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Parametrizations of toric varieties over any field ☆

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Abstract

The columns of an integral matrix D give rise to the toric variety $V_K(I_D)$ and also provide a parametrization of a subset of $V_K(I_D)$, the so-called toric set $\Gamma_K(D)$. We completely determine the toric set $\Gamma_K(D)$ over any field. We provide conditions under which $V_K(I_D)$ is fully parametrized by the columns of D, that means $\Gamma_K(D) = V_K(I_D)$. In particular, we prove that normal toric varieties over any field are always fully parametrized by the columns of an appropriate matrix.

Keywords: Toric sets; Toric varieties; Integral matrices; Parametrization

1. Introduction

Let $K[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$ over any field K. Given an $m \times n$ matrix $D = (b_{i,j})$ with integer entries and no zero columns, the associated *toric ideal* I_D is the kernel of the K-algebra homomorphism

$$\phi : K[x_1, \ldots, x_n] \to K\left[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}\right]$$

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given by

$$\phi(x_i) = t_1^{b_{1,i}} \cdots t_m^{b_{m,i}} \quad \text{for all } i = 1, \ldots, n.$$

The $t_1, \ldots, t_m$ are called *parameters*. Often we shall use the abbreviation $\mathbf{t}^{\mathbf{b}_i}$ instead of $t_1^{b_{1,i}} \cdots t_m^{b_{m,i}}$, where $\mathbf{b}_i = (b_{1,i}, \ldots, b_{m,i})$ for $1 \leqslant i \leqslant n$. The toric ideal I_D is binomial, i.e. it is generated by all the binomials $x_1^{u_1} \cdots x_n^{u_n} - x_1^{v_1} \cdots x_n^{v_n} \in K[x_1, \ldots, x_n]$, where $u_i \in \mathbb{Z}_{\geqslant 0}$, $v_i \in \mathbb{Z}_{\geqslant 0}$ and the $(u_1 - v_1, \ldots, u_n - v_n)$ runs over all vectors in the nullspace $\ker_{\mathbb{Z}}(D) = \{\mathbf{w} \in \mathbb{Z}^n \mid D\mathbf{w}^t = \mathbf{0}^t\}$ of D, see [17]. The set

$$V_K(I_D) = \left\{(u_1, \ldots, u_n) \in K^n \;\middle|\; F(u_1, \ldots, u_n) = 0,\ \forall F \in I_D\right\}$$

of zeroes of I_D is called *toric variety*, which is not necessarily normal.

The *toric set* $\Gamma_K(D)$ is the subset of $V_K(I_D)$ parametrized by the columns of the matrix D, i.e. it is the set of points of the affine space K^n which can be expressed in the form

$$\left(\mathbf{t}^{\mathbf{b}_1}, \ldots, \mathbf{t}^{\mathbf{b}_i}, \ldots, \mathbf{t}^{\mathbf{b}_n}\right)$$

for some t_i in K. Note that different integral matrices, with the property that their nullspaces coincide, give rise to the same toric ideal and toric variety, but may have different toric sets.

Parametrizations of toric varieties are important either for theoretical reasons, see [1–3,6,7, 11,17,18], or for applications, for example, in Computer Aided Geometric Design, see [4,13]. In most cases a full parametrization of the toric variety is needed, i.e. the toric set to be equal with the toric variety. The fact that the toric set can be a proper subset of the toric variety was noted for the first time by E. Reyes, R. Villarreal and L. Zárate in [16]. Where also they gave conditions under which a toric set is identical with the toric variety. Their approach is based on the notion of the Smith normal form of an integral matrix D, that there are unimodular integral matrices $U = (u_{ij})$ and $Q = (q_{ij})$ of orders m and n, respectively, such that

$$UDQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0),$$

where s is the rank of D and the integers $\lambda_1, \ldots, \lambda_s$ are the invariant factors of D, that is, λ_k divides λ_{k+1} and $\lambda_k > 0$ for all k, see [14]. They proved that $\Gamma_K(D) = V_K(I_D)$ if and only if the following two conditions are satisfied:

(1) If $(y_j) \in V_K(I_D)$ and $y_j \neq 0$ for all j, then $\prod_{j \in \{1,\ldots,n\}} y_j^{q_{ji}}$ has a λ_i-root in K for every $i \in \{1, \ldots, s\}$.
(2) $V_K(I_D, x_j) \subset \Gamma_K(D)$ for every $j \in \{1, \ldots, n\}$.

Later we studied in [12] the problem of when a toric variety is fully parametrized, over an algebraically closed field, using the torus action on the toric variety and expressing the toric set as a certain union of toric orbits. We proved that, over an algebraically closed field, every toric variety is fully parametrized by the columns of an appropriate matrix. In this article we combine both techniques to arrive at Theorem 3.3, which provides a necessary and sufficient condition for the equality of a toric set with the toric variety over any field. After that, using the notion of strongly saturated semigroups, we prove that normal toric varieties are always fully parametrized

by the columns of appropriate matrices over any field. Usually, toric varieties studied in Algebraic Geometry are assumed to be normal, see [8,9,17]. Nevertheless, except from Section 4, all our results hold for any toric variety.

The paper is organized as follows. In Section 2, we consider the action of the algebraic torus $(K^*)^m$ on the K^n, where $K^* = K - \{0\}$. This action decomposes the toric variety into toric orbits. When K is not algebraically closed there may be infinitely many toric orbits. It is proved, see Theorem 2.3, that the toric set $\Gamma_K(D)$ is the union of finitely many toric orbits $\mathcal{O}_D(P_{\mathcal{F}})$, where $\mathcal{F} \in \Omega_D$ and Ω_D is the meet-subsemilattice of the face lattice of the cone $\mathbb{Q}_+D$ generated by the faces $\mathcal{F}_{\mathbf{r}_1}(D), \ldots, \mathcal{F}_{\mathbf{r}_m}(D)$ corresponding to the rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ of D.

In Section 3 we introduce the notion of a face complete matrix to study necessary and sufficient conditions for the equality $V_K(I_D) = \Gamma_K(D)$. It is shown that if the toric set $\Gamma_K(D)$ coincides with $V_K(I_D)$, then D is a face complete matrix. The converse is not true. Also we state and prove the main theorem of this section, Theorem 3.3, which asserts that:

The toric set $\Gamma_K(D)$ coincides with the toric variety $V_K(I_D)$ if and only if

(1) D is face complete,

(2) for every face $\mathcal{F}$ of $\mathbb{Q}_+D$ and a point $(y_j) \in V_K(I_D) \cap (K^*)^{\mathbb{E}_{\mathcal{F}}}$, $\prod_{j \in \mathbb{E}_{\mathcal{F}}} y_j^{q_{ji}^{\mathcal{F}}}$, $1 \leqslant i \leqslant s_{\mathcal{F}}$, has a $\lambda_i^{\mathcal{F}}$-root in K where $s_{\mathcal{F}}$ is the dimension of the face $\mathcal{F}$, $\lambda_i^{\mathcal{F}}$ are the invariant factors of the submatrix $D_{\mathcal{F}}$ of D, $q_{ji}^{\mathcal{F}}$ the elements of a matrix $Q_{\mathcal{F}}$ satisfying $U_{\mathcal{F}} D_{\mathcal{F}} Q_{\mathcal{F}} = \operatorname{diag}(\lambda_1^{\mathcal{F}}, \ldots, \lambda_{s_{\mathcal{F}}}^{\mathcal{F}}, 0, \ldots, 0)$ and $(K^*)^{\mathbb{E}_{\mathcal{F}}}$ is the cell corresponding to the face $\mathcal{F}$. The cell $(K^*)^{\mathbb{E}_{\mathcal{F}}}$ is the subset $\{(q_1, \ldots, q_n) \in K^n \mid q_i \neq 0 \text{ if } \mathbf{b}_i \in \mathcal{F},\ q_i = 0 \text{ if } \mathbf{b}_i \notin \mathcal{F}\}$ of the affine space K^n.

