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Existence and Uniqueness of the Periodic Orbits of a Class of Cylinder Equations

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In this paper, we investigate the global behaviors of a class of cylinder equations and obtain certain sufficient conditions for the existence and uniqueness of the periodic orbits.

1. INTRODUCTION

Consider the Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

or equivalently

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - g(x), \end{aligned} \tag{1.1}$$

where f and g are continuous functions on $\mathbb{R}$. As we know, there have been many studies on the existence and uniqueness of the limit cycles for

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the planar system (1.1) (cf. [6], for instance). If the functions f and g are periodic in x of period 2π, then Eq. (1.1) defines a system on the cylinder

$$H: 0 \le x \le 2\pi, \qquad |y| < \infty,$$

and it may have a non-zero-homotopic orbit on H. The cylinder system of this type can be found in the oscillating theory of applied sciences. Some special forms of Eq. (1.1) have been studied qualitatively and numerically (see [2, 4] and [3, pp. 449–473]). However, studies for the global behavior of the general cylinder system (1.1) are relatively few. In 1959, Sansone [4] investigated the existence of non-zero-homotopic periodic orbits of Eq. (1.1) and proved that if $f(x) > 0$ and $g(x) < 0$ for all x, then Eq. (1.1) has a periodic orbit of this type. In this paper, we are concerned with the global behavior of Eq. (1.1) where both f and g may have roots and obtain several new results on the existence and uniqueness of the periodic orbits. Finally, as an application, we provide an example (Example 2.3) in which we discuss completely the global behavior for a particular equation of the form (1.1) appearing in [3, p. 449].

2. THE MAIN RESULTS AND PROOF

Throughout this paper we will always suppose that the continuous f and g are 2π-periodic in x.

Let

$$H^+: 0 \le x \le 2\pi, \qquad y > 0$$

$$H^-: 0 \le x \le 2\pi, \qquad y < 0.$$

Then

$$H = \text{cl.}\,(H^+ \cup H^-).$$

From (1.1) we have

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y}, \qquad y \ne 0. \tag{2.1}$$

Clearly, there exists a constant $M > 0$ such that

$$\left|\frac{dy}{dx}\right| = \left|-f(x) - \frac{g(x)}{y}\right| \le M \qquad \text{for } 0 \le x \le 2\pi \text{ and } |y| \ge 1.$$

It follows that for a given $\delta > 0$, there exists $N > 0$ such that for any solution $(x(t, y_0), y(t, y_0))$ of (1.1) satisfying

$$x(0, y_0) = 0, \qquad y(0, y_0) = y_0, \qquad |y_0| \ge N$$

it holds that for a unique t_0 with $y_0 t_0 > 0$,

$$x(t_0, y_0) = 2\pi, 0 < x(t, y_0) < 2\pi, \text{ and } |y(t, y_0)| \geq \delta \quad \text{for } 0 < t < t_0 \operatorname{sgn} y_0.$$

Hence, if $|y_0| \geq N$, the orbit of the solution $(x(t, y_0), y(t, y_0))$ for $0 \leq t \leq t_0 \operatorname{sgn} y_0$ can be represented as

$$y = Y(x, y_0) \qquad \text{for } 0 \leq x \leq 2\pi$$

with the property

$$|Y(x, y_0)| \geq \delta \qquad \text{for } 0 \leq x \leq 2\pi. \tag{2.2}$$

Integrating (2.1) from 0 to x, we have

$$Y(x, y_0) = y_0 - F(x) - G_1(x, y_0), \tag{2.3}$$

where

$$F(x) = \int_0^x f(u)\, du, \qquad G_1(x, y_0) = \int_0^x \frac{g(u)}{Y(u, y_0)}\, du. \tag{2.4}$$

From (2.2) thru (2.4), we have immediately

LEMMA 2.1. *It holds uniformly for $0 \leq x \leq 2\pi$ that*

$$\lim_{|y_0| \to \infty} |Y(x, y_0)| = \infty.$$

Using Lemma 2.1, we can prove

THEOREM 2.1. *Let*

$$A = \int_0^{2\pi} f(x)\, dx, \qquad B = \int_0^{2\pi} g(x)\, dx, \qquad \textit{and} \qquad C = \int_0^{2\pi} F(x) g(x)\, dx.$$

Then

(i) $\lim_{|y_0| \to \infty} (Y(2\pi, y_0) - y_0) = -A$;
(ii) $\lim_{|y_0| \to \infty} (Y^2(2\pi, y_0) - y_0^2) = -2B$ if $A = 0$;
(iii) $\lim_{|y_0| \to \infty} (Y^3(2\pi, y_0) - y_0^3) = -3C$ if $A = B = 0$.

