Existence of periodic solutions for second-order differential equations with singularities and the strong force condition

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Abstract

We prove the existence of periodic solutions for the equation

\[ u'' + f(u)u' + g(t, u) = e(t), \]  

where the nonlinearity \( g \) has a repulsive singularity at the origin. In previous papers dealing with this kind of problem it is usually assumed a nonintegrability condition on \( g \) near the origin. We provide a weaker condition that substitutes the nonintegrability of \( g \). If \( f \equiv 0 \) the existence of subharmonic solutions is proved utilizing a variational method and when \( f \neq 0 \) we prove the existence of a periodic solution using topological degree theory.

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1. Introduction

This paper deals with the problem of finding periodic solutions, and in particular subharmonic solutions for the forced oscillator with differential equation

\[ u'' + f(u)u' + g(t, u) = e(t), \]  

(2)

where for \( a = -\infty \) or \( a = 0 \) we suppose that \( f: ]a, +\infty[ \to \mathbb{R} \) is continuous and \( g: \mathbb{R} \times ]a, +\infty[ \to \mathbb{R} \) is \( T \)-periodic in its first variable and of \( L^1 \)-Carathéodory type, that is: \( g(\cdot, x) \) is measurable for each \( x \in ]a, +\infty[ \), \( g(t, \cdot) \) is continuous for a.e. \( t \in \mathbb{R} \), and for every \( a < s < S \) there exists \( h_{s,S} \in L^1_{\text{loc}}(\mathbb{R}) \) in such a way that \( |g(t, u)| \leq h_{s,S}(t) \) for all \( u \in [s, S] \) and a.e. \( t \in \mathbb{R} \). Moreover \( e: \mathbb{R} \to \mathbb{R} \) is locally integrable and \( T \)-periodic.

We are mainly interested in singular nonlinearities, that is, we suppose that \( g \) becomes unbounded near the origin. This situation was extensively studied, both, the case with friction (see [7,10–12]) and without (see [1,3,9]). This study was done independently for the case of an attractive force (if \( g(t, x) \to +\infty \) when \( x \to 0^+ \)) and the repulsive one (if \( g(t, x) \to -\infty \) when \( x \to 0^+ \)).

Lazer and Solimini, in a paper of 1987 (see [8]), prove the existence of a periodic solution of the equation

\[ u'' + g(u) = e(t) \]  

(3)

with a repulsive force. They assume a “strong force condition”

\[ \int_0^1 g(u) \, du = -\infty \]  

(4)

and show that there are nonlinearities \( g \), not verifying the strong force condition, for which there are functions \( e \) such that Eq. (3) does not have solutions. Since then, the condition (4) has been considered as necessary for the existence of periodic solutions of (2) and is commonly assumed (see [1,3,5,7,9–12]). The condition (4) has as a consequence that the energy of a solution passing near the origin is arbitrarily large. Keeping in mind the same idea we will show that this condition can be weakened in such a way that it permits some kind of unification with the nonsingular case. Throughout this paper we will assume the existence of a continuous function \( l: ]a, +\infty[ \to \mathbb{R} \) such that \( g(t, u) < l(u) \), for all \( u \in ]a, +\infty[ \) and a.e. \( t \in [0, T] \), verifying the following condition:

\begin{enumerate}
\item[(L1)] There exists a sequence \( ]r_n, r_n'[, \subset ]a, 1[ \) such that \( l(u) < \bar{e} \) for all

\[ u \in \bigcup_{n \in \mathbb{N}} ]r_n, r_n'[, \text{ and } \int_{r_n} \bar{e} \, du \to -\infty. \]

Notice that if \( a = 0 \) this condition is equivalent to (4) when \( l \) has an upper bound near the origin. So we are thinking of examples such as

\[ l(u) = \frac{1}{u^3} \sin \left( \frac{1}{u} \right) \]

that cannot be classified either as repulsive neither as attractive.
In previous papers dealing with this kind of problem (see [1,3,4,7]), the authors usually considered a sign condition on $g(t,x) - \bar{e}$ or else a Landesman–Lazer condition. We replace those conditions by the following condition over $g$, where we denote by $\bar{e} = \frac{1}{T} \int_0^T e(t) dt$ the mean value of $e$:

\begin{equation}
G_1 \text{ There exists } h_+ \in L^1(0,T) \text{ with } \int_0^T h_+ \geq 0 \text{ such that for a.e. } t \in [0,T],
\end{equation}

\[ g(t,x) - \bar{e} \geq h_+(t), \]

for all $x$ greater than a constant $M > 1$.