Finally, in Section 4, we prove that every normal toric variety $V_K(I_D)$ is fully parametrized, over any field, by the columns of an appropriate matrix N. This will be done by finding a face complete matrix N such that all invariant factors of the submatrices $N_{\mathcal{F}}$ are all equal to one, for every face $\mathcal{F}$ of $\mathbb{Q}_+D$. Note that the form of the matrix N is independent of the field K.

2. Rational polyhedral cones, the torus action and the toric set

Let D be an $m \times n$ integral matrix with columns $\mathbf{b}_1^t, \ldots, \mathbf{b}_n^t$. We associate to the toric variety $V_K(I_D) \subset K^n$ the rational convex polyhedral cone $\sigma = \mathbb{Q}_+D$ consisting of all non-negative linear rational combinations of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$. The dimension of σ is equal to the dimension of the $\mathbb{Q}$-vector space

$$\operatorname{span}_{\mathbb{Q}}(\sigma) := \{\lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{Q}\} = \mathbb{Q}D,$$

which is equal to $\operatorname{rank}_{\mathbb{Q}}(D)$. A *face* $\mathcal{F}$ of σ is any set of the form

$$\mathcal{F} = \sigma \cap \left\{\mathbf{x} \in \mathbb{Q}^m\colon\ \mathbf{c}\mathbf{x} = 0\right\},$$

where $\mathbf{c} \in \mathbb{Q}^m$ and $\mathbf{c}\mathbf{x} \geqslant 0$ for all $\mathbf{x} \in \sigma$. Faces of dimension $\operatorname{rank}_{\mathbb{Q}}(D) - 1$ are called *facets*.

The *face lattice* of σ is the poset $L(\sigma)$ of all faces of σ, partially ordered by inclusion. A subset S of $L(\sigma)$ is called a *meet-subsemilattice* if (i) $\sigma \in S$ and (ii) for every finite subset $\{\mathcal{F}_1, \ldots, \mathcal{F}_t\}$ of S $\mathcal{F}_1 \cap \cdots \cap \mathcal{F}_t$ belongs to S. Given a meet-subsemilattice S of $L(\sigma)$, we say that a set

$\{\mathcal{F}_1, \ldots, \mathcal{F}_t\} \subset S$ *generates* S if every element in S, except possibly σ, is the intersection of a subset of $\{\mathcal{F}_1, \ldots, \mathcal{F}_t\}$. The set of all facets of σ generates $L(\sigma)$, see [9].

Two cones $\sigma_1 \subset \mathbb{Q}^l$, $\sigma_2 \subset \mathbb{Q}^m$ are *rationally affine equivalent* if there is a rational affine transformation $\pi : \mathbb{Q}^l \to \mathbb{Q}^m$ such that $\pi|_{\mathrm{span}_{\mathbb{Q}}(\sigma_1)}$ is one-to-one and $\pi(\sigma_1) = \sigma_2$. Note that if the cones σ_1 and σ_2 are rationally affine equivalent, then there is a bijection between the face lattices $L(\sigma_1)$ and $L(\sigma_2)$ preserving the ordering.

The affine space K^n is decomposed into 2^n *coordinate cells*

$$\left(K^*\right)^{\mathbb{E}} := \left\{(q_1, \ldots, q_n) \in K^n \;\middle|\; q_i \neq 0 \text{ for } i \in \mathbb{E},\ q_i = 0 \text{ for } i \notin \mathbb{E}\right\},$$

where $\mathbb{E}$ runs over all subsets of $\{1, \ldots, n\}$. We will use the symbol $P_{\mathbb{E}}$ for the point $(\delta_1^{\mathbb{E}}, \ldots, \delta_n^{\mathbb{E}}) \in K^n$, where $\delta_i^{\mathbb{E}} = 1$ if $i \in \mathbb{E}$ and $\delta_i^{\mathbb{E}} = 0$ if $i \notin \mathbb{E}$.

Let S be a subset of the cone σ, then $\mathbb{E}_S := \{i \in \{1, \ldots, n\} \mid \mathbf{b}_i \in S\}$. Note that $\mathbb{E}_\sigma = \{1, \ldots, n\}$. Given a face $\mathcal{F}$ of σ, we shall denote the point $P_{\mathbb{E}_{\mathcal{F}}}$ by $P_{\mathcal{F}}$. Also for a point $\mathbf{y} = (y_1, \ldots, y_n) \in K^n$ and $\mathbf{t} = (t_1, \ldots, t_m) \in K^m$ we denote by $(\mathbf{yt})_D$ the point $(y_1\mathbf{t}^{\mathbf{b}_1}, \ldots, y_n\mathbf{t}^{\mathbf{b}_n}) \in K^n$. The m-dimensional algebraic torus $(K^*)^m$ acts on the affine n-space K^n via

$$(\mathbf{y}, \mathbf{t}) \mapsto (\mathbf{yt})_D.$$

The orbit of a point $\mathbf{y} \in V_K(I_D)$ is denoted by $\mathcal{O}_D(\mathbf{y})$ and called *toric orbit*. The affine toric variety $V_K(I_D)$ is the union of toric orbits.

When the field K is algebraically closed, the toric orbits are in order-preserving bijection with the faces of σ, see [9,10,15]. The orbit corresponding to the face $\mathcal{F}$ is the orbit of the point $P_{\mathcal{F}}$. In this case the torus action does not depend on the matrix D, i.e. different matrices, with the property that their nullspaces coincide, give the same toric orbits, see [8]. When K is not algebraically closed the torus action depends on the matrix D, see Example 2.4, but it is still true that every point of $V_K(I_D)$ belongs to a cell $(K^*)^{\mathbb{E}_{\mathcal{F}}}$, for a face $\mathcal{F}$ of $\mathbb{Q}_+D$, since K is a subset of its algebraic closure $\bar{K}$. It is possible for the same cell $(K^*)^{\mathbb{E}_{\mathcal{F}}}$ to contain more than one orbit.

Example 2.1. There exist toric varieties over a non-algebraically closed field K and a torus action on them providing infinitely many toric orbits.

