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A Nonlinear Boundary Value Problem*

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1. INTRODUCTION

This paper is concerned with the existence and uniqueness of solutions of the boundary value problem

$$y''(t) + f(t, y(t), y'(t)) = 0 \tag{1.1}$$

$$a_0 y(a) + a_1 y'(a) = c_1 \tag{1.2}$$

$$b_0 y(b) + b_1 y'(b) = c_2, \qquad b > a \tag{1.3}$$

where $f(t, y, y')$ will be taken to satisfy

$$L_1(y_2' - y_1') \leqslant f(t, y, y_2') - f(t, y, y_1') \leqslant L_2(y_2' - y_1'),\ y_2' \geqslant y_1' \tag{1.4}$$

$$f(t, y_2, y') - f(t, y_1, y') \leqslant K(y_2 - y_1),\ y_2 \geqslant y_1 \tag{1.5}$$

The problem treated here is to find the best possible interval $[a, b]$ on which there exists a unique solution of the boundary value problem (1.1), (1.2), (1.3). Such questions for nonlinear differential equations have a long history. Recent contributions are [*1*]–[*4*] and the bibliographies of these papers trace the history of such problems. The main result proved here, Theorem 3.1, includes the existence and uniqueness result given in [*1*], [*2*], and [*4*]. These results correspond to the cases $a_1 = 0$ and either $b_0 = 0$ or $b_1 = 0$ and $a_0 = b_1 = 0$. In [*5*] a monotonicity condition was assumed on $f(t, y, y')$ and existence and uniqueness of solutions of (1.1), (1.2) and (1.3) established over an arbitrary interval. This monotonicity condition corresponds to $K = 0$ in (1.5) and in this case the interval given here is also arbitrary. Theorem 3.1 fails to include the existence and uniqueness result of [*5*] and of Theorem 1 of [*3*] by the assumption of existence and uniqueness, over $[a, b]$, of solutions of the initial value problem for equation (1.1). (The one-sided Lipschitz condition yields uniqueness to the left.)

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Continuability over [a, b] and uniqueness is needed here to define a certain mapping in the proof of Theorem 3.1.

Many of the arguments given here take place in the $y - y'$ plane. In this setting (1.2) and (1.3) are merely straight lines; (1.2) represents an initial line and (1.3) a terminal line. For this reason it is convenient to write these conditions as

$$\ell_1(a) = c_1 \tag{1.2}$$

$$\ell_2(b) = c_2 . \tag{1.3}$$

When, in fact, it is the point set that is of interest, the a and b will be omitted and we will write $\ell_1 = c_1$, etc.

2. Uniqueness

In this section the uniqueness interval will be defined in terms of the constant L_1, L_2, and K. Uniqueness of solutions of (1.1), (1.2) and (1.3) on this interval follows readily from the conditions (1.4) and (1.5) and some comparison lemmas.

Consider

$$y'' + Ly' + Ky = 0 \tag{2.1}$$

$$a_0 y(0) + a_1 y'(0) = 0 \qquad \text{or} \qquad \ell = 0. \tag{2.2}$$

Define $\alpha(L, K, \ell)$ and $-\beta(L, K, \ell)$ as the "time" (t-value) of the next and of the preceding zero of $y'(t)$ for a solution $y(t)$ of (2.1) and (2.2) if such exist, and $+\infty$ and $-\infty$, respectively, otherwise. If $\ell = 0$ is the y axis, both $\alpha(L, K, \ell)$ and $\beta(L, K, \ell)$ are taken to be zero. In the $y - y'$ plane this is just time to traverse the angle between the line $\ell = 0$ and the y axis. Since the equation is linear $\alpha(L, K, \ell)$ and $\beta(L, K, \ell)$ are independent of the initial position on $\ell = 0$ and since the equation has constant coefficients these quantities can be computed explicitly and are independent of the starting time $t = 0$. Since the actual values are never used the computation will be omitted.

