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Family unification with $SO(10)$

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Abstract

Unification based on the group $SO(10)^3 \times S_3$ is studied. Each family has its own $SO(10)$ group, and the S_3 permutes the three families and $SO(10)$ factors. This is the maximal local symmetry for the known fermions. Family unification is achieved in the sense that all known fermions are in a single irreducible multiplet of the symmetry. The symmetry suppresses SUSY flavor changing effects by making all squarks and sleptons degenerate in the symmetry limit. Doublet–triplet splitting can arise simply, and non-trivial structure of the quark and lepton masses emerges from the gauge symmetry, including the "doubly lopsided" form.

In this Letter we propose the idea of family unification based on the group $SO(10) \times SO(10) \times SO(10) \times S_3$, where each family of quarks and leptons transforms as a spinor under its own $SO(10)$, and where the S_3 permutes the three families and the three $SO(10)$ factors.

This structure has several interesting features. First, it may be the only way to achieve family unification [1] satisfactorily in four space–time dimensions. (Attempts to unify families in complex spinors of $SO(4n+2)$ groups have not resulted in realistic models, since these spinors contain families and mirror families when decomposed to the Standard Model symmetry [1]. Family unification can occur in higher dimensions, as in heterotic string theory [2]. For family unification in five, six and other dimensions, see [3].) Here we define family unification to mean that all three families of quarks and leptons including their right-handed neutrinos are contained in a single irreducible representation of the unification group, and that there is a single gauge coupling constant at high scales. (This definition is broader than that often used, in which the three families form an irreducible representation of a *simple* group.) It should be noted that in the scheme we propose here the families are in a reducible representation of $SO(10)^3$, namely $\{(16,1,1) + (1,16,1) + (1,1,16)\}$, but these form an *irreducible* multiplet of the full unification group that includes the S_3 factor.

Second, the group $SO(10) \times SO(10) \times SO(10) \times S_3$ is the largest that can be gauged with 48 fermions forming a complex but anomaly-free set of representations. In that sense, it is the "maximal local symmetry" of the known quarks and leptons, including the right-handed neutrinos. (In [4], the definition of maximal local symmetry also included the condition that the group be simple. By that more restrictive definition, the maximal local symmetry of 48 fermions would be just $SO(10)$.) It is easy to see that the S_3 factor is anomaly free. S_3 makes the gauge couplings of all $SO(10)$ groups equal. As a result, the instanton effects of the three $SO(10)$ groups will also be S_3-invariant, proving that it is anomaly free. (The cyclic permutation group Z_3, which is often considered in such contexts is a subgroup of this S_3. Incidently, in the model we present it might appear that it is possible to gauge additional $U(1)$ factors, where under these $U(1)$'s the fermions are rotated into themselves by a phase factor and the gauge bosons are invariant. However, the only anomaly free part of these $U(1)$'s are the Z_4 centers of the $SO(10)$ groups. It is interesting that the S_3 does not commute with these Z_4, and other $SO(10)$, transformations.)

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Third, the unification of all the known quarks and leptons in a single irreducible multiplet of a local group suppresses SUSY flavor changing processes by making all squark and slepton masses exactly degenerate at the unification scale.

Fourth, the problem of "doublet–triplet splitting" can be solved more easily with this group than in ordinary $SO(10)$ unification by means of the Dimopoulos–Wilczek mechanism [5] also known as the "missing VEV mechanism". That the Dimopoulos–Wiczek mechanism is straightforward to implement in product groups like $SU(5) \times SU(5)$ and $SO(10) \times SO(10)$ was pointed out in [6–8]. The stability of the VEV structure however is non-trivial to achieve in $SO(10) \times SO(10)$ models, as that requires rank reduction of the diagonal $SO(10)$ [9]. As will be shown, the present framework resolves this issue very neatly.

And, finally, the "vertical" group $SO(10)^3$ is also in a sense a "family group", since the three families transform differently under any one of the $SO(10)$ groups. As will be seen later, highly non-trivial patterns emerge in the mass matrices of the quarks and leptons due primarily to the symmetry $SO(10)^3$, though S_3 also plays a role. Some of the patterns that emerge almost automatically in this framework have already been proposed in the literature on purely phenomenological grounds, such as the "doubly lopsided" structure [10].