We also assume a Ahmad–Lazer–Paul type condition (see $(G_2)$ in Section 2). For a proof that these conditions are weaker than the previous ones, we refer to [6].

We study two situations where the strong force condition can be replaced by $(L_1)$. In Section 2 we consider the case without friction and prove the existence of a sequence of subharmonic solutions with arbitrarily large minimal periods. This generalizes some results in [3]. The proof is based on a variational argument. The problem with friction is treated in Section 3, where the existence of periodic solutions is proved, improving a theorem in [7]. The proof uses degree theory. We believe that the same type of generalization can be done in other situations where the strong force condition was used.

2. Multiplicity and subharmonic solutions

Consider Eq. (2) without friction term

\[ u'' + g(t,u) = e(t). \]

We denote by $G(t,u) = \int_1^u g(t,s) ds$ and $L(u) = \int_1^u l(s) ds$, the primitives of $g$ and $l$, respectively. In the following theorem we will consider $a = 0$.

**Theorem 1.** Assume that $g(t,u) \leq l(u)$, for a.e. $t \in [0,T]$ and all $u \in [0,\infty[$, for some continuous function $l : ]0,\infty[ \rightarrow \mathbb{R}$ verifying $(L_1)$ and

\[ L_2 \lim_{x \rightarrow +\infty} \frac{L(x)}{x^2} = 0. \]

Moreover, suppose that $g$ verifies $(G_1)$ and

\[ (G_2) \lim_{x \rightarrow +\infty} \left[ \frac{1}{T} \int_0^T G(t,x) dt - x \bar{e} \right] = +\infty. \]

\[ (G_3) \text{ There exists } h \in L^1_{\text{loc}}(\mathbb{R}) \text{ such that for a.e. } t \in \mathbb{R} \text{ and for every } x \in ]0,1[, \ g(t,x) \leq h(t). \]
Then Eq. (6) has a sequence \((u_k)_{k \in \mathbb{N}}\) of \(kT\)-periodic solutions, in such a way that \(\max u_n\) is unbounded from above. If in addition we have a strict inequality in (5) then the minimal periods tend to infinity.

Concerning the nonsingular case \((a = -\infty)\) we can state a similar theorem.

**Theorem 2.** Assume that \(g(t, u) \leq l(u)\), for a.e. \(t \in [0, T]\) and all \(u \in \mathbb{R}\), for some continuous function \(l : \mathbb{R} \to \mathbb{R}\) verifying (L1) and (L2). Moreover, suppose that \(g\) verifies (G1) and

\[
(G_2)' \quad \lim_{|x| \to +\infty} \left[ \frac{1}{T} \int_0^T G(t, x) \, dt - \bar{e} \right] = +\infty.
\]

\((G_3)' \quad \text{There exists } h \in L^1_{\text{loc}}(\mathbb{R}) \text{ such that for a.e. } t \in \mathbb{R} \text{ and for every } x \in ]\infty, 1[, \ g(t, x) \leq h(t).\]

Then Eq. (6) has a sequence \((u_k)_{k \in \mathbb{N}}\) of \(kT\)-periodic solutions, in such a way that \(\max u_n\) is unbounded from above. If in addition we have a strict inequality in (5) then the minimal periods tend to infinity.

The proof of the last theorem is similar to the proof of Theorem 1, so we will only prove Theorem 1. Notice that subtracting \(\bar{e}\) to both sides of Eq. (2) we can assume, without loss of generality, that \(\bar{e} = 0\).

