

# Linear Widths of Function Spaces Equipped with the Gaussian Measure

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We calculate asymptotics for the linear  $(n, \delta)$ -widths of the Sobolev space  $W_2^r$  equipped with the Gaussian measure  $\mu$  in the  $L_q$ . That is, we consider the quantity

$$\lambda_{n,\delta}(W_2^r, L_q, \mu) = \inf_{G \subset W_2^r, \mu(G) \leq \delta} \lambda_n(W_2^r \setminus G, L_q),$$

where  $\lambda_n(K, X)$  is the linear  $n$ -width of the set  $K$  in the space  $X$ . © 1994 Academic Press, Inc.

## 1. INTRODUCTION

Let  $X$  be a normed linear space and  $W$  a subset of  $X$ . Let  $A$  be a linear operator from  $X$  to  $X$ . Let  $AW$  denote the image of  $W$  under  $A$ . The quantity

$$\lambda(W, A, X) = \sup_{x \in W} \|x - Ax\|_X$$

is called the linear distance of the image  $AW$  from the set  $W$ .

For each  $n = 0, 1, \dots$ , we consider the linear  $n$ -width of the set  $W$  in  $X$ . It is defined by

$$\lambda_n(W, X) = \inf_{\mathcal{L}_n} \inf_{A_n} \lambda(W, A_n, X),$$

where  $\mathcal{L}_n$  runs over all the linear subspaces in  $X$  with dimension at most  $n$  and  $A_n$  runs over all linear operators from  $X$  to  $\mathcal{L}_n$ .

We assume that the set  $W$  is equipped with a Borel field  $\mathcal{B}$ , which consists of the open subsets. Let  $\mu$  be a probability measure defined on  $\mathcal{B}$ . That is,  $\mu$  is a  $\sigma$ -additive nonnegative function on  $\mathcal{B}$  and  $\mu(W) = 1$ .

Let  $\delta \in [0, 1]$  be any given number. We define the linear  $(n, \delta)$ -width of the set  $W$  in the space  $X$  for the measure  $\mu$  as follows. Set

$$\lambda_{n, \delta}(W, X, \mu) = \inf_{G_\delta} \lambda_n(W \setminus G_\delta, X), \tag{1}$$

where  $G_\delta$  runs over all the subsets  $G_\delta \in \mathcal{B}$  with measure  $\mu(G_\delta) \leq \delta$ . The quantity  $\lambda_{n, \delta}$  may be understood as the  $\mu$ -distribution of the best linear approximation on all subsets of  $W$ .

Detailed information about the usual linear widths may be found in [17, 13]. Papers connected with calculating the asymptotics of linear  $n$ -widths include [3, 6, 8, 2].

Quantities similar to (1) were considered in [19]. In the books of Traub and Wozniakowski [18] and Traub *et al.* [19], a different problem connected with the best approximation of the function classes, equipped with measure in a Hilbert space, was investigated. Calculation of  $n$ -widths of the smooth function classes equipped with some given measure are included in [20, 9, 1, 11].

Consider the Hilbert space  $L_2$  of all functions  $x(t)$ ,  $t \in [0, 2\pi]$ , with the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \exp(ikt)$$

and inner product

$$\langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(t) \bar{y}(t) dt \quad (x, y \in L_2).$$

In the space  $L_2$  we define the Veil  $r$ -fractional derivative ( $r \in \mathbb{R}$ )

$$x^{(r)}(t) = \sum_{k=-\infty}^{\infty} (ik)^r c_k \exp(ikt) \quad \left( (ik)^r = |k|^r \exp\left(\frac{\pi i}{2} \text{sign } r\right) \right).$$

In this work we consider the Sobolev space  $W_2^r$  ( $r > 0$ ), which consists of all functions  $x \in L_2$ , with mean value  $c_0 = 0$ , and semi-norm  $\|x\|_{W_2^r} = \langle x^{(r)}, x^{(r)} \rangle$ . The space  $W_2^r$  is a Hilbert space with the inner product defined by  $\langle x, y \rangle_1 = \langle x^{(r)}, y^{(r)} \rangle$ .

We equip  $W_2^r$  with a Gaussian measure  $\mu$  whose mean is zero and whose correlation operator  $C_\mu$  has eigenfunctions  $e_k = \exp(ik \cdot)$  and eigenvalues  $\lambda_k = a |k|^{-s}$  ( $a > 0, s > 1$ ). That is,  $C_\mu e_k = \lambda_k e_k, k \in \mathbb{Z} \setminus \{0\}$ .

In particular on the cylindrical subsets in the space  $W_2^r$  given by

$$G = \{x \in W_2^r : (\langle x, e_n^{(-r)} \rangle_1, \dots, \langle x, e_m^{(-r)} \rangle_1) \in \mathcal{D}\},$$

where  $\mathcal{D}$  is any Borel subset in  $\mathbb{R}^{m-n+1}$  ( $m > n$ ),  $e_k^{(-r)} = (ik)^{-r} \exp(ik(\cdot))$ ,  $k = \pm 1, \pm 2, \dots$ , is an orthonormal system in  $W'_2$ , and the measure  $\mu(G)$  is equal to

$$\mu(G) = \prod_{k=-n}^m (2\pi\lambda_k)^{-1/2} \int_{\mathcal{D}} \exp\left(-\frac{1}{2} \sum_{k=n}^m \lambda_k^{-1} u_k^2\right) du_n \cdots du_m. \quad (2)$$

Detailed information about Gaussian measures may be found in the books of Kuo [5] and Traub *et al.* [19].