We will describe a supersymmetric $SO(10)^3 \times S_3$ model that illustrates some of the possibilities of the idea. The quarks and leptons are in the representation $\{(16,1,1) + (1,16,1) + (1,1,16)\}$, which we shall denote $(16,1,1) + cyclic$ for short. The Higgs doublets of the Standard Model are contained in the "fundamental" Higgs multiplet $(10,1,1) + cyclic$. Two kinds of Higgs multiplets are needed to do breaking of $SU(3)^3 \times S_3$ to the Standard Model and give superheavy mass to the right-handed neutrinos. We take these to be the "bifundamental" Higgs multiplet $(10,10,1) + cyclic$ and the "bispinor" Higgs multiplet $(16,16,1) + cyclic$ (plus the conjugate bispinor multiplet $(\overline{16},\overline{16},1)$).

The Standard Model group is contained within the "diagonal $SO(10)$" of the three factor $SO(10)$ groups. Under this diagonal $SO(10)$ subgroup, the bifundamentals contain the representations $\mathbf{1}+\mathbf{45}+\mathbf{54}$, while the bispinors contain $\mathbf{10}+\mathbf{126}+\mathbf{120}$. It is well known that a Higgs field in the bifundamental representation of a group $G \times G$ can break it to the diagonal subgroup G. Similarly, a set of bifundamentals can break $G \times G \times G$ to the diagonal subgroup of the three factor groups. In our model, there is a minimum of the scalar potential where the vacuum expectation values (VEVs) of the bifundamentals break $SO(10)^3$ all the way down to a diagonal $SU(3)_c \times SU(2)_L \times U(1) \times U(1)$, as will be seen. The bispinors, whose VEVs give mass to the right-handed neutrinos, break the extra $U(1)$ to give the Standard Model group.

The quark and lepton multiplets will be denoted ψ_a, $a = 1,2,3$, where $\psi_1 \equiv (16,1,1)$, $\psi_2 \equiv (1,16,1)$, and $\psi_3 \equiv (1,1,16)$. The fundamental Higgs fields will be denoted H_a, where $H_1 \equiv (10,1,1)$, etc. The bifundamental Higgs fields will be denoted Ω_{ab}, where $\Omega_{12} \equiv (10,10,1)$, etc. A second set of bifundamentals will also be needed, and will be denoted Ω'_{ab}. And the bispinors will be denoted Δ_{ab} and $\bar{\Delta}_{ab}$, where $\Delta_{12} \equiv (16,16,1)$ and $\bar{\Delta}_{12} \equiv (\overline{16},\overline{16},1)$, etc. The indices a and b are not $SO(10)$ indices (which we suppress) but merely labels that indicate which $SO(10)$ groups the multiplets transform non-trivially under. These labels are permuted under the S_3 group.

Only two renormalizable terms are allowed by the gauge symmetry in the Yukawa superpotential of the quarks and leptons, namely

$$W_{\rm Yuk} = Y(\psi_1\psi_1 H_1 + \psi_2\psi_2 H_2 + \psi_3\psi_3 H_3) + Y'(\psi_1\psi_2\bar{\Delta}_{12} + \psi_2\psi_3\bar{\Delta}_{23} + \psi_3\psi_1\bar{\Delta}_{31}). \tag{1}$$

The Higgs superpotential can be written in an obvious notation as $W_{\rm Higgs} = W_H + W_\Omega + W_\Delta + W_{H\Omega} + W_{\Omega\Delta} + W_{H\Delta}$. Let us focus first on W_Ω. If there is only one set of bifundamentals Ω_{ab}, then the most general renormalizable form of W_Ω consistent with symmetry is

$$W_\Omega = \frac{1}{2}M(\mathrm{tr}\,\Omega_{12}\Omega_{21} + \mathrm{tr}\,\Omega_{23}\Omega_{32} + \mathrm{tr}\,\Omega_{31}\Omega_{13}) + \lambda(\mathrm{tr}\,\Omega_{12}\Omega_{23}\Omega_{31}). \tag{2}$$