We start by defining a truncation that will be used throughout the proof of the theorem. Given a function \(g : \mathbb{R} \times ]0, +\infty[ \to \mathbb{R}\), \(L^1\)-Carathéodory and \(0 < r < 1 < R\), let us define \(g_{r, R} : \mathbb{R}^2 \to \mathbb{R}\) by

\[
gr_{r, R}(t, x) = \begin{cases} 
g(t, r) + \sqrt{1 - r} - \sqrt{1 - x} & \text{if } x < r, \\
g(t, x) & \text{if } r \leq x \leq R, \\
g(t, R) - \sqrt{R} + \sqrt{x} & \text{if } x > R. 
\end{cases} \tag{7}
\]

Notice that \(g_{r, R}\) is still a \(L^1\)-Carathéodory function. We define

\[
G_{r, R}(t, u) = \int_0^u g_{r, R}(t, s) \, ds.
\]

Now, consider the following problem:

\[
u''(t) + g_{r, R}(t, u(t)) = e(t), \tag{8}
\]

that we will refer in the future as \((8)_{r, R}\). If a function \(u\) is a periodic solution of \((8)_{r, R}\) that verifies \(u(t) \in [r, R]\) for every \(t \in \mathbb{R}\), then \(u\) is also a solution of (2). We are thus interested in obtaining some estimates for the solutions of \((8)_{r, R}\) and this is the aim of the next lemma.

**Lemma 1.** Suppose that \(g\) verifies the conditions of Theorem 1. Then for each \(k \in \mathbb{N}\) there exists \(0 < r_k < 1 < R_k\) such that every \(kT\)-periodic solution of \((8)_{r_k, R_k}\) satisfies \(r_k < u(t) < R_k, \forall t \in \mathbb{R}\), so it is a \(kT\)-periodic solution of (2).
Proof. Notice that using $(L_2)$ we can take a sequence $R_n > M$ such that
\[ R_n \to +\infty \quad \text{and} \quad \frac{l(R_n)}{R_n} \to 0. \quad (9) \]
Otherwise there exists constants $\epsilon > 0$ and $A > 0$ such that $l(u) > \epsilon u$, for every $u > A$, which gives a contradiction with $(L_2)$.

We argue by contradiction. Suppose there exists $k \in \mathbb{N}$ in such a way that for each $n \in \mathbb{N}$ there exists $u_n$ a $kT$-periodic solution of $(8)_{r_n,R_n}$ which satisfies
\[ \{u_n(t) : t \in \mathbb{R}\} \nsubseteq [r_n, R_n]. \quad (10) \]
Integration of $(8)_{r_n,R_n}$ gives
\[ \int_0^{kT} g_{r_n,R_n}(t, u(t)) \, dt = 0, \quad (11) \]
for every $n \in \mathbb{N}$. We first prove the following

Claim 1. There exists $K$ such that $u_n(t) < K$ for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$.

In fact if this is not the case there exists a sequence, that we still note by $u_n$, in such a way that
\[ \max u_n \to +\infty \quad (12) \]
as $n$ tends to infinity.

As a consequence of (10) and (11) there exists $t_n^1 \in [0, kT]$ such that $u_n(t_n^1) < M$. Indeed, if there exists $n_0 \in \mathbb{N}$ such that $\min u_{n_0} \geq M$ then, due to (10), we have $\{t \in \mathbb{R} : u_{n_0}(t) > R_{n_0}\} \neq \emptyset$ and now, using $(G_1)$,
\[ \int_0^{kT} g_{r_{n_0},R_{n_0}}(t, u_{n_0}(t)) \, dt \]
\[ \geq \int_{M \leq u_{n_0}(t) \leq R_{n_0}} h_+(t) \, dt + \int_{u_{n_0}(t) > R_{n_0}} [h_+(t) - \sqrt{R_{n_0} + u_{n_0}(t)}] \, dt > 0, \]
which contradicts (11). We conclude that for large $n$ there exists $[\alpha_n, \beta_n]$ with $(\beta_n - \alpha_n) \leq kT$ and $t_n^2 \in [\alpha_n, \beta_n]$, so that $\max u_n(t) = u_n(t_n^2)$ and verifying
\[ u_n(\alpha_n) = M = u_n(\beta_n), \quad (13) \]
\[ M \leq u_n(t) \leq u_n(t_n^2), \quad \forall t \in [\alpha_n, \beta_n]. \quad (14) \]
Defining
\[ v_n'(t) = -g_{r_n,R_n}(t, u_n(t)) \quad \text{and} \quad v_n(\alpha_n) = u_n'(\alpha_n), \]
the integration of (8) gives

\[ u_n'(t) = v_n(t) + \int_{\alpha_n}^t e(s)\,ds. \] (15)