Consider the Banach space  $L_q$ ,  $1 \leq q \leq \infty$ , which consists of all function  $x$  on  $[0, 2\pi]$  with norm

$$\|x\|_{L_q} = \left( \int_0^{2\pi} |x(t)|^q dt \right)^{1/q}.$$

It is known that if  $r > 1/2 - 1/q$ , then the space  $W'_2$  is compactly embeddable in the space  $L_q$  (see, e.g., [12]).

Let  $c, c_i, c'_i, i=0, 1, \dots$  be positive constants depending solely upon the parameter  $r, q, a$ , and  $s$ . For two positive functions  $a(y)$  and  $b(y)$ ,  $y \in \mathcal{D}$ , we write  $a(y) \asymp b(y)$  or  $a(y) \ll b(y)$  if there exist constants  $c_1, c_2$ , or  $c$  such that  $c_1 \leq a(y)/b(y) \leq c_2$  or respectively  $a(y) \leq cb(y)$  for all  $y \in \mathcal{D}$ .

The aim of this paper is to calculate the asymptotics of the linear  $(n, \delta)$ -widths  $\lambda_{n, \delta}(W'_2, L_q, \mu)$ . Note that the two-sided estimation for  $\lambda_{n, \delta}(W'_2, L_2, \mu)$  may be obtained from the work of Traub *et al.* [19].

**THEOREM 1.** *Let  $2 \leq q < \infty$ ,  $r > 1/2 - 1/q$ ,  $s > 1$ ,  $a > 0$ . The linear  $(n, \delta)$ -widths of  $W'_2$  with measure  $\mu$  in the space  $L_q$  satisfy the asymptotics*

$$\lambda_{n, \delta}(W'_2, L_q, \mu) \asymp \frac{1 + n^{-1/q} \sqrt{\ln \delta^{-1}}}{n^{r + (s-1)/2}},$$

for any  $\delta \in (0, \frac{1}{2}]$ .

We denote the unit ball in  $W'_2$  by  $BW'_2 = \{x \in W'_2 : \|x\|_{W'_2} \leq 1\}$ . The following inequality (see [19, p. 469]) holds for the measure of the unit ball

$$\mu(BW'_2) > 1 - 5 \exp\left(-\frac{1}{2 \operatorname{trace} C_\mu}\right), \quad (3)$$

where  $\operatorname{trace} C_\mu = \sum_{k=-\infty}^{\infty} \lambda_k = 2a \sum_{k=1}^{\infty} k^{-s}$ . Therefore for all  $a \in I_s \equiv (0, (4 \ln 5 \zeta(s)^{-1})]$ ,  $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$  we have from (3) the inequality  $\mu(BW'_2) \geq c > 0$ . We always assume  $a \in I_s$ .

We define the conditional measure by

$$\mu'(G) = \frac{\mu(G \cap BW'_2)}{\mu(BW'_2)} \quad (G \in \mathcal{B}).$$

We may view  $\mu'$  as a probability measure defined on the sets  $Q$  of the field  $\mathcal{B} \cap BW'_2$  and  $\mu'(Q) = \mu(Q)/\mu(BW_2)$ .

**THEOREM 2.** *Let  $2 \leq q < \infty$ ,  $r > 1/2 - 1/q$ ,  $s > 1$ ,  $a \in I_s$ . The following asymptotic equivalence holds for the ball  $BW'_2$  with measure  $\mu'$  in the space  $L_q$ ,*

$$\lambda_{n,\delta}(BW'_2, L_q, \mu') \asymp \min \left\{ \frac{1}{n^{r-1/2+1/q}}, \frac{n^{1/q} + \sqrt{\ln \delta^{-1}}}{(n^{r+(s-1)/2+1/q})} \right\}$$

for any  $\delta \in (0, \frac{1}{2}]$ .

Note that for linear  $n$ -widths of the ball  $BW'_2$  in the space  $L_q$  we have the equality (see [3])

$$\lambda_n(BW'_2, L_q) \asymp \frac{1}{n^{r-1/2+1/q}} \quad (2 \leq q \leq \infty). \tag{4}$$

Comparing Theorem 2 and the asymptotics of (4) shows, in particular, that if we throw out from the class  $BW'_2$  some set  $G$  with measure  $\mu'(G) \leq \exp(-n^{2/q})$ , then we obtain on the remaining set  $BW'_2 \setminus G$  the approximation order  $n^{-r-(s-1)/2}$ . If  $s = 1 + 2\varepsilon$ , where  $\varepsilon$  is an arbitrary positive small number, then the approximation order is  $n^{-r-\varepsilon}$ , which is essentially smaller than (4).

Using Theorem 2 we obtain the asymptotic equivalence for the best approximation on the class  $BW'_2$  in the space  $L_q$  by trigonometric polynomials of degree  $n$ .