Proof. Equation (2.3) gives that

$$Y(2\pi, y_0) = y_0 - A - G_1(2\pi, y_0).$$

Obviously, from (2.4) and Lemma 2.1,

$$\lim_{|y_0|\to\infty} G_1(2\pi, y_0) = 0.$$

Hence, conclusion (i) follows.

Let $A = 0$. From (2.1) we have

$$y\,dy = -f(x)\,y\,dx - g(x)\,dx.$$

Integrating from 0 to x yields that

$$\tfrac{1}{2}\left[Y^2(x, y_0) - y_0^2\right] = -\int_0^x f(u)Y(u, y_0)\,du - \int_0^x g(u)\,du. \quad (2.5)$$

Substitution of (2.3) into the right-hand side of Eq. (2.5) produces that

$$Y^2(x, y_0) - y_0^2 = F^2(x) - 2y_0F(x) - 2G(x) + 2G_2(x, y_0), \quad (2.6)$$

where

$$G(x) = \int_0^x g(u)\,du, \qquad G_2(x, y_0) = \int_0^x f(u)G_1(u, y_0)\,du. \quad (2.7)$$

Since $A = F(2\pi) = 0$, we have from (2.6)

$$Y^2(2\pi, y_0) - y_0^2 = -2B + 2G_2(2\pi, y_0).$$

Now conclusion (ii) is evident since $\lim_{|y_0|\to\infty} G_2(x, y_0) = \infty$ from (2.7) and Lemma 2.1. Similarly to (2.6) we can deduce that

$$\begin{aligned}\tfrac{1}{3}\left[Y^3(x, y_0) - y_0^3\right] &= -\int_0^x fY^2\,du - \int_0^x gY\,du \\ &= 2\int_0^x fG\,du + \int_0^x Fg\,du - y_0^2F(x) - \tfrac{1}{3}F^3(x) \\ &\quad + y_0F^2(x) - y_0G(x) - 2\int_0^x fG_2\,du + \int_0^x gG_1\,du.\end{aligned}$$

Notice that

$$\int_0^x fG\,du = F(x)G(x) - \int_0^x Fg\,du.$$

We have

$$\tfrac{1}{3}\left[Y^3(2\pi, y_0) - y_0^3\right] = -C - 2\int_0^{2\pi} fG_2\,dx + \int_0^{2\pi} gG_1\,dx$$

if $A = B = 0$. Then the last conclusion follows similarly. The proof is completed.

Using Theorem 2.1 we can determine the boundedness of the orbits of Eq. (1.1) on the half cylinder H^+ or H^- if $|A| + |B| + |C| \neq 0$. In most cases this condition is sufficient for the boundedness of solutions. For example, if $A \neq 0$ or $A = 0$, and $B \neq 0$, the behavior of the orbits of Eq. (1.1) for $|y_0|$ large is as shown in Fig. 2.1.

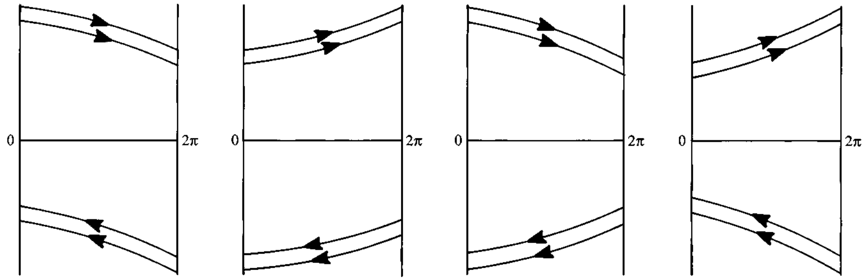
From the first equation of (1.1) it is easy to see that any non-zero-homotopic periodic orbit of (1.1) does not intersect with the x-axis. Thus, we can discuss the existence of such periodic orbits on H^+ or H^-. In what follows we develop our discussion on H^+. First we have

