\delta(t) > 0$, $t \in I_0$, such that for every $\delta$-fine $M$-partition $D = \{(I, \xi)\}$ of $I_0$, we have

$$\left\| (D) \sum f(\xi) \mu(I) - S_f \right\| < \varepsilon.$$ 

We write $(M)\int_{I_0} f = S_f$ and $S_f$ is the McShane integral of $f$ over $I_0$.

$f$ is McShane integrable on a set $E \subset I_0$ if the function $f \cdot \chi_E$ is McShane integrable on $I_0$, where $\chi_E$ denotes the characteristic function of $E$.

We write $(M)\int_E f = (M)\int_{I_0} f \chi_E = F(E)$ for the McShane integral of $f$ on $E$. 

\[ \text{Definition 1.} \] 

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\[ \text{Definition 1.} \]
It is well known that the McShane and the Lebesgue integrals are equivalent. Replacing the term “\(M\)-partition” by “\(K\)-partition” in the definition above we obtain Henstock–Kurzweil integrability and the definition of the Henstock–Kurzweil integral (HK) \(\int_{I_0} f\).

It is clear that if \(f : I_0 \rightarrow X\) is McShane integrable, then it is also Henstock–Kurzweil integrable because every \(K\)-partition is an \(M\)-partition.

The basic properties of the McShane integral and Henstock–Kurzweil integral, for example, linearity, additivity with respect to intervals, etc. can be found in [4–10,12,13,15–21]. We do not present them here. The reader is referred to the above mentioned references for the details.

3. The main results

By Proposition 3.5.4 of [19], the following lemma holds:

**Lemma 2.** If \(f : I_0 \rightarrow X\) is Henstock–Kurzweil (McShane) integrable on \(I_0\), then for each \(x^* \in X^*\), \(x^*(f)\) is Henstock–Kurzweil (McShane) integrable on \(I_0\) and \((HK)\int_{I_0} x^*(f) = x^*((HK)\int_{I_0} f)\).

In [11] it is shown that if a real-function \(f : I_0 \rightarrow R\) is Henstock–Kurzweil integrable on \(I_0\), then there exists a sequence of closed sets \(F_i \subset I_0, i \in \mathbb{N}\), such that \(\bigcup_{i} F_i = I_0\) and \(f\) is Lebesgue integrable on each \(F_i\). By Baire theorem, for each perfect set \(E\) there is \(F_{n_0}\) which contains a portion \(P = E \cap I\) of \(E\) such that \(f\) is McShane integrable on \(P\). So we have

**Lemma 3.** If a real-function \(f : I_0 \rightarrow R\) is Henstock–Kurzweil integrable on \(I_0\), then each perfect set contains a portion on which \(f\) is McShane integrable.

It is also easy to show that the following lemma holds (see [16]):

**Lemma 4.** If \(f : I_0 \rightarrow X\) is Henstock–Kurzweil integrable on \(I_0\), then \(f\) is Henstock–Kurzweil integrable on each subinterval \(I\) of \(I_0\).

By Lemmas 2 and 4, we obtain the following Lemma 5.

**Lemma 5.** Assume that \(f : I_0 \rightarrow X\) is Henstock–Kurzweil integrable on \(I_0\) and that for every \(x^* \in X^*\) the real function \(x^*(f) : I_0 \rightarrow \mathbb{R}\) is McShane integrable.

Then for every interval \(I \subset I_0\) we have

\[
(M)\int_{I} x^*(f) = x^*((HK)\int_{I} f).
\]

**Lemma 6.** Assume that \(f : I_0 \rightarrow X\) is Dunford integrable on \(I_0\) with the indefinite Dunford integral \(\nu\) defined by

\[
\nu(E) = (D)\int_{E} f \in X^{**}.
\]

Assume that \(\nu(J) = (D)\int_{J} f \in X\) for every interval \(J \subset I_0\). Then the following claims are equivalent:

---

\[ \text{HK} \int_{I_0} f \]

\[ x^* \int_{I_0} f \]

\[ (HK) \int_{I_0} f \]

\[ (M) \int_{I} x^*(f) \]

\[ x^*((HK)\int_{I} f) \]

\[ (HK)\int_{I} x^*(f) \]

---

\[ x^*((HK)\int_{I} f) \]
(i) $f$ is Pettis integrable;
(ii) for every sequence $J_i \subset I_0$, $i \in \mathbb{N}$, of nonoverlapping intervals the sum $\sum_{i=1}^\infty \nu(J_i)$ is norm convergent in $X$;
(iii) for every $\varepsilon > 0$ there is $\eta > 0$ such that
$$\|\nu(E)\| = \left\| (D) \int_E f \right\| < \varepsilon$$
provided $E \subset I_0$ is measurable with $\mu(E) < \eta$;
(iv) $\nu$ is countably additive.