Lemma 2.1. *If* $b - a < \alpha(L_2, K, \ell_1) + \beta(L_1, K, \ell_2)$, $a_0 \geqslant 0$, $a_1 \leqslant 0$, $b_i \geqslant 0$, $i = 1, 2$, $| a_0 | + | a_1 | \neq 0$, $| b_0 | + | b_1 | \neq 0$, *then there is at most one solution of the boundary value problem* (1.1), (1.2), (1.3).

Proof. Suppose that $y_1(t)$ and $y_2(t)$ are two different solutions of (1.1), (1.2), (1.3), $y_1(a) \geqslant y_2(a)$. Then, if $z(t) = y_1(t) - y_2(t)$, as long as $z(t) \geqslant 0$, $z(t)$ satisfies

$$z'' + \begin{Bmatrix} L_2 z', & z' \geqslant 0 \\ L_1 z', & z' < 0 \end{Bmatrix} + Kz \geqslant 0, \tag{2.3}$$

$z(t)$ also satisfies the boundary conditions

$$a_0 z(a) + a_1 z'(a) = 0$$
$$b_0 z(b) + b_1 z'(b) = 0$$

A comparison will be made with a solution of

$$x'' + \begin{Bmatrix} L_2 x', & x' \geqslant 0 \\ L_1 x', & x' < 0 \end{Bmatrix} + Kx = 0. \tag{2.4}$$

Converting to polar coordinate, $x = r\cos\theta$, $x' = r\sin\theta$, $z = \rho\cos\omega$, $z' = \rho\sin\omega$, one obtains, for the polar angles of solutions,

$$\theta' = -\sin^2\theta - \begin{Bmatrix} L_2 \sin\theta\cos\theta, & \sin\theta \geqslant 0 \\ L_1 \sin\theta\cos\theta, & \sin\theta < 0 \end{Bmatrix} - K\cos^2\theta$$

$$\omega' \geqslant -\sin^2\omega - \begin{Bmatrix} L_2 \sin\omega\cos\omega, & \sin\omega \geqslant 0 \\ L_1 \sin\omega\cos\omega, & \sin\omega < 0 \end{Bmatrix} - K\cos^2\omega.$$

LEMMA 2.2. *If* $\theta(a) = \omega(a)$, $\theta(t) \leqslant \omega(t)$, $a \leqslant t \leqslant b$.

Proof. This is Lemma 2 of [*1*], except that here we make use of the usual definition of polar angle rather than the angle defined in [*1*].

From Lemma 2.1 it follows that $0 \geqslant \omega(b) \geqslant \theta(b)$. This contradicts the definition of $\alpha(L_2, K, \ell_1) + \beta(L_1, K, \ell_2)$, i.e., $b - a$ is small enough so that no solution of (2.4) starting on $\ell_1 = 0$ at $t = a$ can transverse the angle necessary to reach $\ell_2 = 0$ at $t = b$. Hence uniqueness is established.

A further lemma, a special case of Lemma 3 and remark of [*1*], will be used in what follows and we state it here for reference.

LEMMA 2.3. *If* $x(t)$ *and* $z(t)$ *are nontrivial solutions of* (2.3) *and* (2.4) *with* $x(a) = z(a)$ *and* $x'(a) = z'(a)$, *then,* $z(t) \geqslant x(t)$ *until* $x(t)$ *has a zero to the right of* a. *The inequalities may be reversed throughout.*

Remark. Note that in case $x(t) < 0$, $t > a$, $|z(t)| \leqslant |x(t)|$ until $x(t)$ has a zero. This turns out to be important in one case in the proof of the main theorem.

3. MAIN THEOREM

The principal result of this paper is the following theorem.

THEOREM 3.1. *Let* $f(t, x, y)$ *be continuous, satisfy* (1.4) *and* (1.5), *and suppose solutions of the initial value problem for* (1.1) *at* $t = a$ *exist on* $[a, b]$

and are unique. If $a_0 \geqslant 0$, $a_1 \leqslant 0$, $|a_0| + |a_1| \neq 0$, $b_0 \geqslant 0$, $b_1 \geqslant 0$, $|b_0| + |b_1| \neq 0$ *and if* $b - a < \alpha(L_2, K, \ell_1) + \beta(L_1, K, \ell_2)$ *then there exists a unique solution of the boundary value problem*

$$y'' + f(t, y, y') = 0 \tag{1.1}$$

$$a_0 x(a) + a_1 x'(a) = c_1 \tag{1.2}$$

$$b_0 x(b) + b_1 x'(b) = c_2, \qquad b > a \tag{1.3}$$

for any real c_1, c_2.