Let  $\mathcal{T}_n$  denote the space of trigonometric polynomials of degree  $n$ , i.e.,

$$y(t) = \sum_{k=-n}^n c_k \exp(ikt).$$

Let the  $\delta$ -distance from the class  $BW'_2$  to  $\mathcal{T}_n$  in the space  $L_q$  for the measure  $\mu'$  be defined by

$$E_{n,\delta}(BW'_2, L_q, \mu') = \inf_{\{G: \mu'(G) \leq \delta\}} \sup_{x \in BW'_2 \setminus G} \inf_{y \in \mathcal{T}_n} \|x - y\|_{L_q}.$$

**THEOREM 3.** *If the conditions of Theorem 2 hold, then*

$$E_{n,\delta}(BW'_2, L_q, \mu) \asymp \min \left\{ \frac{1}{n^{r-1/2+1/q}}, \frac{n^{1/q} + \sqrt{\ln \delta^{-1}}}{n^{r+(s-1)/2+1/q}} \right\}.$$

For  $\delta = 0$ , this result is known (see [12])

$$E_n(BW'_2, L_q) \asymp \frac{1}{n^{r-1/2+1/q}}. \quad (5)$$

Comparing the asymptotics (5) and Theorem 3 shows, in particular, that on the set  $BW'_2 \setminus G$ , where  $G$  is some set of  $BW'_2$  measure  $\mu'(G) \leq \exp(n^{-2/q})$ , we can obtain an approximation order essentially smaller than (5).

The proofs of Theorems 1–3 use discretisation techniques (see [7]). This method is based on the reduction of the calculation of the widths of a given class, to the calculation of widths of finite-dimensional set equipped with the Gaussian measure.

## 2. THE ESTIMATION OF LINEAR $(n, \delta)$ -WIDTHS OF FINITE-DIMENSIONAL SETS

In this section we calculate the linear  $(n, \delta)$ -widths in the space  $\mathbb{R}^m$  equipped with the Gaussian measure in the  $l^m_q$ -norm.

Let  $l^m_p$  denote the  $m$ -dimensional normed space consisting of vectors  $x = (x_1, \dots, x_m) \in \mathbb{R}^m$  with norm

$$\|x\|_p = \begin{cases} \left( \sum_{i=1}^m |x_i|^p \right)^{1/p}, & 1 \leq p < \infty \\ \max_{1 \leq i \leq m} |x_i|, & p = \infty. \end{cases}$$

Let  $B^m_\rho = \{x \in l^m_p : \|x\|_p \leq \rho\}$  be the ball of radius  $\rho$  in  $l^m_p$ . Set  $B^m_\rho = B^m_\rho(1)$ .

In the space  $\mathbb{R}^m$  we consider the Gaussian measure  $\gamma = \gamma_m$ , which is defined as

$$\gamma(G) = (2\pi)^{-m/2} \int_G \exp\left(-\frac{1}{2} \sum_{i=1}^m x_i^2\right) dx_1 \cdots dx_m,$$

where  $G$  is any Borel set in  $\mathbb{R}^m$ . Obviously,  $\gamma(\mathbb{R}^m) = 1$ . We use the following measure estimates for balls (see, e.g., [19]). Namely,

$$\gamma(B^m_2(c\sqrt{m})) \leq \frac{1}{2}, \quad \gamma(B^m_2(\rho)) \geq 1 - 5 \exp\left(-\frac{\rho^2}{2m}\right), \quad (6)$$

where  $c$  is an absolute constant, and  $\rho$  any positive number.

Consider the linear  $(n, \delta)$ -widths, with measure, in the  $l_q^m$ -norm, namely

$$\lambda_{n, \delta}(\mathbb{R}^m, l_q^m, \gamma) = \inf_{G_\delta} \inf_{A_n} \sup_{x \in \mathbb{R}^m \setminus G_\delta} \|x - A_n x\|_q.$$

where  $G_\delta$  runs over all Borel sets in  $\mathbb{R}^m$  with measure  $\gamma(G_\delta) \leq \delta$ ,  $A_n$  runs over all linear operators on  $\mathbb{R}^m$  with rank at most  $n$ , and  $\delta \in [0, 1]$ .

**THEOREM 4.** *Let  $2 \leq q < \infty$ ,  $m \geq 2n > 0$ , and  $\delta \in (0, 1/2]$ . Then*

$$\lambda_{n, \delta}(\mathbb{R}^m, l_q^m, \gamma) \asymp m^{1/q} + \sqrt{\ln \delta^{-1}}.$$

We first prove two auxiliary lemmas. We use some known estimations. Namely, if  $u > 1/\sqrt{2}$ , then (see [15])

$$\int_u^\infty e^{-t^2} dt \leq \frac{1}{2u} e^{-u^2}, \quad \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt \leq 1 - \frac{1}{2\sqrt{\pi}} e^{-u^2}. \quad (7)$$

**LEMMA 1.** *Let  $2 \leq q < \infty$  and  $\delta \in [0, \frac{1}{2}]$ . For some constant  $c_0$  depending only on  $q$ , we have*

$$\gamma(x \in \mathbb{R}^m: \|x\|_q \geq c_0(m^{1/q} + \sqrt{\ln \delta^{-1}})\} \leq \delta. \quad (8)$$

*Proof.* Let  $f(x) = \|x\|_q$ . Then  $|f(x) - f(y)| \leq \|x - y\|_2$ ; hence  $f$  is a Lipschitz map on  $\mathbb{R}^m$  with Lipschitz constant  $\sigma = 1$ . By the Maurey-Pisier inequality, see [14], we have for all  $t > 0$ ,

$$\gamma(f - Ef \geq t) \leq \exp(-t^2/2\sigma) = \exp(-t^2/2). \quad (9)$$

But by Kahane's inequality, see [4],  $E_f \asymp (Ef^q)^{1/q} \asymp q^{1/2} m^{1/q}$ . Therefore for some absolute constant  $c > 0$ .

$$\gamma(f \geq t + cq^{1/2} m^{1/q}) \leq \exp(-t^2/2), \quad (10)$$

and taking  $t = (\ln \delta^{-1})^{1/2}$  completes the proof.