Consider the 2×3 matrix

$$D = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

and the toric variety $V_{\mathbb{Q}}(I_D) \subset \mathbb{Q}^3$. The toric ideal $I_D \subset \mathbb{Q}[x_1, x_2, x_3]$ is generated by the binomial $x_3^2 - x_1x_2$. The points of the toric variety are in the four cells $(K^*)^{\emptyset}$, $(K^*)^{\{1\}}$, $(K^*)^{\{2\}}$, $(K^*)^{\{1,2,3\}}$. To each square-free non-zero integer a correspond three distinguished orbits $\mathcal{O}_D(a, 0, 0), \mathcal{O}_D(0, a, 0), \mathcal{O}_D(a, a, |a|)$ on $V_{\mathbb{Q}}(I_D)$. Every point of $V_{\mathbb{Q}}(I_D)$, except from $(0, 0, 0)$, belongs to exactly one of those for an appropriate square-free $a \in \mathbb{Z}$.

Definition 2.2. We associate to every row $\mathbf{r}_i = (b_{i,1}, \ldots, b_{i,n})$ of the matrix D a face $\mathcal{F}_{\mathbf{r}_i}(D)$ of σ in the following way:

(i) if $\mathbf{r}_i$ has at least one negative entry we define $\mathcal{F}_{\mathbf{r}_i}(D) = \sigma$,

(ii) if every entry of $\mathbf{r}_i$ is non-negative we define $\mathcal{F}_{\mathbf{r}_i}(D)$ to be the face of σ defined by $\mathbf{c}_i = (0, 0, \ldots, 1, \ldots, 0)$, where the 1 is in the ith-position.

Note that $\mathcal{F}_{\mathbf{r}_i}(D)$ is a face since $\mathbf{c}_i\mathbf{b}_j = b_{i,j} \geqslant 0$. Also if every entry of $\mathbf{r}_i$ is positive, then $\mathcal{F}_{\mathbf{r}_i}(D) = \{\mathbf{0}\}$. The orbit corresponding to the face $\mathcal{F}_{\mathbf{r}_i}(D)$ is the orbit of the point $P_{\mathcal{F}_{\mathbf{r}_i}(D)}$ with coordinates $g_{i,j} = 0$ if $b_{i,j} > 0$ and $g_{i,j} = 1$ if $b_{i,j} = 0$, since $b_{i,j} > 0$ implies that $\mathbf{b}_j \notin \mathcal{F}_{\mathbf{r}_i}(D)$ and $b_{i,j} = 0$ implies that $\mathbf{b}_j \in \mathcal{F}_{\mathbf{r}_i}(D)$.

For projective toric varieties determined by lattice polytopes the parameters t_i, such that the corresponding faces $\mathcal{F}_{\mathbf{r}_i}(D)$ are facets of the cone σ, are called facet variables, see [4,5]. A more algebraic approach to the same idea is what K. Eto calls lattice divisors, see [7].

The following theorem determines the toric set of any integral matrix D.

Theorem 2.3. *Let K be any field and D an $m \times n$ integral matrix with rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$. The toric set is a union of finitely many toric orbits*

$$\Gamma_K(D) = \bigcup_{\mathcal{F} \in \Omega_D} \mathcal{O}_D(P_{\mathcal{F}}),$$

where Ω_D is the meet-subsemilattice of the face lattice $L(\sigma)$ of the cone $\sigma = \mathbb{Q}_+ D$ generated by the faces $\mathcal{F}_{\mathbf{r}_1}(D), \ldots, \mathcal{F}_{\mathbf{r}_m}(D)$.

Proof. Let $\{\mathbf{b}_1^t, \ldots, \mathbf{b}_n^t\}$ be the set of columns of D. First we will prove that $\Gamma_K(D) \subset \bigcup_{\mathcal{F} \in \Omega_D} \mathcal{O}_D(P_{\mathcal{F}})$. Let $\mathbf{y} \in \Gamma_K(D)$, then $\mathbf{y} = (\mathbf{u}^{\mathbf{b}_1}, \ldots, \mathbf{u}^{\mathbf{b}_n})$ for some $\mathbf{u} = (u_1, \ldots, u_m) \in K^m$. Set $S_{\mathbf{u}} = \{i \mid u_i = 0\}$. Since $\mathbf{y} \in V_K(I_D)$, it belongs to a cell $(K^*)^{\mathbb{E}_{\mathcal{F}}}$ for a face $\mathcal{F}$ of σ. Consider the point $P_{\mathcal{F}} = (\delta_1^{\mathbb{E}_{\mathcal{F}}}, \ldots, \delta_n^{\mathbb{E}_{\mathcal{F}}})$ in K^n. Note that if $\mathbf{u}^{\mathbf{b}_i} \neq 0$ then $\delta_i^{\mathbb{E}_{\mathcal{F}}} = 1$, while $\mathbf{u}^{\mathbf{b}_i} = 0$ implies that $\delta_i^{\mathbb{E}_{\mathcal{F}}} = 0$. Let $\mathbf{v} = (v_1, \ldots, v_m) \in (K^*)^m$ with coordinates $v_i = u_i$ if $i \notin S_{\mathbf{u}}$ and $v_i = 1$ if $i \in S_{\mathbf{u}}$. Then $\mathbf{y} = (P_{\mathcal{F}}\mathbf{v})_D$, since if $\mathbf{u}^{\mathbf{b}_j} \neq 0$ then $\mathbf{u}^{\mathbf{b}_j} = \mathbf{v}^{\mathbf{b}_j}$ and therefore $\delta_j^{\mathbb{E}_{\mathcal{F}}} = 1$ while if $\mathbf{u}^{\mathbf{b}_j} = 0$ then $\delta_j^{\mathbb{E}_{\mathcal{F}}} = 0$. Thus $\mathbf{y}$ belongs to the $\mathcal{O}_D(P_{\mathcal{F}})$. It remains to show that $\mathcal{F} \in \Omega_D$.

Note that $\mathbf{b}_j \in \mathcal{F}$ if and only if $\mathbf{u}^{\mathbf{b}_j} \neq 0$ if and only if $b_{j,i} = 0$ for all $i \in S_{\mathbf{u}}$. Thus $\mathcal{F} = \bigcap_{i \in S_{\mathbf{u}}} \mathcal{F}_{\mathbf{r}_i}(D)$, which implies that $\mathcal{F} \in \Omega_D$. Consequently $\mathbf{y} \in \bigcup_{\mathcal{F} \in \Omega_D} \mathcal{O}_D(P_{\mathcal{F}})$.