Remark. $|a_0| + |b_0| = 0$ cannot occur for then (1.2) and (1.3) are lines parallel to the y axis and $\alpha(L_1, K, \ell_1) = 0$ and $\beta(L_2, K, \ell_2) = 0$ making $b > a$ impossible.

The proof of the main theorem will depend strongly on a comparison with solutions of the piecewise linear equation

$$x'' + \begin{Bmatrix} L_2 x', & x' \geqslant 0 \\ L_1 x', & x' < 0 \end{Bmatrix} + Kx = 0. \tag{3.1}$$

$$(x(a), x'(a)) \in \ell_1 = 0. \tag{3.2}$$

We first establish a lemma.

LEMMA 3.1. *Let* $M > 0$ *be arbitrary. There is a solution of* (3.1) *and* (3.2) *such that the inequality*

$$b_0 x(b) + b_1 x'(b) > M$$

holds. Similarly, there is a solution such that

$$b_0 x(b) + b_1 x'(b) < -M.$$

Proof. Let $x_0(t)$ be a nontrivial solution of

$$x'' + L_2 x' + Kx = 0$$

with initial conditions on $\ell_1(a) = 0$. Let c be the first point after a such that $x_0'(c) = 0$. (If there is no such c, the inequalities are trivially established.) Let $x_1(t)$ be the solution of

$$x'' + L_1 x' + Kx = 0$$

$$x(c) = x_0(c),\ x'(c) = 0.$$

Then

$$x(t) = \begin{cases} x_0(t), & a \leqslant t \leqslant c \\ x_1(t), & c \leqslant t \leqslant b \end{cases} \tag{3.3}$$

is a solution of (3.1) and $\bar{x}(t) = Ax(t)$ is also a solution of (3.1), (3.2) and

$$b_0\bar{x}(b) + b_1\bar{x}'(b) = A(b_0x(b) + b_1x'(b)) = AH.$$

Since $b - a < \alpha(L_2, K, \ell_1) + \beta(L_1, K, \ell_2)$, $x(t)$ may not cross $\ell_2 = 0$ at $t \leqslant b$. Thus $H = b_0x_1(b) + b_1x_1'(b) = \ell_2(b) \neq 0$ and the lemma is established by an appropriate choice of A.

Proof of Theorem 3.1. Let $y_1(t)$ be an arbitrary solution of (1.1), (1.2). Suppose that $b_0y_1(b) + b_1y_1'(b) = d < c_2$ (a similar proof follows if $d > c_2$). Let $x(t)$ be a solution of (3.1), (3.2) such that $b_0x(b) + b_1x'(b) > c_2 - d$. The existence of such a solution is guaranteed by Lemma 3.1.

Let $y_2(t)$ be a solution of (1.1) with initial conditions on $\ell_1 = c_1$ with $y_2(a) = x(a) + y_1(a)$. $x(a) > 0$ since $c_2 - d > 0$. (A similar argument can be constructed if $x(a) < 0$. Note the remark after Lemma 2.3.) Let $z(t) = y_2(t) - y_1(t)$. Then $z(a) = x(a) > 0$, $(z(a), z'(a)) \in \ell_1 = 0$, and $z(t)$ satisfies

$$z'' + \begin{Bmatrix} L_2z', & z' \geqslant 0 \\ L_1z', & z' < 0 \end{Bmatrix} + Kz \geqslant 0$$

as long as $z(t) \geqslant 0$, (which is at least $[a, b]$). Let $\theta(t)$ be the polar angle for $x(t)$ and $\omega(t)$ the polar angle for $z(t)$. From Lemmas 2.1 and 2.3 it follows that $\omega(b) \geqslant \theta(b)$ and $z(b) \geqslant x(b)$.