This is Proposition 2B of [6] for the case of $I_0 \subset \mathbb{R}^m$. Note that in [6] D.H. Fremlin and J. Mendoza only proved above lemma for the case of $I_0 \subset \mathbb{R}^1$. In fact, it also holds if the interval $I_0$ is in $\mathbb{R}^m$.

**Lemma 7.** Suppose that the Banach space $X$ contains no copy of the space $c_0$. Assume that $f : I_0 \to X$ is Henstock–Kurzweil integrable on $I_0$ and Dunford integrable on $I_0$ as well.

Then for every open set $G \subset I_0$ there exists $x_G \in X$ such that

$$\text{(M)} \int_G x^*(f) = x^*(x_G)$$

for every $x^* \in X^*$.

**Proof.** Given $\lambda$ such that $0 < \lambda < 1$ an interval $I$ in $\mathbb{R}^m$ is called $\lambda$-regular if

$$r(I) = \frac{\mu(I)}{d(I)^m} > \lambda,$$

($r(I)$ is the regularity of the interval $I$) and $d(I) = \sup\{|x - y|; \ x, y \in I\}$, $|x - y| = \max\{|x_1 - y_1|, \ldots, |x_m - y_m|\}$, and $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m)$.

Suppose that $G$ is an open subset of $I_0$.

For $t \in G$ let $\delta(t) > 0$ be such that $B(t, \delta(t)) \subset G$.

Let $0 < \lambda < 1$ be fixed. Define

$$\Phi = \left\{ I \subset I_0, I \text{ is an interval; } t \in I \subset B(t, \delta(t)), r(I) > \lambda, t \in G \right\}.$$

Then $\Phi$ is a Vitali cover of $G$ and if $I \in \Phi$ then $I \subset G$.

By the Vitali covering theorem (see, e.g., [13, Proposition 9.2.4]), there is a sequence $E_n, n \in \mathbb{N}$ ($E_n$ is the finite union of nonoverlapping intervals belonging to $\Phi$), such that

$$\mu(G \setminus E_n) < \frac{1}{n},$$

i.e., $\mu(G \setminus E_n) \to 0$ for $n \to \infty$ and $E_n \subset G$ for any $n \in \mathbb{N}$.

Denote $E_0 = \bigcup_{n=1}^\infty E_n$. Since $G \setminus E_0 \subset G \setminus E_n$ for every $n \in \mathbb{N}$ we have $\mu(G \setminus E_0) \leq \mu(G \setminus E_n) < \frac{1}{n}$ for every $n \in \mathbb{N}$ and consequently $\mu(G \setminus E_0) = 0$. This yields $\mu(E_0) = \mu(G)$.

Let us set $F_n = \bigcup_{i=1}^n E_i$. Then clearly $F_n \nearrow E_0$ for $n \to \infty$ and for every $n \in \mathbb{N}$ the set $F_n$ can be expressed as a finite union of nonoverlapping intervals in $\mathbb{R}^m$. 

Set $F_0 = \emptyset$ and define $K_n = F_n \setminus F_{n-1}^\circ$ where $F_{n-1}^\circ$ is the interior of the set $F_{n-1}$. We have $E_0 = \bigcup_{n=1}^{\infty} K_n$, $K_n \cap K_l^\circ = \emptyset$ for $n \neq l$ and again $K_n$ can be expressed as a finite union of nonoverlapping intervals in $\mathbb{R}^m$, i.e.

$$K_n = \bigcup_{i=1}^{p_n} I_i^n,$$

while $\{I_i^n; i = 1, \ldots, p_n, n \in \mathbb{N}\}$ forms an at most countable system of nonoverlapping intervals contained in $E_0$.