Conversely consider a point $\mathbf{y} \in \mathcal{O}_D(P_{\mathcal{F}})$ for some $\mathcal{F} \in \Omega_D$. Then $\mathbf{y} = (P_{\mathcal{F}}\mathbf{u})_D$, for a $\mathbf{u} = (u_1, \ldots, u_m) \in K^m$, and also, from the definition of Ω_D, there is a set $S \subset \{1, \ldots, m\}$ such that $\mathcal{F} = \bigcap_{i \in S} \mathcal{F}_{\mathbf{r}_i}(D)$. Setting $t_i = 1$ if $i \notin S$ and $t_i = 0$ if $i \in S$, we have that $P_{\mathcal{F}} = (\mathbf{t}^{\mathbf{b}_1}, \ldots, \mathbf{t}^{\mathbf{b}_n})$ and therefore $\mathbf{y} = ((\mathbf{tu})^{\mathbf{b}_1}, \ldots, (\mathbf{tu})^{\mathbf{b}_n}) \in \Gamma_K(D)$, where $\mathbf{tu} = (t_1u_1, \ldots, t_mu_m) \in K^m$. $\square$

Theorem 2.3 says that the toric set depends on the matrix D in two respects. First, the rows of the matrix D determine what orbits belong to the toric set. In [12] there are plenty of examples of different matrices, that define the same toric variety, and their toric sets are different. Second, the columns of the matrix determine the torus action which affects the orbits $\mathcal{O}_D(P_{\mathcal{F}})$ and consequently the toric set.

Example 2.4. We return to Example 2.1. Except from $\{\mathbf{0}\}$ the cone $\mathbb{Q}_+ D$ has two proper faces, namely $\mathcal{F}_1 = \mathbb{Q}_+(0, 2)$ and $\mathcal{F}_2 = \mathbb{Q}_+(2, 0)$. Note that $\mathcal{F}_1 = \mathcal{F}_{\mathbf{r}_1}(D)$, $\mathcal{F}_2 = \mathcal{F}_{\mathbf{r}_2}(D)$ and $\mathcal{F}_1 \cap \mathcal{F}_2 = \{\mathbf{0}\}$. Therefore the meet-subsemillatice generated by the faces $\mathcal{F}_{\mathbf{r}_1}(D)$, $\mathcal{F}_{\mathbf{r}_2}(D)$ is the whole face lattice $L(\mathbb{Q}_+ D)$. According to Theorem 2.3 the toric set $\Gamma_{\mathbb{Q}}(D)$ is the union of the four orbits $\mathcal{O}_D(0, 0, 0)$, $\mathcal{O}_D(1, 0, 0)$, $\mathcal{O}_D(0, 1, 0)$, $\mathcal{O}_D(1, 1, 1)$.

Consider also the matrix

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The cones $\sigma_1 = \mathbb{Q}_+ M$, $\sigma_2 = \mathbb{Q}_+ D$, are rationally affine equivalent, since the projection $\pi : \mathbb{Q}^3 \to \mathbb{Q}^2$ which is given by $\pi(q_1, q_2, q_3) = (q_1, q_2)$ satisfies:

$$\pi|_{\mathrm{span}_{\mathbb{Q}}(\sigma_1)} \text{ is one-to-one} \quad \text{and} \quad \pi(\sigma_1) = \sigma_2.$$

Also the nullspaces of D, M coincide, so they define the same toric variety, i.e. $V_{\mathbb{Q}}(I_D) = V_{\mathbb{Q}}(I_M)$. For the matrix M we have that $\mathcal{F}_{\mathbf{r}_1}(M) = \mathbb{Q}_+(0, 2, 1)$, $\mathcal{F}_{\mathbf{r}_2}(M) = \mathbb{Q}_+(2, 0, 1)$ and $\mathcal{F}_{\mathbf{r}_3}(M) = \sigma_1$. Therefore, again from Theorem 2.3, the toric set $\Gamma_{\mathbb{Q}}(M)$ consist of the toric orbits $\mathcal{O}_M(0, 0, 0)$, $\mathcal{O}_M(1, 0, 0)$, $\mathcal{O}_M(0, 1, 0)$, $\mathcal{O}_M(1, 1, 1)$. But the orbits $\mathcal{O}_M(P_{\mathcal{F}})$ and $\mathcal{O}_D(P_{\mathcal{F}})$ are generally different, for example,

$$\mathcal{O}_M(1, 1, 1) = \bigcup_{a \in \Lambda} \mathcal{O}_D\big(a, a, |a|\big) \supsetneqq \mathcal{O}_D(1, 1, 1),$$

where Λ is the set of all square free non-zero integers.

3. Toric sets and face complete matrices

In this section we will provide equivalent conditions for a toric variety $V_K(I_D)$ to coincide with the toric set $\Gamma_K(D)$.

Definition 3.1. A face complete matrix is an $m \times n$ integral matrix D with rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ such that: For every facet $\mathcal{F}$ of $\mathbb{Q}_+ D$ there exists $i \in \{1, \ldots, m\}$ with $\mathcal{F}_{\mathbf{r}_i}(D) = \mathcal{F}$.

In [12] it was proved, when K is algebraically closed, that $V_K(I_D) = \Gamma_K(D)$ if and only if the matrix D is face complete. Generally, letting K be any field, we have the next proposition.

Proposition 3.2. *If $V_K(I_D) = \Gamma_K(D)$ for some field K, then the matrix D is face complete.*

Proof. Let $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ be the set of rows of D. If $\mathcal{F}$ is a face of the cone $\mathbb{Q}_+ D$, then the point $P_{\mathcal{F}}$ belongs to $V_K(I_D)$ and therefore, from Theorem 2.3, we conclude that $L(\mathbb{Q}_+ D)$ is a subset of Ω_D. The face lattice $L(\mathbb{Q}_+ D)$ is generated by the facets of $\mathbb{Q}_+ D$, so every facet is an intersection of a subset of $\{\mathcal{F}_{\mathbf{r}_1}(D), \ldots, \mathcal{F}_{\mathbf{r}_m}(D)\}$. But the facets are maximal elements of $L(\mathbb{Q}_+ D)$, therefore for every facet $\mathcal{F}$ of $\mathbb{Q}_+ D$ there exists $i \in \{1, \ldots, m\}$ with $\mathcal{F}_{\mathbf{r}_i}(D) = \mathcal{F}$. $\square$

The converse of Proposition 3.2 is not true in general. In Example 2.1 the matrix D is face complete, but the toric set $\Gamma_{\mathbb{Q}}(D)$ does not coincide with the toric variety $V_{\mathbb{Q}}(I_D)$.

Given an $m \times n$ integral matrix D with columns $\mathbf{b}_1^t, \ldots, \mathbf{b}_n^t$, we associate to every face $\mathcal{F} \neq \{\mathbf{0}\}$ of $\sigma = \mathbb{Q}_+ D$ a submatrix $D_{\mathcal{F}}$ of D with columns all the vectors $\mathbf{b}_i^t$ such that $\mathbf{b}_i \in \mathcal{F}$. The matrix D_σ coincides with D. We say that a has a λ-root in the field K if the equation $x^\lambda = a$ has a root in K. The next theorem generalizes Theorem 2.3 in [16] taking into account all toric orbits and not only the "big" orbit, i.e. $\mathcal{O}_D(P_\sigma)$.