It will be shown next that $b_0z(b) + b_1z'(b) > c_2 - d$. The reader may find Figure 1 helpful. It is convenient to use terms "above" and "below", $\ell_2 = c$ to describe certain regions; in case $\ell_2 = c$ is parallel to the y' axis, substitute right and left, respectively. Note also that the line $\ell_2 = c_3$ is above $\ell_2 = c_4$ if $c_3 > c_4$ since $\ell_2 = c$ has nonpositive slope.

The point $(x(b), x'(b))$ is at the intersection of the ray $\theta = \theta(b)$ and a line $\ell_2 = c_3$, $c_3 > c_2 - d$ by the choice of $x(t)$. The point $(z(b), z'(b))$ is to the right of the line $z = x(b)$ and has polar angle $\omega(b) \geqslant \theta(b)$. Hence $(z(b), z'(b))$ is above the line $\ell_2 = c_3$, i.e.,

$$b_0z(b) + b_1z'(b) \geqslant c_3 .$$

From this it follows that

$$b_0y_2(b) + b_1y_2'(b) \geqslant b_0y_1(b) + b_1y_1'(b) + c_2 - d$$
$$> d + c_2 - d = c_2 .$$

Thus we have a solution $y_2(t)$ which is above the line $\ell_2 = c_2$ at $t = b$ and a solution $y_1(t)$ which is below $\ell_2 = c_2$ at $t = b$. The desired result will follow from continuity with respect to initial conditions.

Let T be the mapping of the $y - y'$ plane which takes a point (y_0, y_0') into $(y(b), y'(b))$ where $y(a) = y_0$, $y'(a) = y_0'$, and $y(t)$ is a solution of (1.1).

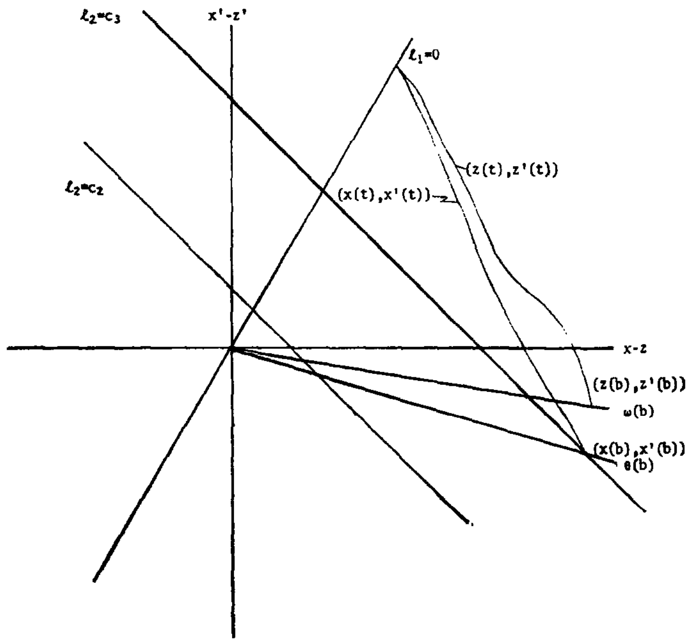

FIG. 1

This mapping is continuous ([6], p. 23 and p. 59) and, in particular, takes $\ell_1 = c_1$ into a connected set. Since a point of $\ell_1 = c_1$ goes into a point above $\ell_2 = c_2$ and a point below $\ell_2 = c_2$, there is a point on $\ell_1 = c_1$, i.e., there is a set of initial conditions satisfying (1.2), whose image is on $\ell_2 = c_2$, i.e., satisfies (1.3). Thus existence of a solution of (1.1), (1.2), (1.3) is established.

Uniqueness was established in Lemma 2.1. The result is best possible in the sense that if $b - a = \alpha(L_2, K, \ell_1) + \beta(L_1, K, \ell_2)$ then there are infinitely many solutions of (3.1) satisfying $\ell_1 = 0$, $l_2 = 0$.

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