The order of labels ab on Ω_{ab} is significant. If $(\Omega_{ab})^{ij}$ is the ij element of the 10×10 matrix Ω_{ab}, then the row index i belongs to the group $SO(10)_a$ and the column index j belongs to the group $SO(10)_b$. This superpotential gives rise to the equations of motion $\Omega_{21} = (\lambda/M)\Omega_{23}\Omega_{31}$, $\Omega_{32} = (\lambda/M)\Omega_{31}\Omega_{12}$, and $\Omega_{13} = (\lambda/M)\Omega_{12}\Omega_{23}$, which, of course, are S_3 permuted versions of each other. Doublet–triplet splitting by means of the Dimopoulos–Wilczek mechanism would require that the VEV of at least one of these bifundamentals had the form $\langle\Omega\rangle = \mathrm{diag}(a\ a\ a\ 0\ 0) \otimes i\tau_2$ (corresponding to the generator B–L of the diagonal $SO(10)$ and the generator $\mathrm{diag}(a\ a\ a\ 0\ 0)$ of the diagonal $U(5)$). However, it is easily seen from the three equations of motion, that if any one of the Ω_{ab} has a VEV of this form, all three of them would have a similar form (in some basis), i.e., would vanish in the lower 4×4 block and therefore a subgroup $SO(4)^3 = (SU(2)_L \times SU(2)_R)^3$ would remain unbroken by the bifundamentals. And even after breaking by the bispinor Higgs VEVs there would still remain an unbroken $(SU(2)_L)^3$. Thus the form in Eq. (2) is too simple to give the bifundamental VEVs the Dimopoulos–Wilczek form and also break the symmetry down to the Standard Model group at low energies.

Satisfactory breaking can happen if there are two sets of bifundamentals, Ω_{ab} and Ω'_{ab}, where Ω'_{ab} is odd under a Z_2 parity and Ω_{ab} is even. Then the most general renormalizable form of W_Ω is

$$\begin{aligned} W_\Omega = {} & \frac{1}{2}M(\mathrm{tr}\,\Omega_{12}\Omega_{21} + \mathrm{tr}\,\Omega_{23}\Omega_{32} + \mathrm{tr}\,\Omega_{31}\Omega_{13}) \\ & + \frac{1}{2}M'(\mathrm{tr}\,\Omega'_{12}\Omega'_{21} + \mathrm{tr}\,\Omega'_{23}\Omega'_{32} + \mathrm{tr}\,\Omega'_{31}\Omega'_{13}) \\ & + \lambda(\mathrm{tr}\,\Omega_{12}\Omega_{23}\Omega_{31}) + \lambda'(\mathrm{tr}\,\Omega_{12}\Omega'_{23}\Omega'_{31} \\ & + \mathrm{tr}\,\Omega'_{12}\Omega_{23}\Omega'_{31} + \mathrm{tr}\,\Omega'_{12}\Omega'_{23}\Omega_{31}). \end{aligned} \tag{3}$$

The equations of motion then become: $\Omega_{ba} = -(\lambda/M) \times \Omega_{bc}\Omega_{ca} + (\lambda'/M)\Omega'_{bc}\Omega'_{ca}$ and $\Omega'_{ba} = -(\lambda'/M') \times (\Omega'_{bc}\Omega_{ca} + \Omega_{bc}\Omega'_{ca})$. These equations have many interesting solutions. The one that seems phenomenologically most interesting has VEVs of the following form:

$$\Omega_{12} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad \Omega_{23} = \Omega_{31} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$
$$\Omega'_{12} = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Omega'_{23} = \Omega'_{31} = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}, \tag{4}$$

where A and A' are 6×6 matrices and B and B' are 4×4 matrices, given by

$$A = aI_6, \qquad A' = a'I_3 \otimes i\tau_2,$$
$$B = bI_4, \qquad B' = b'I_2 \otimes i\tau_2, \tag{5}$$

and where $a = \frac{M'}{2\lambda'}$, $a' = \sqrt{\frac{MM'}{2\lambda'^2} + \frac{\lambda M'^2}{4\lambda'^3}}$, $b = \frac{M'}{\lambda'}$, and $b' = \frac{\sqrt{MM'}}{\lambda'}$. At this minimum, the bifundamentals break $SO(10)^3$ down to a diagonal $SU(3) \times SU(2) \times U(1) \times U(1)$.