Theorem 3.3. *Let K be a field and $V_K(I_D) \subset K^n$ a toric variety defined by an integral matrix D. Then $V_K(I_D) = \Gamma_K(D)$ if and only if*

(1) *D is face complete,*

(2) *for every face $\mathcal{F}$ of $\mathbb{Q}_+D$ and a point $(y_j) \in V_K(I_D) \cap (K^*)^{\mathbb{E}_\mathcal{F}}$, $\prod_{j\in\mathbb{E}_\mathcal{F}} y_j^{q_{ji}^\mathcal{F}}$, $1 \leqslant i \leqslant s_\mathcal{F}$, has a $\lambda_i^\mathcal{F}$-root in K where $s_\mathcal{F}$ is the dimension of the face $\mathcal{F}$, $\lambda_i^\mathcal{F}$ are the invariant factors of $D_\mathcal{F}$, $q_{ji}^\mathcal{F}$ are the elements of a matrix $Q_\mathcal{F}$ satisfying $U_\mathcal{F} D_\mathcal{F} Q_\mathcal{F} = \operatorname{diag}(\lambda_1^\mathcal{F}, \ldots, \lambda_{s_\mathcal{F}}^\mathcal{F}, 0, \ldots, 0)$ and $(K^*)^{\mathbb{E}_\mathcal{F}}$ is the cell corresponding to the face $\mathcal{F}$.*

Proof. Assume first that conditions (1) and (2) hold. It is enough to prove that $V_K(I_D) \subset \Gamma_K(D)$. Let $\mathbf{y} = (y_1, \ldots, y_n) \in V_K(I_D)$, then $\mathbf{y} \in (K^*)^{\mathbb{E}_\mathcal{F}}$ for a face $\mathcal{F}$ of $\mathbb{Q}_+D$. Consider the point $\mathbf{z} = (y_i)_{i\in\mathbb{E}_\mathcal{F}} \in V_K(I_{D_\mathcal{F}}) \subset K^r$, where r is the cardinality of the set $\mathbb{E}_\mathcal{F}$. Then, taking into account that every $y_i \neq 0$ for all $i \in \mathbb{E}_\mathcal{F}$, we have that $\mathbf{z} \in (K^*)^r$. Recall that for the toric ideal $I_{D_\mathcal{F}}$ it holds $I_{D_\mathcal{F}} = I_D \cap K[x_i \mid \mathbf{b}_i \in \mathcal{F}]$, see Proposition 4.13 of [17]. Applying Theorem 2.3 in [16], since $\mathbf{z} \in V_K(I_{D_\mathcal{F}}) \cap (K^*)^r$ and condition (2) is true, we deduce that $\mathbf{z} \in \Gamma_K(D_\mathcal{F})$ and therefore $y_i = \mathbf{t}^{\mathbf{b}_i}$ for every $i \in \mathbb{E}_\mathcal{F}$ and some $t_1, \ldots, t_m$ in K. Moreover, D is face complete and therefore the point $P_\mathcal{F}$ belongs to $\Gamma_K(D)$, which implies that there are $s_1, \ldots, s_m$ in K such that $P_\mathcal{F} = (\mathbf{s}^{\mathbf{b}_1}, \ldots, \mathbf{s}^{\mathbf{b}_n})$. Then the point $\mathbf{y}$ can be expressed as $((\mathbf{st})^{\mathbf{b}_1}, \ldots, (\mathbf{st})^{\mathbf{b}_n})$ and therefore belongs to $\Gamma_K(D)$, where $\mathbf{st} = (s_1t_1, \ldots, s_mt_m) \in K^m$.

Assume now that $V_K(I_D) = \Gamma_K(D)$. Proposition 3.2 assures that D is face complete. Consider now a point $\mathbf{a} = (a_j) \in V_K(I_D) \cap (K^*)^{\mathbb{E}_\mathcal{F}}$, i.e. $a_j \neq 0$ when $j \in \mathbb{E}_\mathcal{F}$ and $a_j = 0$ when $j \notin \mathbb{E}_\mathcal{F}$. Since $V_K(I_D) = \Gamma_K(D)$, we take that $a_j = \mathbf{t}^{\mathbf{b}_j}$ for all $j \in \{1, \ldots, n\}$. Thus using the equality of matrices $D_\mathcal{F} Q_\mathcal{F} = U_\mathcal{F}^{-1} \operatorname{diag}(\lambda_1^\mathcal{F}, \ldots, \lambda_{s_\mathcal{F}}^\mathcal{F}, 0, \ldots, 0)$ we deduce that

$$\prod_{j\in\mathbb{E}_\mathcal{F}} a_j^{q_{ji}^\mathcal{F}} = \prod_{j\in\mathbb{E}_\mathcal{F}} \mathbf{t}^{q_{ji}^\mathcal{F}\mathbf{b}_j} = \left(\mathbf{t}^{\mathbf{u}_i}\right)^{\lambda_i^\mathcal{F}},$$

where $\mathbf{u}_i$ is the i-row of $U_\mathcal{F}^{-1}$. Consequently the $\prod_{j\in\mathbb{E}_\mathcal{F}} a_j^{q_{ji}^\mathcal{F}}$ has a $\lambda_i^\mathcal{F}$-root in K, for every $i \in \{1, \ldots, s_\mathcal{F}\}$. $\square$

The field $\mathbb{R}$ of real numbers is the field most commonly used in geometric modeling and is where it is especially useful for a toric set to equal its toric variety. In the case $K = \mathbb{R}$ we have the following corollary of Theorem 3.3.

Corollary 3.4. *Let $V_\mathbb{R}(I_D) \subset \mathbb{R}^n$ a toric variety defined by an integral matrix D. Then $V_\mathbb{R}(I_D) = \Gamma_\mathbb{R}(D)$ if and only if*

(1) *D is face complete,*

(2) *for every face $\mathcal{F}$ of $\mathbb{Q}_+D$ and a point $(y_j) \in V_\mathbb{R}(I_D) \cap (\mathbb{R}^*)^{\mathbb{E}_\mathcal{F}}$, $\prod_{j\in\mathbb{E}_\mathcal{F}} y_j^{q_{ji}^\mathcal{F}} \geqslant 0$, for all i with $\lambda_i^\mathcal{F}$ even, $1 \leqslant i \leqslant s_\mathcal{F}$, where $s_\mathcal{F}$ is the dimension of the face $\mathcal{F}$, $\lambda_i^\mathcal{F}$ are the invariant factors of $D_\mathcal{F}$, $q_{ji}^\mathcal{F}$ are the elements of a matrix $Q_\mathcal{F}$ satisfying $U_\mathcal{F} D_\mathcal{F} Q_\mathcal{F} = \operatorname{diag}(\lambda_1^\mathcal{F}, \ldots, \lambda_{s_\mathcal{F}}^\mathcal{F}, 0, \ldots, 0)$ and $(K^*)^{\mathbb{E}_\mathcal{F}}$ is the cell corresponding to the face $\mathcal{F}$.*

Example 3.5. Consider the 3×3 matrix

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

and the toric ideal $I_M = (x_3^6 - x_1^3 x_2) \subset K[x_1, x_2, x_3]$. The cone $\sigma_1 = \mathbb{Q}_+ M$ has two facets, namely $\mathcal{H}_1 = \mathbb{Q}_+(0, 6, 3)$ and $\mathcal{H}_2 = \mathbb{Q}_+(2, 0, 1)$. Also $\mathcal{H}_1 = \mathcal{F}_{\mathbf{r}_1}(M)$ and $\mathcal{H}_2 = \mathcal{F}_{\mathbf{r}_2}(M)$. Therefore M is face complete.