Since $\bigcup_{n=1}^{p} K_n \subset E_0, p \in \mathbb{N}$, we have $\sum_{n=1}^{p} \mu(K_n) = \mu(\bigcup_{n=1}^{p} K_n) \leq \mu(E_0) = \mu(G) \leq \mu(I_0) < \infty$.

Given $x^* \in X^*$ the real function $x^*(f)$ is McShane integrable on $I_0$ and therefore it is also Lebesgue integrable on $I_0$.

Hence the Lebesgue integral $\int_{I_0} x^*(f)$ exists and

$$\int_{I_0} x^*(f) = \int_{E_0} x^*(f)$$

because $\mu(G \setminus E_0) = 0$ and $E_0 \subset G$.

Further we have

$$(M) \int_{G} x^*(f) = (M) \int_{E_0} x^*(f)$$

$$= (M) \int_{\bigcup_{n=1}^{\infty} K_n} x^*(f) = (M) \int_{\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{p_n} I_i^n} x^*(f)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{p_n} (M) \int_{I_i^n} x^*(f) < \infty.$$
\[
\sum_{n=1}^{\infty} \sum_{i=1}^{p_n} x^* \left( (HK) \int_{I_i^n} f \right)
\]

of real numbers is unconditionally, and therefore also absolutely, convergent.

Since \( X \) contains no copy of \( c_0 \), by the Bessaga–Pełczynski theorem [3, p. 22] the series
\[
\sum_{n=1}^{\infty} \sum_{i=1}^{p_n} x^* \left( (HK) \int_{I_i^n} f \right)
\]
and for \( N \to \infty \) the left-hand side of this equality converges to \( x^*(x_G) \) while the right-hand side converges to \( (M) \int_G x^*(f) \). This yields \( (M) \int_G x^*(f) = x^*(x_G) \) for every \( x^* \in X^* \) and the proof is complete. \( \square \)

**Lemma 8.** Suppose that the Banach space \( X \) contains no copy of the space \( c_0 \). Assume that \( f : I_0 \to X \) is Henstock–Kurzweil integrable on \( I_0 \) and that for every \( x^* \in X^* \) the real function \( x^*(f) : I_0 \to \mathbb{R} \) is McShane integrable.

Then for every closed set \( H \subset I_0 \) there exists an element \( x_H \in X \) such that
\[
(M) \int_H x^*(f) = x^*(x_H)
\]
for every \( x^* \in X^* \).

**Proof.** If \( H \subset I_0 \) is closed then \( I_0 \setminus H \) is open and for every \( x^* \in X^* \) we have
\[
x^* \left( (M) \int_{I_0} f \right) = \int_{I_0} x^*(f) = \int_{I_0} x^*(f) + \int_{I_0 \setminus H} x^*(f) = \int_{I_0 \setminus H} x^*(f) + x^*(x_{I_0 \setminus H}),
\]
where for the open set \( I_0 \setminus H \) the element \( x_{I_0 \setminus H} \in X \) is given by Lemma 7.

Hence
\[
\int_H x^*(f) = x^* \left( (M) \int_{I_0} f - x_{I_0 \setminus H} \right)
\]
and we put \( x_H = (M) \int_{I_0} f - x_{I_0 \setminus H} \in X \). \( \square \)

The next statement is a corollary of Lemma 6.

**Corollary 9.** Suppose that the Banach space \( X \) contains no copy of the space \( c_0 \). Assume that \( f : I_0 \to X \) is Henstock–Kurzweil integrable on \( I_0 \) and that for every \( x^* \in X^* \) the real function \( x^*(f) : I_0 \to \mathbb{R}^1 \) is McShane integrable. Then for every \( \varepsilon > 0 \) there is \( \eta > 0 \) such that
\[
\left\| (D) \int_E f \right\| < \varepsilon
\]
provided \( E \subset I_0 \) is measurable with \( \mu(E) < \eta \).
Proof. By Lemma 4, for every interval \( J \subset I_0 \) we have \( v(J) = (D) \int_J f = (HK) \int_J f \in X \).