To illustrate some of the possibilities, we mention a few other solutions of the many that exist: (i) One can have the same form as in Eqs. (4) and (5) but with $A' = a'I_6$, $a' = \sqrt{\frac{MM'}{2\lambda'^2} - \frac{\lambda M'^2}{4\lambda'^3}}$. This breaks $SO(10)^3$ down to $SU(4) \times SU(2) \times U(1) \times U(1)$. (ii) One can have the same form as in Eqs. (4) and (5) but with $B' = b'I_4$, $b' = \frac{\sqrt{+MM'}}{\lambda'}$. This breaks the symmetry down to $SU(3) \times SU(2) \times SU(2) \times U(1)$. (iii) One can have the same form as in Eqs. (4) and (5) but with *both* the substitutions of cases (i) and (ii). This would break the group only down to the Pati–Salam group $SU(4) \times SU(2) \times SU(2)$. (iv) There are solutions where all three of the Ω_{ab} have the form that Ω_{12} has in Eq. (4) and all three of the Ω'_{ab} have the form that Ω'_{12} has in Eq. (4). (v) There are solutions where the matrices have different forms than shown in Eq. (4), for example having different rank than 4, 6, or 10. Some of the unbroken groups that can result are $SU(5) \times U(1)$, $SU(4) \times U(1) \times U(1)$, $SO(8) \times SO(2)$. A complete analysis of all the minima would be rather lengthy.

The form in Eq. (4) is a useful one for the purposes of doublet–triplet splitting, as it can lead to a single pair of Higgs doublets being light. To see this, consider $W_{H\Omega}$, whose most general renormalizable form consistent with symmetry is

$$W_{H\Omega} = \lambda_{H\Omega}(H_1\Omega_{12}H_2 + H_2\Omega_{23}H_3 + H_3\Omega_{31}H_1). \tag{6}$$

Similar terms with Ω'_{ab} are ruled out by the Z_2 parity. If the explicit mass term $(H_1H_1 + H_2H_2 + H_3H_3)$ is forbidden (or suppressed to be of order the weak scale) by symmetry (for example a softly broken discrete symmetry or an R symmetry) then the mass matrix of the color-triplets and weak-doublets in H_a have the form

$$M_3 = \begin{pmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

It is apparent that only a single pair of doublet Higgs fields (namely those in H_3) remain light, as needed for gauge-coupling unification. Therefore, only H_3 will get a weak-interaction-breaking VEV, and because of the form of the Yukawa superpotential given in Eq. (1) only the third family of quarks and leptons will get mass. Below it will be shown that higher dimension operators can generate other entries in the quark and lepton mass matrices, allowing non-zero (but presumably smaller) masses for the other families and CKM mixing. It is interesting that the $SO(10)^3 \times S_3$ symmetry and a choice of minimum consistent with a single pair of light Higgs doublets leads to a natural hierarchy wherein one family is heavier than the others.

In order to give mass to the right-handed neutrinos the bispinors $\bar{\Delta}_{ab}$ must receive non-vanishing VEVs such that both spinors of the bispinor point in the Standard Model-singlet direction. One possible superpotential which achieves this is given below.

$$W_\Delta = \lambda_\Delta\big\{\big(\Delta_{12}\bar{\Delta}_{12} - M^2\big)S_{12} + \big(\Delta_{23}\bar{\Delta}_{23} - M^2\big)S_{23} + \big(\Delta_{31}\bar{\Delta}_{31} - M^2\big)S_{31}\big\}, \tag{8}$$

where S_{ab} are $SO(10)$ singlets. This superpotential admits a solution where all three Δ_{ab}'s and $\bar{\Delta}_{ab}$'s have equal VEVs along their respective Standard Model singlet directions. Higher dimensional operators of the form $(\Delta\bar{\Delta})^2/M_*$ will have to be introduced to give masses to all pseudo-Goldstone bosons from these fields. We assume that the mass scale M_* in this expansion is slightly above the GUT scale, say by a factor of 3–5, which would make the expansion trustable. In this case the pseudo-Goldstone bosons will acquire masses of order a few times below the GUT scale from these non-renormalizable couplings. Since these particles are not much below the GUT scale their effects on the gauge coupling evolution are likely to be small. In particular, the gauge couplings will still stay perturbative at the GUT scale and will remain so upto the scale M_*.