THEOREM 2.2. *Suppose that* $A > 0$. *If* $g(x) \leq 0$ *for all* x *and*

$$f(x) < 0 \qquad \text{when } g(x) = 0, \tag{2.8}$$

then Eq. (1.1) *has exactly one periodic orbit on* H^+ *and no periodic orbit on* H^-.

Proof. We first consider the case where $g(x) < 0$ for all x. Then, $\dot{y}|_{y=0} = -g(x) > 0$. It follows from Theorem 2.1 that Eq. (2.1) has at least



(a) A > 0 (b) A < 0 (c) A = 0, B = 0 (d) A = 0, B < 0

FIGURE 2.1

one 2π-periodic solution of H^+. For any 2π-periodic solution $Y(x, y_0)$ of Eq. (2.1), we have from (2.3)

$$A + \int_0^{2\pi} \frac{g(x)}{Y(x, y_0)} dx = 0. \tag{2.9}$$

Note that $Y(x, y_0) \neq Y(x, y_0^1) \neq 0$ for $y_0 \neq y_0^1$. It follows from (2.9) and $g(x) < 0$ that there is a unique y_0 such that (2.9) is satisfied. Hence, we have proved the theorem if $g(x) < 0$.

Next, we consider the general case. From the above discussion, for any $\varepsilon > 0$, the perturbed system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - (g(x) - \varepsilon) \end{aligned} \tag{2.10}$$

has a unique periodic orbit

$$\Gamma_\varepsilon : y = Y(x, y_0(\varepsilon)) > 0$$

with $Y(0, y_0(\varepsilon)) = Y(2\pi, y_0(\varepsilon)) = y_0(\varepsilon)$. Note that Γ_ε is stable. We have

$$\oint_{\Gamma_\varepsilon} -f(x)\, dt \leq 0. \tag{2.11}$$

Let

$$Y(x, y_0(0)) = \lim_{\varepsilon \to 0} Y(x, y_0(\varepsilon)). \tag{2.12}$$

Then $y = Y(x, y_0(0))$ $(0 \leq x \leq 2\pi)$ is a closed invariant curve of Eq. (1.1) on H. We now prove that

$$Y(x, y_0(0)) > 0 \qquad \text{for all } x \in [0, 2\pi]. \tag{2.13}$$

If this is not the case, then without loss of generality, we may assume that, $Y(0, y_0(0)) = y_0(0) = 0$. Note that $\dot{y}|_{y=0} = -g(x)$. It follows easily that $g(0) = 0$. That is, the origin is a critical point of Eq. (1.1). Thus, the curve $y = Y(x, y_0(0))$ represents a singular closed orbit Γ_0 of Eq. (1.1) on H. Obviously from (2.12), $\lim_{\varepsilon \to 0} \Gamma_\varepsilon = \Gamma_0$. Therefore from (2.8) we have

$$\oint_{\Gamma_\varepsilon} -f(x)\, dt \to +\infty \qquad \text{as } \varepsilon \to 0,$$

which contradicts (2.11). Then (2.13) follows. Therefore, we obtain a periodic orbit $y = Y(x, y_0(0))$ of Eq. (1.1). The uniqueness of periodic orbits follows from (2.9). This finishes the proof.

EXAMPLE 2.1. From Theorem 2.2, the system

$$\dot{x} = y,$$
$$\dot{y} = -(1 - 2\cos x)y - \cos x + 1$$

has a unique non-trivial periodic orbit on H.