**LEMMA 2.** *Let  $\delta \in [0, e^{-1}]$ . For any vector  $z \in \mathbb{R}^m$ , we have*

$$\gamma(x: |(x, z)| \geq c'_0 \|z\|_2 \sqrt{\ln \delta^{-1}}) \geq \delta,$$

where  $(\cdot, \cdot)$  is the standard inner product, and  $c'_0$  an absolute constant.

*Proof.* Since  $\gamma$  is an invariant measure with respect to orthogonal transformation in the space  $\mathbb{R}^m$ , it suffices to prove the lemma for the vector  $z^* = (\|z\|_2, 0, \dots, 0)$ . Using inequality (7) we have

$$\gamma(x: |(x, z^*)| \geq \|z\|_2 \sqrt{\ln \delta^{-1}}) \geq \frac{1}{2} \sqrt{\frac{\delta}{\pi}}.$$

The lemma now follows.

*Proof of Theorem 4.* We first prove the upper bound for the linear  $(n, \delta)$ -widths. Consider the set in  $\mathbb{R}^m$ :

$$Q_\delta = \{x \in \mathbb{R}^m: \|x\|_q \geq c_0(m^{1/q} + \sqrt{\ln \delta^{-1}})\}.$$

From Lemma 1 we have the estimation  $\gamma(Q_\delta) \leq \delta$ . Therefore

$$\lambda_{n, \delta} \equiv \lambda_{n, \delta}(\mathbb{R}^m, l_q^m, \gamma) \leq \sup_{x \in \mathbb{R}^m \setminus Q_\delta} \|x\|_q \leq c_0(m^{1/q} + \sqrt{\ln \delta^{-1}}).$$

To prove the lower bound, let  $\varepsilon$  be any positive number. We define a linear operator  $T$  with rank at most  $n$  and a set  $G \subset \mathbb{R}^m$  with measure  $\gamma(G) \leq \delta$ , for which

$$\lambda_{n, \delta} \geq \sup_{x \in \mathbb{R}^m \setminus G} \|x - Tx\|_q - \varepsilon. \quad (11)$$

We can describe the operator  $T$  by

$$Tx = \sum_{k=1}^n (x, u_k) v_k \quad (x \in \mathbb{R}^m),$$

where  $u_k, v_k$  are vectors in  $\mathbb{R}^m$ . We have for  $1/q + 1/q' = 1$

$$\begin{aligned} \|x - Tx\|_q &= \max_{y \in B_q^n} (x - Tx, y) = \max_y \left( y - \sum_{k=1}^n (y, v_k) u_k, x \right) \\ &\geq \max_{1 \leq i \leq m} \left| \left( e_i - \sum_{k=1}^n (e_i, v_k) u_k, x \right) \right|, \end{aligned} \quad (12)$$

where  $e_i$  is the  $i$ th unit vector. Let  $z_i = e_i - \sum_{k=1}^n (e_i, v_k) u_k$ . Consider the set

$$H = \bigcup_{i=1}^m H_i, \quad H_i = \{x \in \mathbb{R}^m: |(x, z_i)| \geq c'_0 \sqrt{\frac{1}{2} \ln \delta^{-1}}\},$$

We know (see [16]) that for any vectors  $u_k, v_k, k = 1, \dots, n$ , there exists an index  $i_0$  such that  $\|z_{i_0}\|_2 \geq 1/\sqrt{2}$ . Therefore from Lemma 2

$$\gamma(H) \geq \gamma(H_{i_0}) \geq \gamma(x: |(x, z_{i_0})| \geq c'_0 \|z_{i_0}\|_2 \sqrt{\ln \delta^{-1}}) > \delta. \quad (13)$$

From inequalities (12) and (13) and since  $\gamma(G) \leq \delta$ , we have

$$\sup_{x \in \mathbb{R}^m \setminus G} \|x - Tx\|_q \geq c'_0 \sqrt{\frac{1}{2} \ln \delta^{-1}}.$$

From here and inequality (11), letting  $\varepsilon \rightarrow 0$ , we obtain

$$\lambda_{n, \delta} \geq c'_0 \sqrt{\frac{1}{2} \ln \delta^{-1}}. \tag{14}$$

We obtain one more lower estimate for  $\lambda_{n, \delta}$ . Using Hölder's inequality we have

$$\lambda_{n, \delta} \geq m^{1/q-1/2} \lambda_{n, \delta}(\mathbb{R}^m, l_2^m, \gamma) = m^{1/q-1/2} \inf_{G: \gamma(G) \leq \delta} \sup_{x \in \mathbb{R}^m \setminus G} p(x),$$

where  $p(x) = (\sum_{i=1}^{m-n} x_i^2)^{1/2}$ . From inequality (6) it follows that

$$\gamma(p(x) \geq c \sqrt{m}) = \gamma_{m-n}(\mathbb{R}^{m-n} \setminus B_2^{m-n}(c \sqrt{m-n})) \geq \frac{1}{2}.$$

Therefore, for any  $\delta < \frac{1}{2}$  using  $m \geq 2n$ , we obtain

$$\lambda_{n, \delta} \geq m^{1/q-1/2} c \sqrt{m-n} \geq \frac{c}{\sqrt{2}} m^{1/q}. \tag{15}$$

From inequalities (14) and (15) we obtain the lower estimate for  $\lambda_{n, \delta}$ , and Theorem 4 is proved.