An interesting feature of this VEV structure is that from Eq. (1), it will generate a Majorana right-handed neutrino mass matrix which has equal entries in the off-diagonals, and zero entries along the diagonals (as in M_3 of Eq. (7)). That will result in two degenerate ν^c fields, which may be relevant for resonant leptogenesis. Higher dimensional operators can split the exact degeneracy of the two ν^c fields. It will be necessary to assume that the S_2 interchange symmetry between the first two families (a subgroup of the original S_3) is broken by such higher dimensional operators in order to split the masses of the first two family quarks and leptons. Using the same assumption for the ν^c fields, we find that operators such as $(\psi_1\psi_1\bar{\Delta}_{12}\bar{\Delta}_{13}\Delta_{23}/M_*^2)$ will split the near–degenerate ν^c masses by an amount $(M_G/M_*)^2$ which gives roughly $(\Delta M/M) \sim 10^{-3}$, in the right range for resonant leptogenesis.

There are only two renormalizable terms allowed in $W_{\Omega\Delta}$ by symmetry, namely $(\Delta_{ab}\Delta_{ab}\Omega_{ab} + cyclic)$ and $(\bar{\Delta}_{ab}\bar{\Delta}_{ab}\Omega_{ab} + cyclic)$. These terms give sufficient coupling between the bifundamental Higgs sector and the bispinor Higgs sector to prevent any uneaten Goldstone bosons. (With insufficient coupling between two kinds of Higgs, Goldstone bosons can arise that correspond to relative rotations of their VEVs.) It is in this regard that the present model fares better than the usual $SO(10)$ models, where new mechanisms should come in to stabilize the VEV structure [11].

Turning to higher dimension operators, one finds that there are only a few quartic operators allowed by the $SO(10)^3 \times S_3$ symmetry in the Higgs superpotential. Some of these (such as $(H_a^2 \Omega_{bc}^2 + cyclic)$) can be constructed by multiplying pairs of the invariant quadratic operators H_a^2, Ω_{ab}^2, and $\Delta_{ab}\bar{\Delta}_{ab}$ (or by taking such products and contracting the gauge indices differently). In addition, there are the following five types of invariant quartic operators: $O_1 \equiv (H_a H_b \Omega_{ca}\Omega_{cb} + cyclic)$; $O_2 \equiv (H_a H_b \Delta_{ab}^2 + cyclic)$ and $\bar{O}_2 \equiv (H_a H_b \bar{\Delta}_{ab}^2 + cyclic)$; $O_3 \equiv (\Omega_{ab}\Omega_{ac}\Delta_{bc}^2 + cyclic)$ and $\bar{O}_3 \equiv (\Omega_{ab}\Omega_{ac}\bar{\Delta}_{bc}^2 + cyclic)$; $O_4 \equiv (\Delta_{ab}^4 + cyclic)$ and $\bar{O}_4 \equiv (\bar{\Delta}_{ab}^4 + cyclic)$; and $O_5 \equiv (\Delta_{ab}\Delta_{bc}\bar{\Delta}_{ca}H_b + cyclic)$ and $\bar{O}_5 \equiv (\bar{\Delta}_{ab}\bar{\Delta}_{bc}\Delta_{ca}H_b + cyclic)$. The operators of type O_1, O_3, and $\bar{O}_3$ can exist with the product of bifundamentals being $\Omega\Omega$, $\Omega\Omega'$, or $\Omega'\Omega'$, as far as the symmetry $SO(10)^3 \times S_3$ is concerned; which of these is actually allowed in the superpotential depends on the Z_2 parity assignments of the fields.

There are only two types of quartic operators allowed by $SO(10)^3 \times S_3$ in the Yukawa superpotential, namely $O_{Y1} \equiv (\psi_a \psi_a H_b \Omega_{ab} + cyclic)$, $O_{Y2} \equiv (\psi_a \psi_b \Omega_{ab}\Delta_{ab} + cyclic)$. Again, in both cases $SO(10)^3 \times S_3$ permits such operators with either Ω or Ω', whereas some will not be allowed by Z_2 parity.