In the case that g has different signs. The simplest case is that g has exactly two roots on $[0, 2\pi)$. We will consider this case in the rest of this paper. Without loss of generality we can assume that there exists $x_0 \in (0, 2\pi)$ such that

$$g(0) = 0, (x - x_0)g(x) > 0 \qquad \text{for } x \neq x_0, x \in (0, 2\pi). \quad (2.14)$$

THEOREM 2.3. *Suppose that*
(i) $A > 0$, $B \leq 0$ *or* $A = B = 0$, $C > 0$;
(ii) $g \in C^1$ *and* $g'(0) \neq 0$;
(iii) $F(x) < F(y)$ *for all* $0 < y < x_0 < x < 2\pi$ *satisfying* $G(x) = G(y)$.

Then Eq. (1.1) *has a periodic orbit on* H^+.

Proof. From condition (i) and Theorem 2.1 we have $Y(2\pi, y_0) < y_0$ for $y_0 > 0$ sufficiently large. Hence, in order to prove the existence of a periodic orbit on H, it suffices to prove that

$$Y(2\pi, y_0) > y_0 \qquad \text{for } y_0 > 0 \text{ small.} \quad (2.15)$$

By condition (ii), the critical points $(0,0)$ and $(2\pi, 0)$ are saddle points of Eq. (1.1) (they are the same point on H). Making the so-called Lienard transformation $v = y + F(x)$, we have from Eq. (1.1)

$$\dot{x} = v - F(x), \qquad \dot{v} = -g(x). \quad (2.16)$$

Further, noting (2.14) we introduce the Filippov transformation

$$Z = \int_{x_0}^{x} g(u)\, du = G(x) - G(x_0) \quad (2.17)$$

so that Eq. (2.16) becomes

$$\frac{dZ}{dv} = F_i(Z) - v, \qquad 0 \leq Z \leq Z_i, i = 1, 2, \quad (2.18)$$

where $F_2(Z) = F(x_i(Z))$, $i = 1, 2$, $Z_1 = G(2\pi) - G(x_0)$, $Z_2 = G(0) - G(X_0)$, and $x_2(Z) \in [0, x_0]$ for $0 \leq Z \leq Z_2$, and $x_1(Z) \in [x_0, 2\pi]$ for $0 \leq Z \leq Z_1$ are inverse functions of (2.17). Note that

$$Z_1 - Z_2 = G(2\pi) - G(0) = B.$$

It follows from condition (i) that

$$Z_1 \leq Z_2. \tag{2.19}$$

Then, it is easy to see that condition (iii) is equivalent to

$$F_1(Z) < F_2(Z) \qquad \text{for } 0 < Z < Z_1. \tag{2.20}$$

By (2.20) and applying the comparison theorem (see [5]) of ordinary differential equations to (2.18) we can prove that Eq. (1.1) has no non-trivial zero-homotopic periodic orbits or separatrix loops (see [1] and [6, Chapter 5]). Therefore, there are three cases on the relative positions of separatrice of Eq. (1.1) on H^+ as shown in Fig. 2.2.

Clearly, in the case shown in Fig. 2.2(c), inequality (2.15) holds. Hence, we need to rule out the possibilities shown in Figs. 2.2 (a) and 2.2 (b). If the case shown in Fig. 2.2 (a) occurs, then for the phase portrait of separatrice of the planar system (2.16) we have the case shown in Fig. 2.3 (a).

Let $P = (2\pi, A)$, and Q be the intersection point of the line $x = x_0$ with the separatrix connecting the origin O and P. By applying the comparison theorem to (2.18), it follows from (2.20) that the image of the orbit segment $\overset{\frown}{O\ Q}$ under the transformation (2.17) must lie below that of $\overset{\frown}{Q\ P}$, and hence, $Z_2 < Z_1$, (cf. Fig. 2.3 (b)). This contradicts (2.19). In the same way, a contradiction arises in the case shown in Fig. 2.2 (b). This finishes the proof.