Assume that \( J_i \subset I_0, i \in \mathbb{N}, \) is a sequence of nonoverlapping intervals. Then the Henstock–Kurzweil integral \( (HK) \int_J f \in X \) exists for every \( i \in \mathbb{N} \) and by Lemma 5 we have \( x^*((HK) \int_{J_i} f) = (M) \int_{J_i} x^*(f) \) for \( i \in \mathbb{N} \).

It is easy to see, by McShane integrability of \( x^* f \) on \( I_0 \), that the series \( \sum_{i=1}^{\infty} x^*((HK) \int_{J_i} f) \) of real numbers is absolutely convergent.

Since \( X \) contains no copy of \( c_0 \), by the Bessaga–Pelczynski theorem [3, p. 22] the series \( \sum_{i=1}^{\infty} (HK) \int_{J_i} f = \sum_{i=1}^{\infty} (D) \int_{J_i} f \) is unconditionally convergent in norm to an element \( x \in X \).

Then (ii) of Lemma 6 is satisfied and therefore we obtain the corollary. \( \square \)

Theorem 10. Suppose that the Banach space \( X \) contains no copy of the space \( c_0 \). Assume that \( f : I_0 \rightarrow X \) is Henstock–Kurzweil integrable on \( I_0 \) and that \( f \) is Dunford integrable on \( I_0 \). Then \( f : I_0 \rightarrow X \) is Pettis integrable.

Proof. It follows at once from Lemma 8, Corollary 9 and Lemma 6. \( \square \)

Let us note that if the function \( f : I_0 \rightarrow X \) is Henstock–Kurzweil integrable, by Lemma 2, for each \( x^* \in X^* \) the real function \( x^*(f) \) is Henstock–Kurzweil integrable on \( I_0 \) and if we further assume that \( x^*(f) \) is McShane integrable on a subset \( H \) of \( I_0 \), then \( x^*(f) \) is Henstock–Kurzweil integrable on \( G = I_0 \setminus H \) and \( (HK) \int_G x^*(f) = (HK) \int_{I_0} x^*(f) - (M) \int_H x^*(f) \).

Now we introduce the following concept:

Property (P). We say that a Henstock–Kurzweil integrable function \( f : I_0 \rightarrow X \) satisfies the property (P) if for every open subset \( G \) of \( I_0 \) there is a family \( \{I_n\} \) of nonoverlapping intervals \( I_n \) such that \( G = \bigcup_{n=1}^{\infty} I_n \), for each \( x^* \in X^* \) the function \( x^*(f) \) is McShane integrable on \( H = I_0 \setminus G \) and equality

\[
(HK) \int_G x^*(f) = (HK) \int_{I_0} x^*(f) - (M) \int_H x^*(f) = \sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f)
\]

holds, where the series of the right hand is absolutely convergent.

The following is an example of function defined on an compact interval of \( \mathbb{R}^m, m > 1 \), taking values in a Banach space which contains no copy of \( c_0 \) and satisfying Property (P).

Example. Let \( X \) be an infinite-dimensional Banach space and contain no copy of \( c_0 \). Suppose that a series \( \sum_{n=1}^{\infty} x_n \) in \( X \) is unconditionally convergent and not absolutely convergent. For every positive integers \( i, j \) let

\[
I_{i,j} = \left( \frac{1}{i+1}, \frac{1}{i} \right) \times \left( \frac{1}{j+1}, \frac{1}{j} \right) \subset \mathbb{R}^2.
\]

The set \( \{I_{i,j}\}_{i,j=1}^{\infty} \) is countable, we denote its elements by \( I_1, I_2, \ldots, I_n, \ldots \) for convenience.

Define a function \( f : [0, 1] \times [0, 1] \rightarrow X \) by \( f(t) = \frac{1}{\mu(I_{i,j})} x_n \) for \( t \) in \( I_n \) and \( f(t) = 0 \) for all other values of \( t \). Obviously, the function \( f \) is measurable. By Proposition 2.3.3 and Theorem 6.2.1 of [19], \( f \) is Henstock–Kurzweil integrable.
Since $X$ contains no copy of $c_0$, then for each $x^* \in X^*$, $\sum_{n=1}^{\infty} |x^*(x_n)|$ is convergent. Hence,

$$\int_{I_0} |x^*(f)| = \sum_{n=1}^{\infty} \int_{I_n} |x^*(f)| = \sum_{n=1}^{\infty} |x^*(x_n)| < \infty.$$ 