Let $\mathbf{a} = (0, a_2, 0) \in V_K(I_M) \cap (K^*)^{\mathbb{E}_{\mathcal{H}_1}}$. Here

$$U_{\mathcal{H}_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M_{\mathcal{H}_1} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}, \qquad Q_{\mathcal{H}_1} = (1) \quad \text{and} \quad \lambda_1^{\mathcal{H}_1} = 3.$$

So $\mathbf{a} \in \Gamma_K(D)$ if and only if a_2 has a 3-root in K. For every point $\mathbf{a} = (a_1, 0, 0) \in V_K(I_M) \cap (K^*)^{\mathbb{E}_{\mathcal{H}_2}}$ we have that a_1 always has a $\lambda_1^{\mathcal{H}_2}$-root in K, since $\lambda_1^{\mathcal{H}_2} = 1$. Finally, consider a point $\mathbf{a} = (a_1, a_2, a_3) \in V_K(I_M) \cap (K^*)^{\mathbb{E}_{\sigma_1}}$, i.e. $a_i \neq 0$ for every $i \in \{1, 2, 3\}$. In this case

$$U_{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}, \qquad M_{\sigma_1} = M, \qquad Q_{\sigma_1} = \begin{pmatrix} 3 & -4 & 3 \\ 1 & -1 & 1 \\ -5 & 7 & -6 \end{pmatrix} \quad \text{and} \quad \lambda_1^{\sigma_1} = \lambda_2^{\sigma_1} = 1.$$

Obviously $a_1^3 a_2 a_3^{-5}$, $a_1^{-4} a_2^{-1} a_3^7$ always have 1-roots in K. Consequently, from Theorem 3.3, we take that $\Gamma_K(M) = V_K(I_M)$, only in the case that all the elements of K have cube roots, for example, when K is the field of real numbers $\mathbb{R}$ or $\mathbb{F}_3$, while the toric set does not coincide with the toric variety if there are elements in the field K without cube roots, for example, when $K = \mathbb{F}_7$.

Condition (1) in Theorem 3.3 can always be achieved, as we will show in the next proposition.

Proposition 3.6. *Let K be any field. For every $m \times n$ integral matrix D there exists a face complete matrix M such that $V_K(I_M) = V_K(I_D)$ and the cones $\sigma_1 = \mathbb{Q}_+ D$, $\sigma_2 = \mathbb{Q}_+ M$ are rationally affine equivalent.*

Proof. Let $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ be the set of rows and $\{\mathbf{b}_1^t, \ldots, \mathbf{b}_n^t\}$ the set of columns of D. Let $\{\mathcal{F}_1, \ldots, \mathcal{F}_s\}$ be the set of facets of σ_1. If

$$\{\mathcal{F}_1, \ldots, \mathcal{F}_s\} \subseteq \left\{\mathcal{F}_{\mathbf{r}_1}(D), \ldots, \mathcal{F}_{\mathbf{r}_m}(D)\right\},$$

then D is a face complete matrix. Suppose that

$$\{\mathcal{F}_1, \ldots, \mathcal{F}_s\} \nsubseteq \left\{\mathcal{F}_{\mathbf{r}_1}(D), \ldots, \mathcal{F}_{\mathbf{r}_m}(D)\right\}$$

and say $\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_k}$ are the facets not in $\{\mathcal{F}_{\mathbf{r}_1}(D), \ldots, \mathcal{F}_{\mathbf{r}_m}(D)\}$. For every $i \in \{1, \ldots, k\}$ the $\mathcal{F}_{j_i}$ is a facet of σ_1, so

$$\mathcal{F}_{j_i} = \sigma_1 \cap \left\{\mathbf{y} \in \mathbb{Q}^m \;\middle|\; \mathbf{d}_i \mathbf{y} = 0\right\}$$

for some $\mathbf{d}_i = (d_{i,1}, \ldots, d_{i,m}) \in \mathbb{Q}^m$ with

$$\mathbf{d}_i \mathbf{y} \geqslant 0, \quad \forall \mathbf{y} \in \mathbb{Q}_+ D.$$

From the matrix D we create a new $(m+k) \times n$ matrix M, by adding the k new rows $\mathbf{c}(\mathcal{F}_{j_i}) = (g_i \mathbf{d}_i \mathbf{b}_1, g_i \mathbf{d}_i \mathbf{b}_2, \ldots, g_i \mathbf{d}_i \mathbf{b}_n)$ to D, where g_i is a common denominator of $\mathbf{d}_i \mathbf{b}_1, \mathbf{d}_i \mathbf{b}_2, \ldots, \mathbf{d}_i \mathbf{b}_n$ and $1 \leqslant i \leqslant k$. We have that

$$\mathbf{c}(\mathcal{F}_{j_i}) = g_i d_{i,1} \mathbf{r}_1 + \cdots + g_i d_{i,m} \mathbf{r}_m,$$

so $\ker_{\mathbb{Z}}(M) = \ker_{\mathbb{Z}}(D)$ and therefore $V_K(I_M) = V_K(I_D)$. Consider the projection

$$\pi : \mathbb{Q}^{m+k} \to \mathbb{Q}^m,$$

defined by

$$\pi(u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+k}) = (u_1, \ldots, u_m).$$

It is easy to see that π is a rational linear transformation such that $\pi|_{\mathrm{span}_{\mathbb{Q}}(\sigma_2)} = \mathrm{span}_{\mathbb{Q}}(\sigma_1)$. Thus the cones σ_1, σ_2 are rationally affine equivalent and therefore $\mathcal{H}$ is a facet of σ_2 if and only if $\pi(\mathcal{H})$ is a facet of σ_1. Let $(\mathbf{b}'_1)^t, \ldots, (\mathbf{b}'_n)^t$ be the columns of M. Note that, for every $q \in \{1, \ldots, n\}$, $\mathbf{b}_q \in \mathcal{F}_{j_i}$ if and only if $\mathbf{b}'_q \in \mathcal{F}_{\mathbf{c}(\mathcal{F}_{j_i})}(M)$. This implies that $\pi(\mathcal{F}_{\mathbf{c}(\mathcal{F}_{j_i})}(M)) = \mathcal{F}_{j_i}$, for every $i = 1, \ldots, k$. Also $\pi(\mathcal{F}_{\mathbf{r}_q}(M)) = \mathcal{F}_{\mathbf{r}_q}(D)$ for every $q = 1, \ldots, m$. Since $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are all the facets of σ_1, we take that the set