The function \( v_n(.) + \int_0^t h_+(s)\,ds \) is decreasing in \([\alpha_n, \beta_n]\). Since by (G1) and the definition of \( v_n \) we have \( v_n'(t) + h_+(t) = -g_{r_n,R_n}(t,u_n(t)) + h_+(t) \leq 0, \forall t \in [\alpha_n, \beta_n] \). Using (15) we conclude that

\[ \max u_n - M \leq kT(v_n(\alpha_n) + k\|e - h_+\|_1). \] (16)

Since we assumed that \( \max u_n \to +\infty \), from (16) we see that \( v_n(\alpha_n) > k\|e\|_1 \) for all large \( n \). On the other hand, by (15), \( v_n(t_{n3}) \leq k\|e\|_1 \).

It will be convenient to define a similar truncature for \( l \). For each \( 0 < r < 1 < R \) we let

\[ l_{r,R}(x) = \begin{cases} l(r) + \sqrt{1-r} - \sqrt{1-x} & \text{if } x < r, \\ l(x) - \sqrt{R} + \sqrt{x} & \text{if } r \leq x \leq R, \\ l(R) - \sqrt{R} + \sqrt{x} & \text{if } x > R, \end{cases} \]

and \( L_{r,R}(x) = \int_1^x l_{r,R}(s)\,ds \). By (G1) it is clear that \( l(x) \geq 0 \) for \( x > M \). For all \( t \in [\alpha_n, t_{n3}] \), using (15) and the fact that \( g_{r_n,R_n} \leq l_{r_n,R_n} \) we see that

\[
\frac{d}{dt} \left[ L_{r_n,R_n}(u_n(t)) + \frac{1}{2}(v_n(t) - k\|e\|_1)^2 \right] = l_{r_n,R_n}(u_n(t))\left( v_n(t) + \int_{\alpha_n}^t e(s)\,ds \right) - g_{r_n,R_n}(t,u_n(t))(v_n(t) - k\|e\|_1) \\
\geq l_{r_n,R_n}(u_n(t))(v_n(t) - k\|e\|_1) - g_{r_n,R_n}(t,u_n(t))(v_n(t) - k\|e\|_1) \geq 0.
\]

We conclude that \( L_{r_n,R_n}(u_n(.)) + \frac{1}{2}(v_n(.)) - k\|e\|_1)^2 \) is increasing in \([\alpha_n, t_{n3}] \). So, for all large \( n \) we have

\[ L(M) + \frac{1}{2}(v_n(\alpha_n) - k\|e\|_1)^2 \leq L_{r_n,R_n}(u_n(t_{n3})). \] (17)

For each \( 1 < M < R_n \leq x \) we have \( l(x) \geq 0 \) and so

\[
L_{r_n,R_n}(x) = L(R_n) + (x - R_n)(l(R_n) - \sqrt{R_n}) + \frac{2}{3}x^{3/2} - \frac{2}{3}R_n^{3/2} \\
\leq L(x) + xl(R_n) + \frac{1}{2}R_n^{3/2} + \frac{2}{3}x^{3/2} \\
\leq L(x) + x^2\frac{l(R_n)}{R_n} + x^{3/2}.
\]

Considering the definition of \( l_{r_n,R_n} \) we see that the last estimation also holds for \( M < x < R_n \). Using (L1), (9) and the above inequalities we see that for every \( \epsilon > 0 \) there exists \( C_\epsilon \) and \( n_0 \) in such a way that

\[ L_{r_n,R_n}(x) \leq \epsilon x^2 + C_\epsilon, \] (18)
for all $n > n_0$ and $M < x$. By (16)–(18), for every $n > n_0$,
\[
L(M) + \frac{1}{2}(v_n(\alpha_n) - k\|e\|_1)^2 \leq \varepsilon(u_n(t_n^3))^2 + C\varepsilon
\]
\[
\leq \varepsilon(kT(v_n(\alpha_n) + k\|e - h + 1\|) + M)^2 + C\varepsilon,
\]
since $v_n(\alpha_n) \to +\infty$, we get a contradiction for small $\varepsilon$, when $n \to +\infty$. This ends the proof of Claim 1.