It is easy to verify that for every open subset $G$ of $I_0$ with $G = \bigcup_{k=1}^{\infty} J_k$ and $\{J_k\}$ is a family of nonoverlapping intervals, for each $x^* \in X^*$ the function $x^*(f)$ is McShane integrable on $H = I_0 \setminus G$ and the equality

$$\int_G x^*(f) = \int_{I_0} x^*(f) - (M) \int_H x^*(f) = \sum_{k=1}^{\infty} \int_{J_k} x^*(f)$$

holds. Therefore, $f$ satisfies Property (P).

**Lemma 11.** Suppose that the Banach space $X$ contains no copy of the space $c_0$ and $f : I_0 \to X$ is Henstock–Kurzweil integrable on $I_0$. Assume that for each $x^* \in X^*$ the function $x^*(f)$ is McShane integrable on a closed subset $H$ of $I_0$ and $f$ satisfies Property (P). Then there exists an element $x_H \in X$ such that

$$(M) \int_H x^*(f) = x^*(x_H)$$

for each $x^* \in X^*$.

**Proof.** For each $x^* \in X^*$, the real function $x^*(f)$ is Henstock–Kurzweil integrable on $I_0$ and by the McShane integrability of $x^*(f)$ on $H$, $x^*(f)$ is Henstock–Kurzweil integrable on the open set $G = I_0 \setminus H$. It follows from Property (P) that for each $x^* \in X^*$, $\sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f)$ is absolutely convergent and

$$(HK) \int_G x^*(f) = \sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f) < \infty. \quad (2)$$

On the other hand, Lemma 4 shows that $f$ is Henstock–Kurzweil integrable on each subinterval $I_n$, $n \in \mathbb{N}$, and

$$(HK) \int_{I_n} x^*(f) = x^*\left((HK) \int_{I_n} f\right)$$

for each $x^* \in X^*$ and every $n \in \mathbb{N}$.

So, for each $x^* \in X^*$, (2) can be written as follows:

$$(HK) \int_G x^*(f) = \sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f) = \sum_{n=1}^{\infty} x^*\left((HK) \int_{I_n} f\right) < \infty \quad (3)$$

and $\sum_{n=1}^{\infty} x^*((HK) \int_{I_n} f)$ is absolutely convergent.

Since $X$ contains no copy of $c_0$, by the Bessaga–Pelczynski theorem [3, p. 22] the series $\sum_{n=1}^{\infty} (HK) \int_{I_n} f$ is unconditionally convergent in norm to an element $x_G \in X$ and
\[
\begin{align*}
(HK) \int_G x^*(f) &= \sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f) = \sum_{n=1}^{\infty} x^* \left( (HK) \int_{I_n} f \right) \\
&= x^* \left( \sum_{n=1}^{\infty} (HK) \int_{I_n} f \right) = x^*(x_G).
\end{align*}
\]

Hence,
\[
(M) \int_H x^*(f) = (HK) \int_{I_0} x^*(f) - (HK) \int_G x^*(f) = x^* \left( (HK) \int_{I_0} f - x_G \right).
\]

Denote \( x_H = (HK) \int_{I_0} f - x_G \), then \( x_H \in X \) and \( (M) \int_H x^*(f) = x^*(x_H) \). The lemma is proved. □

**Remark.** For 1-dimensional sense (\( I_0 \subset \mathbb{R}^1 \)), Property (P) in Lemma 11 can be removed. Because if \( I_0 \subset \mathbb{R}^1 \), \( G = I_0 \setminus H \) is an open set in \( I_0 \) and further let \( \{I_n\} \) be an enumeration of the intervals contiguous to \( H \). Then
\[
I_0 = H \cup G = H \cup \left( \bigcup_{n=1}^{\infty} I_n \right) \quad \text{and} \quad \mu(G) = \sum_{n=1}^{\infty} \mu(I_n).
\]

Theorem 15.10 of [9] (or Theorem 1 of [18]) shows that
\[
(HK) \int_G x^*(f) = (HK) \int_{I_0} x^*(f) - (M) \int_H x^*(f) = \sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f)
\]
and \( \sum_{n=1}^{\infty} (HK) \int_{I_n} x^*(f) \) is absolutely convergent. Therefore, Property (P) automatically holds. Hence, we have the following Corollary 12.