$$\left\{\mathcal{F}_{\mathbf{r}_1}(M), \ldots, \mathcal{F}_{\mathbf{r}_m}(M), \mathcal{F}_{\mathbf{c}(\mathcal{F}_{j_1})}(M), \ldots, \mathcal{F}_{\mathbf{c}(\mathcal{F}_{j_k})}(M)\right\}$$

contains all the facets of σ_2. □

4. Saturated groups and normal toric varieties

Given an $m \times n$ integral matrix D with columns $\mathbf{b}_1^t, \ldots, \mathbf{b}_n^t$, we shall denote by $\mathbb{Z}D$ the subgroup $\{l_1 \mathbf{b}_1 + \cdots + l_n \mathbf{b}_n \colon l_1, \ldots, l_n \in \mathbb{Z}\}$ of $\mathbb{Z}^m$ spanned by $\mathbf{b}_1, \ldots, \mathbf{b}_n$. The *saturation* of $\mathbb{Z}D$, denoted by $\mathrm{Sat}(\mathbb{Z}D)$, is the group $\mathbb{Z}^m \cap \mathbb{Q}D := \{\mathbf{u} \in \mathbb{Z}^m \mid d\mathbf{u} \in \mathbb{Z}D \text{ for some } d \in \mathbb{Z}^* = \mathbb{Z} - \{0\}\}$. If $\lambda_1, \ldots, \lambda_s$ are the invariant factors of D, then there is a $\mathbb{Z}$-base $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ of $\mathrm{Sat}(\mathbb{Z}D)$ such that $\{\lambda_1 \mathbf{v}_1, \ldots, \lambda_s \mathbf{v}_s\}$ is a $\mathbb{Z}$-base of $\mathbb{Z}D$, see [14]. When $\mathbb{Z}D = \mathrm{Sat}(\mathbb{Z}D)$, i.e. the invariant factors of D equal 1, the group $\mathbb{Z}D$ is called *saturated*. First we will give a sufficient condition for the equality $\Gamma_K(D) = V_K(I_D)$, based on the notion of strongly saturated groups.

Definition 4.1. The group $\mathbb{Z}D$ is called strongly saturated if $\mathbb{Z}D_{\mathcal{F}}$ is saturated for every face $\mathcal{F}$ of the cone $\mathbb{Q}_+ D$.

Theorem 4.2. *If D is a face complete matrix and $\mathbb{Z}D$ is strongly saturated, then $\Gamma_K(D) = V_K(I_D)$ over any field K.*

Proof. From the definitions we have that $\mathbb{Z}D_{\mathcal{F}} = \mathrm{Sat}(\mathbb{Z}D_{\mathcal{F}})$, for every face $\mathcal{F}$ of $\mathbb{Q}_+ D$. Thus all the invariant factors $\lambda_i^{\mathcal{F}}$ of $D_{\mathcal{F}}$ equal 1 and therefore, from Theorem 3.3, the equality $\Gamma_K(D) = V_K(I_D)$ holds. □

The following lemma will be useful in the proof of Theorem 4.6.

Lemma 4.3. *For every toric variety $V_K(I_D)$ defined by an integral matrix D, there exists a face complete matrix M such that*

(i) $V_K(I_D) = V_K(I_M)$,
(ii) $\mathbb{Z}M$ *is saturated.*

Proof. From Proposition 3.6 there exists a face complete matrix N such that $V_K(I_D) = V_K(I_N)$ and the cones $\mathbb{Q}_+D$, $\mathbb{Q}_+N$ are rationally affine equivalent. Since $\mathbb{Z}N^t \subset \mathrm{Sat}(\mathbb{Z}N^t)$ and $\mathrm{rank}(\mathbb{Z}N^t) = \mathrm{rank}(\mathrm{Sat}(\mathbb{Z}N^t))$, there is a $\mathbb{Z}$-base $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ of $\mathrm{Sat}(\mathbb{Z}N^t)$ such that $\{\lambda_1\mathbf{u}_1, \dots, \lambda_s\mathbf{u}_s\}$ is a $\mathbb{Z}$-base of $\mathbb{Z}N^t$. From the matrix N we create a new matrix M, by adding the rows $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ to N. Note that both M and N are integral matrices with the same nullspace. Therefore $I_M = I_N = I_D$. Using the fact that $\mathbb{Z}N^t \subset \mathrm{Sat}(\mathbb{Z}N^t)$ we take the equalities

$$\mathbb{Z}M^t = \mathbb{Z}N^t + (\mathbb{Z}\mathbf{u}_1 + \cdots + \mathbb{Z}\mathbf{u}_s) = \mathbb{Z}N^t + \mathrm{Sat}\big(\mathbb{Z}N^t\big) = \mathrm{Sat}\big(\mathbb{Z}N^t\big).$$

Thus $\mathrm{Sat}(\mathbb{Z}M^t) = \mathrm{Sat}(\mathrm{Sat}(\mathbb{Z}N^t))$, while

$$\mathrm{Sat}\big(\mathrm{Sat}\big(\mathbb{Z}N^t\big)\big) = \mathrm{Sat}\big(\mathbb{Z}N^t\big).$$

So $\mathbb{Z}M^t = \mathrm{Sat}(\mathbb{Z}M^t)$. But the matrices M, M^t have the same invariant factors, which implies that $\mathbb{Z}M = \mathrm{Sat}(\mathbb{Z}M)$. $\square$

Example 4.4. According to Lemma 4.3 every toric variety is defined by a matrix M which is face complete and the group $\mathbb{Z}M$ is saturated. But there are examples that the group $\mathbb{Z}M$ is not strongly saturated and $\Gamma_K(M) \neq V_K(I_M)$ over some field K.

Consider the matrix

$$N = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

and the toric ideal $I_N = (x_3^4 - x_1^2x_2) \subset \mathbb{R}[x_1, x_2, x_3]$. The cone $\mathbb{Q}_+N$ has two facets, namely $\mathcal{H}_1 = \mathbb{Q}_+(0, 4, 2)$ and $\mathcal{H}_2 = \mathbb{Q}_+(2, 0, 1)$. Also $\mathcal{H}_1 = \mathcal{F}_{\mathbf{r}_1}(N)$ and $\mathcal{H}_2 = \mathcal{F}_{\mathbf{r}_2}(N)$. Therefore N is face complete. The invariant factors of N are $\lambda_1 = \lambda_2 = 1$. Consequently $\mathbb{Z}N$ is saturated. In addition $\mathbb{Z}N$ is not strongly saturated, since $\mathbb{Z}N_{\mathcal{H}_1} \neq \mathrm{Sat}(\mathbb{Z}N_{\mathcal{H}_1})$. But

$$\Gamma_{\mathbb{R}}(N) \neq V_{\mathbb{R}}(I_N),$$

since $(0, -1, 0) \in V_{\mathbb{R}}(I_N)$ and $(0, -1, 0) \notin \Gamma_{\mathbb{R}}(N)$. In fact there is no integral matrix M such that $V_{\mathbb{R}}(I_M) = V_{\mathbb{R}}(I_D)$ and $\Gamma_{\mathbb{R}}(M) = V_{\mathbb{R}}(I_D)$. Suppose that there is a matrix M with the above properties and let $\mathbf{b}_1^t$, $\mathbf{b}_2^t$, $\mathbf{b}_3^t$ be the three columns of M. We have that $4\mathbf{b}_3 = 2\mathbf{b}_1 + \mathbf{b}_2$, so every coordinate of $\mathbf{b}_2$ is even, which implies that the point $(0, -1, 0)$ of $V_{\mathbb{R}}(I_D)$ is never in the toric set. Thus there are examples of toric varieties that can never be fully parametrized.