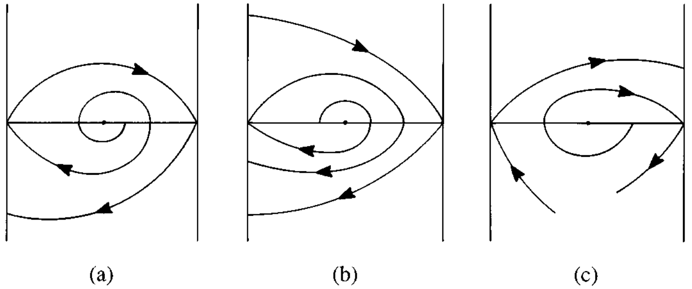


FIGURE 2.2

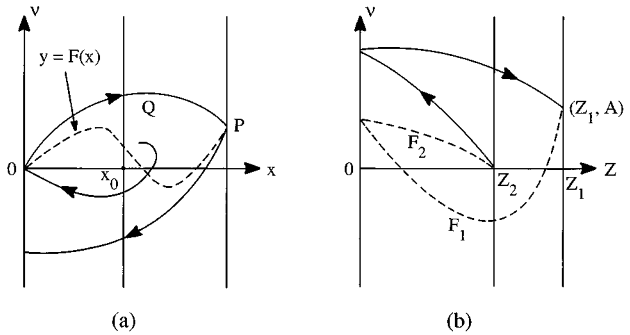


FIGURE 2.3

Remark 2.1. From the proof of Theorem 2.3, we know that under conditions (ii) and the case shown in (iii), Fig. 2.2 (c) must appear if $B \leq 0$. Similarly, we can prove that under the same conditions (ii) and (iii), the case shown in Fig. 2.4 (a) must occur on H^- if $B \geq 0$. Especially, the case shown in Fig. 2.4 (b) occurs if $B = 0$.

Therefore, it follows from Theorem 2.1 immediately

COROLLARY 2.1. *Suppose that conditions* (ii) *and* (iii) *of Theorem* 2.3 *hold*.

(i) *If* $A > 0$, $B \geq 0$, *then Eq*. (1.1) *has a periodic orbit on* H^-.

(ii) *If* $A > 0$, $B = 0$, *or* $A = B = 0$, $C > 0$, *then Eq*. (1.1) *has two non-zero-homotopic orbits on* H.

Remark 2.2. If, instead of condition (iii) of Theorem 2.3, it holds that

(iii) $F(x) > F(y)$ for all $0 < y < x_0 < x < 2\pi$ satisfying $G(x) = G(y)$, then we have Fig. 2.5.

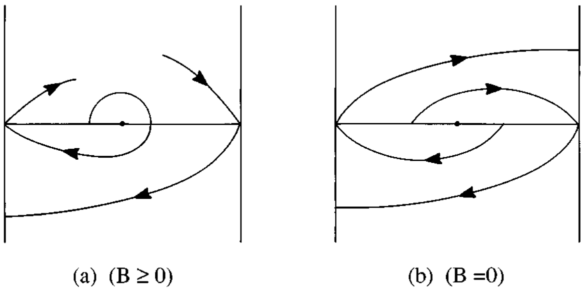


FIGURE 2.4

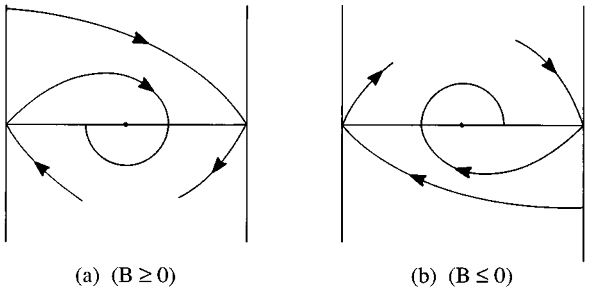


FIGURE 2.5

Remark 2.3. Suppose that $B \leq 0$, let $y_0 \in (0, x_0)$ satisfy $G(y_0) - G(x_0) \geq Z_1$. If $F(x) < F(y)$ for all $y_0 < y < x_0 < x < 2\pi$, then (2.20) (and therefore, condition (iii) of Theorem 2.3) holds.

EXAMPLE 2.2. Consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(\cos x + \lambda) y + \sin\left(x + \frac{\pi}{8}\right) + \sin\frac{\pi}{8}, \end{aligned} \tag{2.21}$$

where λ is a constant. Using Theorem 2.3 and Remark 2.3, we can prove that if $0 < \lambda \leq 2\sqrt{2}/3\pi$, then (2.21) has a periodic orbit on H^+.