Consider in the space  $\mathbb{R}^m$  the Gaussian measures with parameter  $\alpha$  given by

$$\bar{\gamma} = \bar{\gamma}_m = \left(\frac{\alpha}{2\pi}\right)^{m/2} \int_G \exp\left(-\frac{\alpha}{2} \sum_{i=1}^m x_i^2\right) dx_1 \cdots dx_m.$$

Set  $B = B_2^m(\sqrt{m})$ . From inequality (6) for  $\alpha > \alpha_0 \equiv 1 + 2 \ln 10$  we have

$$\bar{\gamma}(B) \geq \frac{1}{2}.$$

Define on  $\mathbb{R}^m$  the conditional measure concentrated on the ball  $B$ , i.e.,

$$\bar{\gamma}'(G) = \frac{\bar{\gamma}(G \cap B)}{\bar{\gamma}(B)} \quad (G \subset \mathbb{R}^m).$$

Obviously  $\bar{\gamma}'(B) = 1$ . Therefore  $\bar{\gamma}'$  is a probability measure defined on the Borel subsets of the ball  $B$  and

$$\bar{\gamma}'(G) = \frac{\bar{\gamma}(G)}{\bar{\gamma}(B)} \quad (G \subset B)$$

LEMMA 2A. *If  $\delta \in [\exp(-m/2), 1/2]$  and  $z \in \mathbb{R}^m$ , then*

$$\bar{\gamma}'(x \in B: |(x, z)| \geq c' \|z\|_2 \sqrt{\frac{1}{2} \ln \delta^{-1}}) \geq \delta, \tag{16}$$

where  $c'$  is an absolute constant.



Indeed from the fact that  $\bar{\gamma}'$  is an invariant measure with respect to orthogonal transposition, it follows that

$$\begin{aligned} g &\equiv \bar{\gamma}(x \in \mathbb{R}^m: |(x, z)| \geq \|z\|_2 \sqrt{\ln \delta^{-1}}, \|x\|_2 \leq \sqrt{m}) \\ &= 2 \left( \frac{\alpha}{2\pi} \right)^{m/2} \int_{\sqrt{\ln \delta^{-1}}}^{\sqrt{m}} e^{-(\alpha/2)x_1^2} dx_1 \\ &\quad \times \int_{B_2^{m-1}(\sqrt{m-x_1^2})} \exp\left(-\frac{\alpha}{2}(x_2^2 + \dots + x_m^2)\right) dx_2 \dots dx_m. \end{aligned}$$

Further, from the definitions of  $\delta$  and  $\alpha$  we have

$$\begin{aligned} g &\geq 2 \left( \frac{\alpha}{2\pi} \right)^{m/2} \int_{\sqrt{\ln \delta^{-1}}}^{\sqrt{3/2 \ln \delta^{-1}}} e^{-(\alpha/2)x_1^2} dx_1 \\ &\quad \times \int_{B_2^{m-1}(\sqrt{m-3/2 \ln \delta^{-1}})} \exp\left(-\frac{\alpha}{2}(x_2^2 + \dots + x_m^2)\right) dx_2 \dots dx_m \\ &\geq \sqrt{\frac{2\alpha}{\pi}} \delta^{3\alpha/4} \cdot \bar{\gamma}_{m-1}(B_2^{m-1}(c_1 \sqrt{m})), \end{aligned} \quad (17)$$

where  $c_1 = \frac{1}{2}$ . From inequality (6) it follows that  $\bar{\gamma}_{m-1}(B_2^{m-1}(c_1 \sqrt{m})) \geq c'_0$ , where  $c'_0$  is an absolute constant. Hence using inequality (17) and the definition of  $\alpha$  we have

$$\bar{\gamma}'(x \in B: |(x, z)| \geq \|z\|_2 \sqrt{\ln \delta^{-1}}) \geq g \geq \sqrt{\frac{2\alpha}{\pi}} c'_0 \delta^{3\alpha/4} \geq c_2 \delta^{3\alpha/4}.$$

Inequality (16) now follows.

**COROLLARY 1.** *Let  $2 \leq q < \infty$  and  $\alpha > \alpha_0$ . For the linear  $(n, \delta)$ -widths of the ball  $B$ , with measure  $\bar{\gamma}'$ , in the space  $l_q^m$  we have*

$$\lambda_{n, \delta}(B, l_q^m, \bar{\gamma}') \asymp \min\{\sqrt{m}, \alpha^{-1/2}(m^{1/q} + \sqrt{\ln \delta^{-1}})\}, \quad (18)$$

where  $m \geq 2n$  and  $\delta \in [0, \frac{1}{2}]$ .

The upper estimate in (18) follows directly from Theorem 4 and the obvious inequality  $\lambda_{n, \delta}(B, l_q^m, \bar{\gamma}') \leq \sqrt{m}$ . The lower estimate in (18) for  $\delta \geq \delta_0$ ,  $\delta_0 = \exp(-\alpha m)$ , repeats the proof of the lower estimate in Theorem 4. However, we must use Lemma 2A rather than Lemma 2. Then we have

$$\lambda_{n, \delta}(B, l_q^m, \bar{\gamma}') \geq c\alpha^{-1/2}(m^{1/q} + \sqrt{\ln \delta^{-1}}). \quad (19)$$

If  $\delta < \delta_0$ , then from (19) we have

$$\lambda_{n, \delta}(B, l_q^m, \bar{\gamma}') \geq \lambda_{n, \delta_0}(B, l_q^m, \bar{\gamma}') \geq c \sqrt{m}. \tag{20}$$

From the inequalities (19) and (20) we obtain (18).