**Corollary 12.** Suppose that the Banach space \( X \) contains no copy of the space \( c_0 \) and \( f : I_0 \to X \) (\( I_0 \subset \mathbb{R}^1 \)) is Henstock–Kurzweil integrable on \( I_0 \). Assume that \( H \) is a closed set in \( I_0 \) and for each \( x^* \in X^* \) the real function \( x^*(f) \) is McShane integrable on \( H \). Then there exists an element \( x_H \in X \) such that
\[
(M) \int_H x^*(f) = x^*(x_H)
\]
for each \( x^* \in X^* \).

**Remark.** In the proof of Lemma 11, (1) is a key form. For the multidimensional Henstock–Kurzweil integral, (1) does not automatically hold, because from the Henstock–Kurzweil integrability of the function \( f \) on the interval \( I_0 \) one can not deduce that \( f \) is Henstock–Kurzweil integrable on a measurable subset of \( I_0 \) (see [19]). Therefore, taking notice of Theorem 10, we have the following Corollary 13.

**Corollary 13.** Suppose that the Banach space \( X \) contains no copy of the space \( c_0 \) and \( H \) is a closed set in \( I_0 \) (\( I_0 \subset \mathbb{R}^m \)). Assume that \( f \) is Henstock–Kurzweil integrable on \( H \) and further for
each $x^* \in X^*$ the real function $x^*(f)$ is McShane integrable on $H$. Then there exists an element $x_H \in X$ such that

$$(M) \int_H x^*(f) = x^*(x_H)$$

for each $x^* \in X^*$.

**Proof.** By Theorem 10 $f \chi_H$ is Pettis integrable on $I_0$ and therefore $f$ is Pettis integrable on $H$, so there is an element $x_H \in X$ such that $(M) \int_H x^*(f) = x^*(x_H)$ for each $x^* \in X^*$. □

**Theorem 14.** Suppose that the Banach space $X$ contains no copy of the space $c_0$ and $f : I_0 \to X$ is Henstock–Kurzweil integrable on $I_0$. Assume that $f$ satisfies Property (P) and that $f$ is Dunford integrable on a measurable set $E_0 \subset I_0$.

Then $f$ is Pettis integrable on $E_0$.

**Proof.** The Dunford integrability of $f$ on the measurable set $E_0 \subset I_0$ shows that $x^*(f)$ is McShane integrable on $E_0$. By Lemma 2, $x^*(f)$ is Henstock–Kurzweil integrable on $I_0$ for each $x^* \in X^*$. It follows that $x^*(f)$ is Henstock–Kurzweil integrable on $I_0 \setminus E_0$ and

$$(HK) \int_{I_0 \setminus E_0} x^*(f) = (HK) \int_{I_0} x^*(f) - (M) \int_{E_0} x^*(f).$$

To prove that $f$ is Pettis integrable on $E_0$, we have to show that for every measurable $E \subset E_0$ there is $x_E \in X$ such that $(M) \int_E x^*(f) = x^*(x_E)$ for each $x^* \in X^*$.

In fact, for every measurable subset $E \subset E_0$ and for each $x^* \in X^*$, the McShane integrability of $x^*(f)$ on $E_0$ implies that $x^*(f)$ is McShane integrable on $E \subset E_0$. For each $n \in \mathbb{N}$, there is a sequence of closed subsets $H_n \subset E$ such that

$H_n \subset H_{n+1}$, \quad $\mu(E \setminus H_n) < \frac{1}{n}$

and

$\mu\left(E \setminus \bigcup_{n=1}^{\infty} H_n\right) = 0$.

By the absolutely continuity of the Lebesgue integral we have

$$(M) \int_E x^*(f) = \lim_{n \to \infty} (M) \int_{H_n} x^*(f).$$

It follows from Lemma 11 that there exist $x_{H_n} \in X$, $n = 1, 2, \ldots$, such that

$$(M) \int_{H_n} x^*(f) = x^*(x_{H_n})$$

and

$$(M) \int_E x^*(f) = \lim_{n \to \infty} (M) \int_{H_n} x^*(f) = \lim_{n \to \infty} x^*(x_{H_n}).$$
for each \( x^* \in X^* \).