**Claim 2.** There exists $t_n^4$ such that $r_n' < u_n(t_n^4)$.

Let us suppose that $u_n(t) \leq r_n', \forall t \in \mathbb{R}$. Since $g$ is negative in $[r_n, r_n']$ we obtain
\[
\int_0^{kT} g_{r_n, R_n}(t, u_n(t)) \, dt = \int_{0 < u_n(t) < r_n} g(t, r_n) + \sqrt{1 - r_n} - \sqrt{1 - u_n(t)} \, dt
\]
\[
+ \int_{r_n \leq u_n(t) \leq r_n'} g(t, u_n(t)) \, dt < 0,
\]
which is in contradiction with (11) and proves Claim 2.

**Claim 3.** There exists $C_2 > 0$, such that $\|u_n'\|_{\infty} \leq C_2$.

Using Claim 1, $(G_3)$, (11) and the $L^1$-Carathéodory properties of $g$, we obtain
\[
\int_{u_n(t) < 1} |g_{r_n, R_n}(t, u_n(t))| \, dt \leq \int_{u_n(t) < 1} \left[ -g_{r_n, R_n}(t, u_n(t)) + h(t) \right] \, dt + k\|h\|_{L^1}
\]
\[
\leq \int_{1 \leq u_n(t) \leq K} \left| g_{r_n, R_n}(t, u_n(t)) \right| \, dt + 2k\|h\|_{L^1} \leq C_1,
\]
for all sufficiently large $n \in \mathbb{N}$. So we can conclude that
\[
\int_0^{kT} \left| g_{r_n, R_n}(t, u_n(t)) \right| \, dt \leq 2C_1.
\]
Let $a_n \in [0, kT]$ be such that $u_n'(a_n) = 0$. Then
\[
\|u_n'\|_{\infty} = \max_{t \in [0, kT]} \left| \int_{a_n}^t [e(t) - g_{r_n, R_n}(t, u_n(t))] \, ds \right| \leq k\|e\|_{L^1(0, T)} + 2C_1 = C_2.
\]
Finally we will reach the desired contradiction. For all large $n$ we have $R_n > K$, so by (10) there exists $t_n^5 > t_n^4$ such that $u_n(t_n^5) < r_n$. Consider $[t_n^6, t_n^7] \subset [t_n^4, t_n^5]$ in such a way that $t_n^7 - t_n^6 < kT$ and
\[
\begin{align*}
\Phi_k(u) := \int_{0}^{kT} \left[ \frac{1}{2} \dot{u}^2(t) - G_{r_k,R_k}(t, u(t)) + e(t)u(t) \right] dt.
\end{align*}
\]
Suppose that there exists $E > 0$ such that $\max u_n < E$. Since by the assumptions on $g$ we have $g(t, x) < h_1(t)$, for all $x \in ]0, E[$ and a.e. in $t \in [0, T]$, then
\[
\frac{1}{k} \Phi_k(u_k) \geq -C - \|u_k\|_{\infty}\|e\|_1,
\]
which is in contradiction with (22) and completes the proof of the first part of the theorem. The proof of the second part follows the arguments in [4] and [6]. □

3. The equation with friction

In this section we will assume that $a = 0$. Consider the problem of the existence of $T$-periodic solutions for
\[
u'' + f(u)u' + g(t, u) = e(t),\tag{23}
\]
where $f : ]0, +\infty[ \to \mathbb{R}$ and $g : ]0, +\infty[ \to \mathbb{R}$ are continuous and $e \in L^2_{\text{loc}}$ is $T$-periodic. We prove the following

**Theorem 3.** Suppose that there exists $l : ]0, +\infty[ \to \mathbb{R}$ a continuous function and $h \in L^1(0, T)$ such that for all $u > 0$ and for a.e. $t \in [0, T],$
\[
g(t, u) < l(u), \tag{24}
g(t, u) < h(t). \tag{25}
\]
Moreover let us suppose that $l$ verifies $(L_1)$ (with $a = 0$), $g$ verifies $(G_1)$ with strict inequality in (5), and that $f(u) \neq 0$ for all $u \in ]0, +\infty[$. Then the problem (23) has at least one $T$-periodic solution.