### 3. PROOF OF THEOREM 1

First we give a few auxiliary statements. Consider two sequences of integers  $m_0 = 0$ ,  $m_N = 3^{N-1}$  and  $l_0 = 0$ ,  $l_N = \sum_{s=1}^N m_s$ , where  $N = 1, 2, \dots$ . We decompose the integers  $\mathbb{Z}$  on blocks  $\{A_N\}_{N=-\infty}^{\infty}$ , where  $A_0 = \{0\}$ ,  $A_N = \{l_N, \dots, l_{N+1} - 1\}$  for  $N = 1, 2, \dots$ , and  $A_N = -A_{-N}$  for  $N = -1, -2, \dots$ . For negative  $N$ , set  $m_N = m_{-N}$ ,  $l_N = l_{-N}$ . The cardinality of the block  $A_N$  equals  $m_N$ .

For any number  $N$  we denote  $T_N$  the space consisting of the trigonometric series  $y(\cdot) = \sum_{k \in A_N} c_k \exp(ik(\cdot))$ . We define on the space  $T_N$  the norm

$$\|y\|_{q, N} = \left( \sum_{s=0}^{m_N-1} \left| y \left( \frac{2\pi s}{m_N} \right) \right|^q \right)^{1/q}.$$

From the Hardy–Littlewood inequality (see [21, Vol. 2, p. 4]) for any  $1 < q < \infty$  we have

$$\|y\|_{L_q} \asymp m_N^{-1/q} \|y\|_{q, N} \quad (y \in T_N). \tag{21}$$

In particular for  $q = 2$  we have the equality  $\|y\|_{L_2} = m_N^{-1/2} \|y\|_{2, N}$ . From the Marcinkiewicz theorem (see [21]) we have

$$\|y^{(\alpha)}\|_{L_q} \asymp m^\alpha \|y\|_{L_q} \quad (y \in T_N, \alpha \in \mathbb{R}). \tag{22}$$

For every  $N$  we consider, in the space  $W'_2$ , the projection operator

$$P_N: \sum_{k=-\infty}^{\infty} c_k e_k \rightarrow \sum_{k \in A_N} c_k e_k. \tag{23}$$

LEMMA 3. Set

$$\alpha_{N, \delta} = \frac{c'_0(m_N^{1/q} + \sqrt{\ln \delta^{-1}})}{m_N^{r+1/q+(s-1)/2}},$$

where  $c'_0$  is some constant depending just on  $r$  and  $q$ . Then for any  $\delta \in [0, \frac{1}{2}]$  we have

$$\mu(x \in W'_2: \|P_N x\|_{L_q} \geq \alpha_{N, \delta}) \leq \delta.$$

*Proof.* Since

$$P_N x = \sum_{k \in \mathcal{A}_N} \langle x, e_k \rangle e_k = \sum_{k \in \mathcal{A}_N} \langle x^{(-r)}, e_k^{(-r)} \rangle_1 e_k,$$

from (22) and from the definition of Gaussian measure (2) we have

$$\begin{aligned} \mu &\equiv \mu(x: \|P_N x\|_{L_q} \geq \alpha_{N, \delta}) \leq \mu\left(x: \left\| \sum_{k \in \mathcal{A}_N} \langle x, e_k^{(-r)} \rangle_1 e_k \right\|_{L_q} \geq c_1 m_N^r \alpha\right) \\ &= \left( \prod_{k=l_N}^{l_{N+1}-1} 2\pi\lambda_k \right)^{-1/2} \int_{\mathcal{D}} \exp\left(-\frac{1}{2} \sum_{k=l_N}^{l_{N+1}-1} \lambda_k^{-1} y_k\right) dy_{l_N} \cdots dy_{l_{N+1}-1}, \end{aligned}$$

where

$$\mathcal{D} = \left\{ (y_{l_N}, \dots, y_{l_{N+1}-1}) \in \mathbb{R}^{m_N}: \left\| \sum_{k=l_N}^{l_{N+1}-1} y_k e_k \right\|_{L_q} \geq c_1 m_N^r \alpha_{N, \delta} \right\}.$$

Recall that  $\lambda_k = a|k|^{-s}$ . By the substitution  $t_k/\sqrt{\lambda_k} \rightarrow t_k$  and equality (22) we have

$$\mu \leq (2\pi)^{-m_N/2} \int_{\mathcal{D}_1} \exp\left(-\frac{1}{2} \sum_{k=1}^{m_N} y_k\right) dy_1 \cdots dy_{m_N} = \gamma_{m_N}(\mathcal{D}_1), \quad (24)$$

where  $\mathcal{D}_1 = \{y \in \mathbb{R}^{m_N}: \|\sum_{k=1}^{m_N} y_k e_k\|_{L_q} \geq c_2 m_N^{r+s/2}\}$ .