Let \( H_0 = \emptyset \). Then for each \( x^* \in X^* \),

\[
\begin{align*}
(M) \int_{E} x^* (f) &= \lim_{n \to \infty} \left( M \int_{H_n} x^* (f) \right) \\
&= \sum_{n=1}^{\infty} \left[ \left( M \int_{H_n} x^* (f) \right) - \left( M \int_{H_{n-1}} x^* (f) \right) \right] \\
&= \sum_{n=1}^{\infty} \left[ x^*(x_{H_n}) - x^*(x_{H_{n+1}}) \right] = \sum_{n=1}^{\infty} x^*(x_{H_n} - x_{H_{n+1}}).
\end{align*}
\]

Since \( X \) contains no copy of \( c_0 \), by the Bessaga–Pełczynski theorem [3, p. 22], there is \( x_E \in X \) such that \( \sum_{n=1}^{\infty} (x_{H_n} - x_{H_{n+1}}) \) is unconditionally convergent in norm to \( x_E \in X \) and \( (M) \int_{E} x^*(f) = x^*(x_E) \). Hence, \( f \) is Pettis integrable on \( E_0 \) and the proof is complete. \( \square \)

In [4, Theorem 8] D.H. Fremlin proved the following result for the case of an interval \( I_0 \subset \mathbb{R}^1 \).

**Lemma 15.** A function \( f : I_0 \to X \) is McShane integrable on \( I_0 \) if and only if it is Henstock–Kurzweil integrable and Pettis integrable.

Checking Fremlin’s proof it can be seen that it still holds when \( I_0 \) is an interval in \( \mathbb{R}^m \). In fact, Lemma 15 was also proved in [19] for the case of \( I_0 \subset \mathbb{R}^m \).

**Remark.** The theorem in [20, p. 535] points out that a function \( f : I_0 \to X \) is Pettis integrable, then \( f \) is Henstock–Kurzweil integrable. In fact, this result is not correct. Otherwise, suppose that the theorem in [20, p. 535] holds, this means that the Pettis integrable function \( f \) is Henstock–Kurzweil integrable on \( I_0 \). It follows immediately from Lemma 15 that \( f \) is McShane integrable on \( I_0 \). However, the example 3C [6, p. 143] and the example (CH) [21, p. 1184] show that there is a function \( f \) such that \( f \) is Pettis integrable but not McShane integrable.

We come now to our main results.

**Theorem 16.** Suppose that the Banach space \( X \) contains no copy of the space \( c_0 \). Assume that the function \( f : I_0 \to X \) is Henstock–Kurzweil integrable and satisfies Property (P). Then each perfect set contains a portion on which \( f \) is McShane integrable.

**Proof.** Let \( E \) be a perfect set in \( I_0 \) and let \( \Delta = \{ I_n \} \) be the sequence of all open intervals in \( I_0 \) that intersect \( E \) and have rational endpoints. Let \( E_n = E \cap I_n \), \( n = 1, 2, \ldots \). For each pair of positive integers \( m \) and \( n \) let \( E^n_m = \{ x^* \in X^*: \int_{E_n} |x^*(f)| \leq m \} \). Then \( X^* = \bigcup_{m}^{\infty} \bigcup_{n}^{\infty} E^n_m \).

In fact, for each \( m \) and \( n \) we have \( E^n_m \subset X^* \), so \( \bigcup_{m}^{\infty} E^n_m \subset X^* \). On the other hand, for every \( x^* \in X^* \), by Lemma 2, \( x^*(f) \) is Henstock integrable on \( I_0 \). It follows from Lemma 3 that each perfect set \( E \) contains a portion \( P = E \cap I \) on which \( x^*(f) \) is McShane integrable. So there is a \( n_0 \in N \) such that \( I_{n_0} \subset I \) and \( I_{n_0} \in \Delta \). Note that \( x^*(f) \) is McShane integrable on \( P = E \cap I \), then \( x^*(f) \) is McShane integrable on a portion \( E_{n_0} = E \cap I_{n_0} \subset E \cap I \) and therefore there is a \( m_0 \) such that \( \int_{E_{n_0}} |x^*(f)| \leq m_0 \). Therefore, \( x^* \in E^0_{m_0} \) and \( X^* \supset \bigcup_{m}^{\infty} \bigcup_{n}^{\infty} E^n_m \). That is, \( X^* \supset \bigcup_{m}^{\infty} \bigcup_{n}^{\infty} E^n_m \).
Now we prove that each of the sets $E^n_m$ is closed.