We shall denote by $\mathbb{N}D$ the semigroup $\mathbb{N}\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. The semigroup $\mathbb{N}D$ is called *normal* when $\mathbb{N}D = \mathbb{Z}D \cap \mathbb{Q}_+D$.

Proposition 4.5. [17] *The toric variety $V_K(I_D)$ is normal if and only if $\mathbb{N}D$ is normal.*

Theorem 4.6. *Let K be any field and D an integral matrix. If the variety $V_K(I_D)$ is normal, then there is a matrix M such that $V_K(I_M) = V_K(I_D)$ and also $\Gamma_K(M) = V_K(I_D)$.*

Proof. From Lemma 4.3 there is an $m \times n$ face complete matrix M such that $V_K(I_D) = V_K(I_M)$ and $\mathbb{Z}M = \mathrm{Sat}(\mathbb{Z}M)$. Since $V_K(I_D)$ is normal and $V_K(I_D) = V_K(I_M)$, we have, from Proposition 4.5, that $\mathbb{N}M = \mathbb{Z}M \cap \mathbb{Q}_+M$. Thus $\mathbb{N}M = \mathbb{Z}^m \cap \mathbb{Q}M \cap \mathbb{Q}_+M$ or $\mathbb{N}M = \mathbb{Z}^m \cap \mathbb{Q}_+M$ because $\mathbb{Q}_+M \subset \mathbb{Q}M$. Let $\mathcal{F}$ be a face of $\mathbb{Q}_+M$. We will prove that

$$\mathbb{N}M_{\mathcal{F}} = \mathbb{Z}^m \cap \mathbb{Q}_+M_{\mathcal{F}}.$$

Let $\mathbf{y} \in \mathbb{Z}^m \cap \mathbb{Q}_+M_{\mathcal{F}}$. Then $\mathbf{y}$ belongs to $\mathbb{Z}^m \cap \mathbb{Q}_+M = \mathbb{N}M$. This means that there is a relation $\mathbf{y} = u_1\mathbf{b}_1 + \cdots + u_n\mathbf{b}_n$ with $u_1, \ldots, u_n$ natural numbers, where $\mathbf{b}_1^t, \ldots, \mathbf{b}_n^t$ are the columns of M. Multiplying the last relation with a vector $\mathbf{c}_{\mathcal{F}}$ defining $\mathcal{F}$ and using the fact that $\mathbf{c}_{\mathcal{F}}\mathbf{y} = 0$, since $\mathbf{y} \in \mathbb{Q}_+M_{\mathcal{F}}$, we take that $u_i = 0$ for every i such that $\mathbf{b}_i \notin \mathcal{F}$, therefore $\mathbf{y}$ belongs to $\mathbb{N}M_{\mathcal{F}}$. So $\mathbb{N}M_{\mathcal{F}} = \mathbb{Z}^m \cap \mathbb{Q}_+M_{\mathcal{F}}$.

We claim that $\mathbb{Z}M_{\mathcal{F}} = \mathrm{Sat}(\mathbb{Z}M_{\mathcal{F}})$. From the definition we have that

$$\mathbb{Z}M_{\mathcal{F}} \subset \mathrm{Sat}(\mathbb{Z}M_{\mathcal{F}}).$$

Let $\mathbf{x} \in \mathrm{Sat}(\mathbb{Z}M_{\mathcal{F}}) = \mathbb{Z}^m \cap \mathbb{Q}M_{\mathcal{F}}$. Since $\mathbf{x} \in \mathbb{Q}M_{\mathcal{F}}$ there exist a vector $\mathbf{a}$ in $\mathbb{N}M_{\mathcal{F}}$ such that $\mathbf{x} + \mathbf{a} \in \mathbb{Q}_+M_{\mathcal{F}}$. Thus $\mathbf{x} + \mathbf{a} \in \mathbb{Z}^m \cap \mathbb{Q}_+M_{\mathcal{F}} = \mathbb{N}M_{\mathcal{F}}$ and therefore $\mathbf{x} \in \mathbb{Z}M_{\mathcal{F}}$. The preceding discussion yields that $\mathbb{Z}M$ is strongly saturated, so, from Theorem 4.2, the toric variety $V_K(I_D)$ coincides with $\Gamma_K(M)$. $\square$

If d is a positive integer and $U_{m,d}$ is the transpose of the matrix with rows the vectors of

$$A_{m,d} = \{(a_1, \ldots, a_m) \in \mathbb{N}^m \mid a_1 + \cdots + a_m = d\},$$

then $\Gamma_K(U_{m,d})$ is the (m,d)-*Veronese toric set* and $V_K(I_{U_{m,d}})$ is the (m,d)-*Veronese toric variety*. E. Reyes, R. Villarreal, L. Zárate proved that the Veronese toric varieties, over an algebraically closed field, are fully parametrized by the columns of $U_{m,d}$, see [16]. Also Veronese toric varieties are normal and therefore, using Theorem 4.6, we can generalize the previous result over any field.

Corollary 4.7. *A Veronese toric variety is fully parametrized by the columns of an appropriate integral matrix over any field.*

Note that the matrix which fully parametrizes the (m,d)-Veronese toric variety is the transpose of the matrix with rows the vectors of

$$\{(a_1, \ldots, a_m, 1) \in \mathbb{N}^{m+1} \mid a_1 + \cdots + a_m = d\}.$$

Example 4.8. There exist also non-normal toric varieties that satisfy the hypotheses of Theorem 4.2. Consider the matrix

$$D = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{pmatrix}.$$

The matrix D is not face complete, see [12], and not saturated, since the invariant factors of the matrix D are 1, 1, 3. But the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

defines the same toric variety $V_K(I_D)$ and it is face complete, see [12]. For every one of the 13 faces of the cone $\mathbb{Q}_+M$ different from $\{\mathbf{0}\}$ the corresponding matrices $M_{\mathcal{F}}$ have all invariant factors 1, therefore $\mathbb{Z}M$ is strongly saturated. Theorem 4.2 imply that the toric set is the same with the toric variety. If $\mathbf{b}_i^t$ denotes the ith column of the matrix M we have that

$$(1, 1, 1, 1, 1, 1, 1) = \mathbf{b}_1 + \mathbf{b}_5 - \mathbf{b}_6 = \frac{1}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_4 \in \mathbb{Z}M \cap \mathbb{Q}_+M,$$

but it does not belong to $\mathbb{N}M$. Therefore the toric variety $V_K(I_M)$ is not normal.

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