From (21) it follows that the set  $\mathcal{D}_1$  is contained in the set

$$\mathcal{D}_2 = \left\{ y \in \mathbb{R}^{m_N}: \left( \sum_{u=1}^{m_N} \left| \sum_{k=1}^{m_N} y_k e_k \left( \frac{2\pi u}{m_N} \right) \right|^q \right)^{1/q} \geq c_3 m_N^{r+s/2+1/q} \alpha_{N, \delta} \right\}.$$

Since the matrix  $(e_k(2\pi u/m_N)/\sqrt{m_N})_{k, u=1, \dots, m_N}$  is orthogonal, and finite-dimensional Gaussian measure is invariant with respect to orthogonal transposition in the space  $\mathbb{R}^{m_N}$ , we have

$$\gamma_{m_N}(\mathcal{D}_1) \leq \gamma_{m_N}(\mathcal{D}_2) = \gamma_{m_N}(\mathcal{D}_3), \quad (25)$$

where  $\mathcal{D}_3 = \{y \in \mathbb{R}^{m_N}: (\sum_{k=1}^{m_N} |y_k|^q)^{1/q} \geq c_3 m_N^{r+(s-1)/2+1/q} \alpha_{N, \delta}\}$ .

Let  $c'_0 = c_0/c_3$ , where  $c_0$  is the constant from Lemma 1. Using the definition  $\alpha_{N, \delta}$  from Lemma 1, we obtain

$$\gamma_{m_N}(\mathcal{D}_3) = \gamma_{m_N}(y \in \mathbb{R}^{m_N}: \|y\|_q \geq c_0(m_N^{1/q} + \sqrt{\ln \delta^{-1}})) \leq \delta. \quad (26)$$

From inequalities (24)–(26) we obtain Lemma 3.

*Proof of Theorem 1.* We first prove the upper bound for linear  $(n, \delta)$ -widths  $\lambda_{n, \delta}(W'_2, L_q, \mu)$ . Denote  $\delta_N = \delta \cdot 3^{(N' - |N|)}$ ,  $N = \pm N', \pm(N' + 1), \dots$ ,  $N' = [\log n/2]$  (where  $[\ ]$  is the integer part of number and  $\log u = \log_3 u$ ). Consider the sequence of sets

$$G_N = \{x \in W'_2 : \|P_N x\|_{L_q} \geq \alpha_{N, \delta_N}\}.$$

Using Lemma 3 we estimate the measure of the set  $G = \bigcup_{|N| \geq N'} G_N$

$$\mu(G) \leq \sum_{|N| \geq N'} \mu(G_N) \leq \sum_{|N| \geq N'} \delta_N \leq \delta \sum_{|N| \geq N'} 3^{N' - |N|} \ll \delta. \quad (27)$$

Since  $m_N = 3^{|N|}$ , we have

$$\begin{aligned} \sum_{|N| \geq N'} \alpha_{N, \delta_N} &= c'_0 \sum_{|N| \geq N'} \frac{m_N^{1/q} + \sqrt{\ln \delta_N^{-1}}}{m_N^{r+1/q+(s-1)/2}} \\ &\ll \sum_{|N| \geq N'} 3^{-[r+(s-1)/2]|N|} (1 + 3^{-|N|/q} \sqrt{\ln \delta_N^{-1}}) \\ &\ll 3^{-[r+(s-1)/2]N'} (1 + 3^{-N'/q} \sqrt{\ln \delta^{-1}}) \\ &\ll \frac{1 + n^{-1/q} \sqrt{\ln \delta^{-1}}}{n^{r+(s-1)/2}}. \end{aligned} \quad (28)$$

Consider the linear operator  $A_n = \sum_{N=-N'}^{N'} P_N$ . From the inequalities (27) and (28) we have for linear  $(n, \delta)$ -widths the estimate

$$\begin{aligned} \lambda_{n, \delta}(W'_2, L_q, \gamma) &\leq \lambda(W'_2 \setminus G, A_n, L_q) \\ &\ll \sup_{x \in W'_2 \setminus G} \left\| \sum_{|N| \geq N'} P_N x \right\|_{L_q} \\ &\leq \sum_{|N| \geq N'} \alpha_{N, \delta_N} \ll \frac{1 + n^{-1/q} \sqrt{\ln \delta^{-1}}}{n^{r+(s-1)/2}}. \end{aligned}$$

We now prove the lower estimate. Let  $\varepsilon$  be any positive number. Denote by  $A_n$  the linear operator of rank at most  $n$ , and by  $G_\delta$  the set in  $W'_2$  with measure  $\mu(G_\delta) \leq \delta$  such that

$$\lambda_{n, \delta}(W'_2, L_q, \mu) \geq \lambda(W'_2 \setminus G_\delta, A_n, L_q) - \varepsilon. \quad (29)$$

Consider the projection operator in the space  $L_q$  given by  $Q_n: \sum_{k=-\infty}^{\infty} c_k e_k \rightarrow \sum_{k=n}^{3n+1} c_k e_k$ . From the Marcinkewicz theorem we have the inequality  $\|Q_n x\|_{L_q} \leq c_1 \|x\|_{L_q}$  for all  $x \in L_q$ . Further, using (22) and the definition of the Gaussian measure (2) we have

$$\begin{aligned}
& \mu(x \in W_2: \|x - A_n x\|_{L_q} \geq c_1^{-1} \alpha_{N, \delta}) \\
& \geq \mu(\|Q_n x - Q_n A_n x\|_{L_q} \geq \alpha_{n, \delta}) \\
& = \mu\left(\left\|\sum_{k=n}^{3n+1} \langle x, e_k \rangle (e_k - Q_n A_n e_k)\right\|_{L_q} \geq \alpha_{n, \delta}\right) \\
& \geq \mu\left(\left\|\sum_{k=n}^{3n+1} \langle x, e_k^{(-r)} \rangle_1 (e_k - Q_n A_n e_k)\right\|_{L_q} \geq c_2 n^r \alpha_{n, \delta}\right) \\
& = \left(\prod_{k=n}^{3n+1} \frac{1}{2\pi\lambda_k}\right)^{1/2} \int_{\mathcal{D}} \exp\left(-\frac{1}{2} \sum_{k=n}^{3n+1} \lambda_k^{-1} y_k^2\right) dy_n \cdots dy_{3n+1}, \quad (30)
\end{aligned}$$

where

$$\mathcal{D} = \left\{ y = (y_n, \dots, y_{3n+1}) \in \mathbb{R}^{2n+1} : \left\|\sum_{k=n}^{3n+1} y_k (e_k - Q_n A_n e_k)\right\|_{L_q} \geq c_2 n^r \alpha_n \right\}.$$