An example of a nonlinearity to which the last theorem can be applied is the function defined by
\[
g(t, u) = \begin{cases} \min\left\{ \frac{1}{u^3} \sin\left(\frac{1}{u}\right) + 1, \frac{1}{\sqrt{t(t-T)}} \right\}, & \text{if } t \in ]0, T[, \\ g(0, u) = g(T, u) = 0, \end{cases} \tag{26}
\]
and defined by periodicity in $t \in \mathbb{R}$.

**Proof.** Let us start by introducing an auxiliary function $p : ]0, +\infty[ \to ]-\infty, 0[$ that we choose to be continuous, strictly increasing, verifying $p(1) = -1$ and
\[
\int_{r_n}^{r'_n} p(s) ds \to -\infty
\]
as $n$ tends to infinity. Consider the homotopy
\[
u'' + f(u)u' + g_\lambda(t, u) = e_\lambda(t), \tag{27}
\]
where we define $g_\lambda(t,u) = (1 - \lambda)(\bar{e} + 1 + p(u)) + \lambda g(t,u)$ and $e_\lambda(t) = (1 - \lambda)\bar{e} + \lambda e(t)$. Notice that $\bar{e}_\lambda = \bar{e}$ for all $\lambda \in [0,1]$. We can choose functions $l$ and $h$ in such a way that $g_\lambda$ verify (24) and (25) uniformly in $\lambda \in [0,1]$.

Our next purpose is to find a uniform bound for any solution of (27). First we show that for each solution $u$ of (27) there exists $t_0 \in [0,T]$ such that

$$u(t_0) < M. \quad (28)$$

Indeed, if this is not the case then $u(t) \geq M > 1$, $\forall t \in [0,T]$, and by $(G_1)$,

$$\int_0^T g_\lambda(t,u(t)) \, dt \geq (1 - \lambda) \int_0^T \left[ \bar{e} + 1 + p(u(t)) \right] dt + \lambda \int_0^T \left[ h_+(t) + \bar{e} \right] dt$$

$$\geq T(1 - \lambda)\bar{e} + (1 - \lambda) \int_0^T \left[ 1 + p(u(t)) \right] dt + \lambda T \bar{e} > T\bar{e},$$

when $\lambda \neq 1$. If $\lambda = 1$ we also have

$$\int_0^T g_\lambda(t,u(t)) \, dt > \int_0^T \left[ h_+(t) + \bar{e} \right] dt \geq T\bar{e}.$$

On the other hand, the integration of (27) gives

$$\int_0^T g_\lambda(t,u(t)) \, dt = \int_0^T e_\lambda(t) \, dt = T\bar{e} \quad (29)$$

which results in a contradiction. Inequality (28) yields

$$u(t) \leq u(t_0) + \int_0^T |u'(t)| \, dt < M + \sqrt{T} \|u'\|_2. \quad (30)$$

Now, multiplying (27) by $u$ and integrating over $[0,T]$ by parts yields

$$\int_0^T (u'(t))^2 \, dt = \int_0^T \left[ g_\lambda(t,u(t)) - e_\lambda(t) \right] u(t) \, dt. \quad (31)$$

Then, using (25) and the last inequality, we obtain

$$\|u'\|_2^2 \leq \int_0^T |h(t) - e_\lambda(t)| u(t) \, dt \leq C(M + \sqrt{T} \|u'\|_2)$$

which gives a bound for $\|u'\|_2$. Finally by (30) we obtain an upper bound, say $Y$, for all the solutions of (27).
We claim that \( u' \) is also uniformly bounded. To prove this we use a similar trick to the one in the proof of Lemma 1. Using (25), (29) and the bound on \( \|u'\|_2 \) we obtain that, for every solution \( u \) of (27),

\[
\int_{u(t)<1} |g_\lambda(t, u(t))| \, dt \leq \int_{u(t)<1} \left[ -g_\lambda(t, u(t)) + h(t) \right] \, dt + \|h\|_1 \\
\leq \int_{1 \leq u(t) \leq Y} |g_\lambda(t, u(t))| \, dt - T \bar{e} + 2\|h\|_1 \leq C_1.
\]