First, note that the function $\sin(x + \pi/8) + \sin \pi/8$ has exactly two simple roots $-\pi/4$ and π on the interval $[-\pi/4, 2\pi - \pi/4)$. Moving the critical points $(-\pi/4, 0)$ to the origin, we have from (2.21) that

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\left(\cos\left(x - \frac{\pi}{4}\right) + \lambda\right) y + \sin\left(x - \frac{\pi}{8}\right) + \sin\frac{\pi}{8}. \end{aligned}$$

Then, we have

$$f(x) = \cos\left(x - \frac{\pi}{4}\right) + \lambda, \qquad g(x) = -\sin\left(x - \frac{\pi}{8}\right) - \sin\frac{\pi}{8}.$$

It is easy to see that

$$F(x) = \sin\left(x - \frac{\pi}{4}\right) + \lambda x + \frac{1}{\sqrt{2}} \qquad \text{and}$$

$$G(x) = \cos\left(x - \frac{\pi}{8}\right) - \cos\frac{\pi}{8} - x \sin\frac{\pi}{8},$$

and

$$A = 2\pi\lambda, \qquad B = -2\pi \sin\frac{\pi}{8}, \qquad \text{and} \qquad x_0 = \pi + \frac{\pi}{4}.$$

Let $y_0 = \pi - \pi/8$. Then it is direct that $G(y_0) \geq B$. Since $0 < \lambda \leq 2\sqrt{2}/3\pi < \cos \pi/8$ we can examine that

$$f(y_0) < 0 \qquad \text{and} \qquad F(x_0) \geq A. \tag{2.22}$$

Note that f has exactly two simple roots on $[0, 2\pi]$ which belong to $(3\pi/4, y_0) \cup (x_0, 2\pi)$. It follows from (2.22) that $F(y) > F(x_0) > F(x)$ for $y_0 < y < x_0 < x < 2\pi$. Hence, the conclusion follows from Theorem 2.3 and Remark 2.3. For the uniqueness of the periodic orbits we have

THEOREM 2.4. *Suppose that* (2.14) *holds. If*

$$F(x) \geq 0, \not\equiv 0 \qquad \text{for } 0 < x < x_0,$$
$$F(x) \leq A, \not\equiv A \qquad \text{for } x_0 < x < 2\pi,$$

then Eq. (1.1) *has at most one periodic orbit on H^+ or on H^-. Therefore, Eq.* (1.1) *has at most two non-zero-homotopic periodic orbits on H.*

Proof. Let $L: y = y(x) > 0$ be a periodic orbit of Eq. (1.1) on H^+. It suffices to prove that

$$I(L) \equiv \oint_L -f(x)\, dt < 0. \tag{2.23}$$

We have

$$\begin{aligned} I(L) &= \int_0^{2\pi} -\frac{f(x)}{y(x)}\, dx \\ &= \int_0^{x_0} -\frac{f(x)}{y(x)}\, dx + \int_{x_0}^{2\pi} -\frac{f(x)}{y(x)}\, dx \\ &\equiv J_1 + J_2. \end{aligned} \tag{2.24}$$

Put $v(x) = y(x) + F(x)$; then

$$v'(x) = \frac{dy(x)}{dx} + f(x) = -\frac{g(x)}{y(x)}.$$

Hence, $v'(x) < 0$ for $x \in (x_0, 2\pi)$. It follows from our assumption that

$$v(x) - F(x) \geq v(x) - A \geq v(2\pi) - A \qquad \text{for } x \in (x_0, 2\pi).$$

Therefore,

$$\begin{aligned}
J_2 &= \int_{x_0}^{2\pi} \frac{f(x)}{F(x) - v(x)}\, dx \\
&= \int_{x_0}^{2\pi} \frac{(F(x) - v(x))' + v'(x)}{F(x) - v(x)}\, dx \\
&= \ln \frac{v(2\pi) - A}{v(x_0) - F(x_0)} + \int_{x_0}^{2\pi} \frac{-v'(x)}{v(x) - F(x)}\, dx \\
&< \ln \frac{v(2\pi) - A}{v(x_0) - F(x_0)} + \int_{x_0}^{2\pi} \frac{-v'(x)}{v(x) - A}\, dx \\
&= \ln \frac{v(x_0) - A}{v(x_0) - F(x_0)} \le 0,
\end{aligned}$$

since $A \ge F(x_0)$. In the same way, we can prove that $J_1 < 0$. Thus, from (2.24) we have $I(L) < 0$. This shows that any periodic orbit L must be asymptotically stable. hence, it must be unique if it exists on H^+.