With the help of the substitution  $t_k/\sqrt{\lambda_k} \rightarrow t_k$  and equality (21), we obtain

$$\begin{aligned}
& \left(\prod_{k=n}^{3n+1} 2\pi\lambda_k\right)^{-1/2} \int_{\mathcal{D}} \exp\left(-\frac{1}{2} \sum_n^{3n+1} \lambda_k^{-1} y_k\right) dy_n \cdots dy_{3n+1} \\
& \geq \left(\prod_{k=0}^{2n} 2\pi\right)^{-1/2} \int_{\mathcal{D}_1} \exp\left(-\frac{1}{2} \sum_{k=0}^{2n} y_k^2\right) dy_0 \cdots dy_{2n} = \gamma_{2n+1}(\mathcal{D}_1), \quad (31)
\end{aligned}$$

where

$$\mathcal{D}_1 = \left\{ y \in \mathbb{R}^{2n+1} : \left(\sum_{l=0}^{2n} \left|\sum_{k=0}^{2n} y_k [e_k(\theta_l) - (Q_n A_n e_k)(\theta_l)]\right|^q\right)^{1/q} \geq c_3 n^{r+s/2+1} \right\}$$

and  $\theta_l = 2\pi l/(2n+1)$ . Consider the two matrices

$$E = \left(\frac{e_k(\theta_l)}{\sqrt{2n+1}}\right)_{k, l=0, \dots, 2n} \quad \text{and} \quad H = \left(\frac{Q_n A_n e_k(\theta_l)}{\sqrt{2n+2}}\right)_{k, l=0, \dots, 2n}.$$

Then we can write

$$\begin{aligned}
\mathcal{D}_1 & = \{y \in \mathbb{R}^{2n+1} : \|Ey - Hy\|_q \geq c_3 n^{r+(s-1)/2+1/q} \alpha_{n, \delta}\} \\
& = \{y \in \mathbb{R}^{2n+1} : \|Ey - Hy\|_q \geq c_3 c'_0 (n^{1/q} + \sqrt{\ln \delta^{-1}})\}.
\end{aligned}$$

Since  $E$  is an orthogonal matrix and the finite-dimensional Gaussian measure  $\gamma_{2n+1}$  is invariant with respect to orthogonal transposition in the space  $\mathbb{R}^{2n+1}$ , we have

$$\gamma_{2n+1}(\mathcal{D}_1) = \gamma_{2n+1}(\mathcal{D}_2), \quad (32)$$

where

$$\mathcal{D}_2 = \{y \in \mathbb{R}^{2n+1} : \|y - HE^{-1}y\|_q \geq c_3 c'_0 (n^{1/q} + \sqrt{\ln \delta^{-1}})\}.$$

From the definition of the matrix  $H$  it follows that the rank of  $HE^{-1}$  is at most  $n$ . Therefore from Theorem 4, for some constant  $c'_0$ , we have  $\gamma_{2n+1}(\mathcal{D}_2) > \delta$ . Further from (30)–(32) it follows that

$$\mu(x \in W'_2 : \|x - A_n x\|_{L_q} \geq c_1^{-1} \alpha_{n,\delta}) \geq \gamma_{2n+1}(\mathcal{D}_2) > \delta.$$

Hence we obtain

$$\lambda(W'_2 \setminus G_\delta, A_n, L_q) \geq c_1^{-1} \alpha_{n,\delta}.$$

Letting  $\varepsilon \rightarrow 0$  in inequality (29) we obtain the lower estimate. Theorem 1 is proved.

The proof of Theorem 2 is analogous to the proof of Theorem 1. Indeed, the upper estimate for the linear  $(n, \delta)$ -width follows from the inequality

$$\lambda_{n,\delta}(BW'_2, L_q, \mu') \leq \lambda_{n,\delta}(W'_2, L_q, \mu) \leq \frac{1 + n^{-1/q} \sqrt{\ln \delta^{-1}}}{n^{r + (s-1)/2}}$$

and from the known estimate for linear  $n$ -widths (see [3])

$$\lambda_{n,\delta}(BW'_2, L_q, \mu') \leq \lambda_{n,0}(BW'_2, L_q, \mu') \leq n^{-r + 1/2 - 1/q}.$$

The proof of the lower estimate repeats the proof in Theorem 1. But here, instead of Theorem 4 we use Corollary 1.

The proof of Theorem 3 repeats the proof of Theorem 1. But in the lower estimate, instead of the operator  $A_n$ , we consider the zero operator.

We remark that an announcement about results on Kolmogorov  $(n, \delta)$ -widths of the  $W'_2$  spaces with measure  $\mu$  in the  $L_q$ -norm appears in Maiorov [10, 22]. There also appear analogous results for Wiener spaces.

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