Let $x^*$ be a limit point of $E^n_m$ and $\{x^*_k\}$ a sequence in $E^n_m$ that converges to $x^*$. Then the sequence $\{|x^*_k f|\}$ converges pointwise on $I_0$ to the function $|x^*(f)|$ and by Fatou’s lemma we have

$$\int_{E^n_m} |x^*(f)| \leq \liminf_{k \to \infty} \left\{ \int_{E^n_m} |x^*_k f| \right\} \leq m.$$ 

This shows that $x^* \in E^n_m$ and conclude that the set $E^n_m$ is closed.

By the Baire Category Theorem, there exist $M, N, x^*_0, r > 0$ such that $\{x^* : \|x^* - x^*_0\| \leq r\} \subset EN_M$.

For each $x^*$ in $X^*$ with $\|x^*\| \neq 0$, by $x^*_0 \in \{x^* : \|x^* - x^*_0\| \leq r\}, \frac{r}{\|x^*\|} x^* + x^*_0 \in \{x^* : \|x^* - x^*_0\| \leq r\}$, we find that

$$\int_{E^n_m} |x^*(f)| \leq \frac{\|x^*\|}{r} \left\{ \int_{E^n_m} \frac{r}{\|x^*\|} x^*(f) + x^*_0(f) \right\} + \int_{E^n_m} |x^*_0(f)| \right\} \leq 2M \frac{r}{\|x^*\|}.$$ 

Hence, for each $x^*$ in $X^*$ the function $x^*(f)$ is Lebesgue integrable on the portion $E_N = E \cap I_N$. This shows that $f$ is Dunford integrable on $E \cap I_N$.

According to Theorem 14, we obtain that $f$ is Pettis integrable on $E \cap I_N$. It follows from Lemma 15 that $f$ is McShane integrable on $E \cap I_N = P_0$. □

Note that the “perfect set” in Theorem 16 may be replaced by “closed set”, the result still holds.

**Remark.** In the proof of Theorem 16, taking the perfect set $E$ as a subinterval $I$ of $I_0$, i.e., $E = I$, the portion $E_N = I \cap I_N$ of $I$ is still a subinterval $J$ of $I_0$. That is, $J = I \cap I_N$ is a subinterval of $I_0$. Checking the above proof it can be seen that $f$ is Dunford integrable on the subinterval $J$. On the other hand, by Lemma 4, $f$ is always Henstock–Kurzweil integrable on the subinterval $J$ of $I_0$. It follows from Theorem 10 that $f$ is Pettis integrable on $J$. By Lemma 15 $f$ is McShane integrable on $J$. Hence, we obtain the following theorem.

**Theorem 17.** Suppose that the Banach space $X$ contains no copy of the space $c_0$ and $I_0 \subset R^m$. If a function $f : I_0 \to X$ is Henstock–Kurzweil integrable, then there exists a subinterval $J$ of $I_0$ such that $f$ is McShane integrable on $J$.

This is an answer to Karták’s question from [1] for the Banach space case mentioned in the introduction.

Using the Baire Category Theorem and Theorem 9 in [17] (or Theorem 4.16 in [19]), the following theorem can be obtained.

**Theorem 18.** Suppose that the Banach space $X$ contains no copy of the space $c_0$ and that a function $f : I_0 \to X$ is given. Then $I_0$ can be written as a countable union of closed sets $E_n$ such that $f$ is McShane integrable on each $E_n$ if and only if every closed set contains a portion on which $f$ is McShane integrable.

Combining Theorems 16 and 18, we obtain the statement as follows.
**Theorem 19.** Suppose that $X$ contains no copy of the space $c_0$. Assume that a function $f : I_0 \to X$ is Henstock–Kurzweil integrable and satisfies Property (P). Then $I_0$ can be written as a countable union of closed sets $E_n$ such that $f$ is McShane integrable on each $E_n$.

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**References**