If L is situated on H^-: $y(x) < 0$. Then, instead of (2.24), we have

$$\begin{aligned}
I(L) &= \int_{2\pi}^{0} -\frac{f(x)}{y(x)}\, dx \\
&= \int_{0}^{2\pi} \frac{f(x)}{y(x)}\, dx \\
&= \int_{0}^{x_0} \frac{f(x)}{y(x)}\, dx + \int_{x_0}^{2\pi} \frac{f(x)}{y(x)}\, dx \\
&\equiv \hat{J}_1 + \hat{J}_2.
\end{aligned}$$

Since, in this case, we have

$$v'(x) = -\frac{g(x)}{y(x)} > 0 \qquad \text{for } x \in (x_0, 2\pi),$$

we can prove that $\hat{J}_2 < 0$ and $\hat{J}_1 < 0$ using the same method we used previously. Hence, we obtain again that $I(L) < 0$. This proves that L is a unique periodic orbit on H^-. The proof is completed.

From Theorem 2.4 and the discussion of Example 2.2, Eq. (2.21) has a unique periodic orbit on H^+ for $\lambda = 2\sqrt{2}/3\pi$.

COROLLARY 2.2. *Suppose that* $f(x) \le 0$ (*or* ≥ 0) *for all* $x \in \mathbb{R}$. *Then Eq*. (1.1) *has at most one non-zero-homotopic periodic orbit on* H.

Proof. From the proof of Theorem 2.4, Eq. (1.1) has at most one periodic orbit on H^+ or H^-. Hence, it is sufficient to prove that Eq. (1.1) cannot have a periodic orbit on both H^+ and H^- at the same time. Let L: $y = Y(x, y_0)$ be a periodic orbit of Eq. (1.1) on H. Then, from (2.5) we have

$$B = -\int_0^{2\pi} f(x)Y(x, y_0)\, dx.$$

This gives the desired conclusion since both $f(x)$ and $Y(x, y_0)$ keep the constant sign.

EXAMPLE 2.3. Consider the cylinder system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(\mu + \cos x)y + k \sin x, \end{aligned} \tag{2.25}$$

where $k \neq 0$ is a constant and μ is supposed to be a parameter. Equation (10-5) of [3] with $\Lambda_0 = 0$ can be reduced to the above equation. Without loss of generality, we may assume that $k = 1$ in Eq. (2.25). We prove that there exists a critical value $\mu = \mu_0 \in (0, 1)$ such that if and only if $\mu_0 < \mu < 1$ (resp., $0 < \mu < \mu_0$), Eq. (2.25) has a unique non-trivial zero-homotopic periodic orbit on H (resp., precisely two non-zero-homotopic periodic orbits on H). The bifurcation diagrams are shown in Fig. 2.6.

In fact, let

$$f(x) = \mu + \cos x, \qquad g(x) = -\sin x, \qquad x_0 = \pi, \qquad A = 2\pi\mu,$$