So, we conclude that

\[
\int_0^T |g_\lambda(t, u(t))| \, dt \leq 2C_1.
\]

Considering \( a \in [0, T] \) such that \( u'(a) = 0 \), we have

\[
\|u'\|_\infty = \max_{t \in [0,T]} \left| \int_a^t [e(s) - g_\lambda(s, u(s)) - f(u(s))u'(s)] \, ds \right| < C_2,
\]

as claimed.

Our final goal will be to show that there are no solutions of (27) on the boundary of

\[
\Omega = \left\{ u \in C^1([0, T]): \epsilon < u(t) < Y, \|u'\|_\infty < C_2 \right\},
\]

for some \( \epsilon > 0 \). We will argue by contradiction.

Let us suppose that for each \( n \in \mathbb{N} \) we have a solution \( u_n \) of (27) with \( \lambda = \lambda_n \), in such a way that \( \min u_n = u_n(t_n^1) = r_n \). Since \( g_\lambda(t, u) < l(u) < \bar{e} \) for \( u \in [r_n, r_n'] \), (29) shows that there exists a smallest \( t_n^2 \in \mathbb{R} \) such that \( t_n^2 > t_n^1 \) and \( u(t_n^2) = r_n' \). Multiplying (27) by \( u_n' \) and integrating over \( [t_n^1, t_n^2] \) gives us

\[
\left( \frac{u_n'(t_n^2)}{2} \right)^2 + \int_{t_n^1}^{t_n^2} f(u_n')(u_n')^2 \, dt + \int_{t_n^1}^{t_n^2} g_\lambda(t, u_n)u_n' \, dt = \int_{t_n^1}^{t_n^2} e_{\lambda_n}u_n' \, dt.
\]

Using the bounds already obtained we conclude that

\[
\int_{t_n^1}^{t_n^2} g_\lambda(t, u_n)u_n' \, dt
\]

is bounded. On the other hand, the above integral is equal to

\[
(1 - \lambda_n) \left[ (\bar{e} + 1)(r_n' - r_n) + \int_{r_n}^{r_n'} p(s) \, ds \right] + \lambda_n \int_{t_n^1}^{t_n^2} g(t, u_n)u_n' \, dt
\]
\[
\leq (1 - \lambda_n) \int_{r_n}^{r_n'} p(s) \, ds + \lambda_n \int_{l_n}^{l_n'} \left[ l(u_n)(u_n' + C_2) + C_2 g(t, u_n) \right] \, dt + C_3
\]
\[
\leq (1 - \lambda_n) \int_{r_n}^{r_n'} p(s) \, ds + \lambda_n \int_{r_n}^{r_n'} l(s) \, ds + C_4
\]
and the last expression tends to \(-\infty\) as \(n\) tends to infinity. This is a contradiction. We conclude that for \(n\) large enough there are no solutions of (27) with \(\min u_n = r_n\). Thus letting \(\epsilon = r_n\), Eq. (27) has no solution at the boundary of \(\Omega\).

In order to apply coincidence degree, we define the operators

\[
\mathcal{L}: D(\mathcal{L}) \subseteq H^2(0, T) \rightarrow L^2(0, T),
\]
\[
\mathcal{N}: [0, 1] \times \bar{\Omega} \rightarrow L^2(0, T),
\]
where

\[
D(\mathcal{L}) = \{ u \in H^2(0, T): u(0) = u(T), \ u'(0) = u'(T) \}
\]

by

\[
\mathcal{L}u = u'',
\]
\[
\mathcal{N}(\lambda, u) = e^\lambda - f(u)u' - g_\lambda(\cdot, u).
\]

Since the only solution of \(p(u) = -1\) is \(u = 1\), one has (see [10])

\[
D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(\cdot, \cdot), \Omega) = D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}(0, \cdot), \Omega) = \pm \deg(\bar{e} - g_0(u), ]\epsilon, Y[) = 1,
\]
where \(\deg\) denotes the Brower degree. \(\Box\)

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**References**