$$F(x) = \int_0^x f(u)\, du = \mu x + \sin x, \qquad \text{and}$$

$$G(x) = \int_0^x g(u)\, du = \cos x - 1.$$

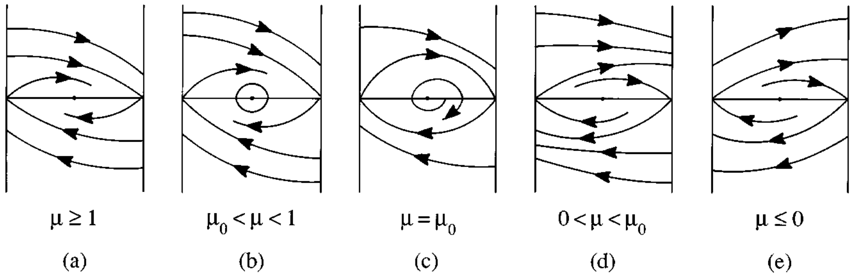


FIGURE 2.6

Then

$$\left(\frac{f}{g}\right)' = \frac{1 + \mu \cos x}{\sin^2 x} > 0 \qquad \text{for } x \in (0, 2\pi), x \neq \pi, \text{ if } |\mu| \leq 1,$$

and $F(x) \geq 0$ ($\leq A$) for $0 < x < \pi$ ($\pi < x < 2\pi$), if $\mu \geq 0$.

It follows from Theorem 2.4 and [7] that

(i) Equation (2.25) has at most one periodic orbit on H^+ or H^- for $\mu \geq 0$.

(ii) Equation (2.25) has at most one non-trivial zero-homotopic periodic orbit on H for $|\mu| \leq 1$.

Further, $G(x) = G(y)$, $0 < y < \pi < x < 2\pi$, implies that $y = 2\pi - x$. Hence, for $\pi < x < 2\pi$, we have

$$F(x) - F(y) = 2\mu(x - \pi) + 2\sin x \begin{cases} \geq 2[(x - \pi) + \sin x] > 0 & \text{if } \mu \geq 1, \\ < 0 & \text{if } \mu \leq 0. \end{cases} \tag{2.26}$$

It follows from [1, 6] that

(iii) Equation (2.25) has no non-trivial zero-homotopic periodic orbits and separatrix loops on H if $\mu \leq 0$ or $\mu \geq 1$.

The case shown in Fig. 2.6 (a) follows from Theorem 2.1, conclusions (i) and (iii) above, and Remark 2.2. Note that the critical point $(\pi, 0)$ is stable (unstable) if $\mu > 1 (< 1)$ since $f'(\pi) = \mu - 1$. For $\mu = 1$, it is stable from (2.26). Therefore, for $\mu < 1$ and $|\mu - 1|$ small, Eq. (2.25) has a stable limit cycle generated from the point $(\pi, 0)$. It is easy to examine that Eq. (2.25) forms a rotated vector field with respect to μ [6, Chapter 3]. Hence, the limit cycle expands with μ decreasing from 1. Observe that the vector field (2.25) is symmetric with respect to the point $(\pi, 0)$: it is invariant under the change of $(x, y) \to (2\pi - x, -y)$. From conclusions (i) and (iii) above, there must be a value $\mu = \mu_0 \in (0, 1)$ such that the cases shown in Figs. 2.6 (b) and 2.6 (c) appear. And then, the cases shown in Figure 2.6 (d) and 2.6 (e) follow from conclusion (ii) above, Theorem 2.1, Corollary 2.1, and the theory of rotated vector fields [6].

REFERENCES

1. Han Maoan, A note on the nonexistence of the limit cycles, *Chinese Sci. Bull.* **23**, (1992), 1950–1953.
2. M. Levi, Qualitative analysis of the periodically forced relaxation oscillations, *Mem. Amer. Math. Soc.* **214**, (1981), 1–147.

3. W. Lindsey, ‘‘Synchronization System in Communication and Control,’’ Prentice−Hall, Englewood Cliffs, NJ, 1972.
4. G. Sansone, Soluzioni Periodiche di Seconda Specie Dell’equazione Del Pendolo Generalizzata, *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nur. Sez. 1 (8)* **5**, (1959), 59−79.
5. G. Sansone and R. Conti, ‘‘Nonlinear Differential Equations,’’ Pergamon Press, New York, 1964.
6. Ye Yiangian, ‘‘Theory of Limit Cycles,’’ Translations of Mathematical Monographs, Vol. 6, Amer. Math. Soc., Providence, RI, 1986.
7. Zhang Zhefen, Proof of the uniqueness theorem of limit cycles of generalized Lienard equations, *Appl. Anal.* **23** (1986), 63−76.