These quartic operators, which may be induced either by Planck-scale physics or by integrating out fields at the unification scale, have interesting consequences. Consider first the operator O_5, which written out is $\Delta_{12}\Delta_{23}\bar{\Delta}_{31}H_2 + \Delta_{23}\Delta_{31}\bar{\Delta}_{12}H_3 + \Delta_{31}\Delta_{12}\bar{\Delta}_{23}H_1$. The second term, which involves H_3, is interesting because it induces weak-scale $SU(2)_L$-breaking VEVs in both Δ_{23} and Δ_{31}.

This happens as follows. If we write out this term in $SO(10)^3$ notation, it has the form $\Delta_{23}\Delta_{31}\bar{\Delta}_{12}H_3 = (1, 16, 16)(16, 1, 16)(\overline{16}, \overline{16}, 1)(1, 1, 10)$. Using $[SU(5) \times U(1)]^3$ notation [12], the first factor (Δ_{23}) has a superlarge VEV in the $(1^0, 1^5, 1^5)$ direction, the second factor (Δ_{31}) has a superlarge VEV in the $(1^5, 1^0, 1^5)$ direction, the third factor ($\bar{\Delta}_{12}$) has a superlarge VEV in the $(1^{-5}, 1^{-5}, 1^0)$ direction, and the last factor (H_3) has a weak-scale VEV in the $(1^0, 1^0, 5^{-2})$ direction. Therefore, there is effectively a linear term for the $(1^0, 1^5, \bar{5}^{-3})$ component of Δ_{23} that arises from this product: $(1^0, 1^5, \bar{5}^{-3}) \cdot \langle(1^5, 1^0, 1^5)\rangle\langle(1^{-5}, 1^{-5}, 1^0)\rangle\langle(1^0, 1^0, 5^{-2})\rangle$. It is easy to see that this will induce a weak-scale VEV in this component. So we may write $\langle\Delta_{23}(1^0, 1^5, \bar{5}^{-3})\rangle \sim M_W$. Similarly there is a linear term for the $(1^5, 1^0, \bar{5}^{-3})$ component of the Δ_{31} coming from the product $\langle(1^0, 1^5, 1^5)\rangle \cdot (1^5, 1^0, \bar{5}^{-3}) \cdot \langle(1^{-5}, 1^{-5}, 1^0)\rangle\langle(1^0, 1^0, 5^{-2})\rangle$. So we may write $\langle\Delta_{31}(1^5, 1^0, \bar{5}^{-3})\rangle \sim M_W$.

These weak-scale VEVs in Δ_{23} and Δ_{31} are interesting, in turn, because they can contribute to quark and lepton masses if the operator O_{Y2} is present (in either its Ω or its Ω' form). If one examines the term $\psi_2\psi_3\Omega'_{23}\Delta_{23} = (1, 16, 1)(1, 1, 16)(1, 10, 10)(1, 16, 16)$, ones sees that it contains $(1^0, \bar{5}^{-3}, 1^0)$ $(1^0, 1^0, 10^1)\langle(1^0, 5^{-2}, \bar{5}^2)\rangle\langle(1^0, 1^5, \bar{5}^{-3})\rangle$. In $SU(5)$ language, this is a contribution to a term of the form $\bar{5}_2 10_3\langle\bar{5}_H\rangle$, i.e., the 10 of the third family times the $\bar{5}$ of the second family. Therefore this operator gives a 23 element of the charged-lepton mass matrix M_L and a 32 element of the down-quark mass matrix M_D. It does not contribute to any other components of these matrices, and it does not contribute to the up-quark mass matrix. This is exactly the kind of entry that is needed in the so-called "lopsided" mass matrix models [13]. If such entries are comparable to the 33 elements of M_D and M_L, then they explain the fact that the 2–3 mixing angle is large for the left-handed leptons (i.e., $U_{\mu 3} \sim 1$) but small for the left-handed quarks (i.e., $V_{cb} \ll 1$). If the relevant quartic terms arise from integrating out fields with mass of order M_{GUT} rather than $M_{P\ell}$, there is no reason that these lopsided mass matrix elements necessarily have to be smaller than the 33 elements, even though the latter arise from cubic terms. (Even with quartic terms generating the 23 and 13 entries, they may be comparable to the 33 entry if $\tan\beta$ is small.) It should be noted that in most published lopsided models the operator that gives the lopsided entries $(M_D)_{32}$ and $(M_L)_{23}$ is such that these entries are equal in magnitude. It is important that they be at least approximately equal to reproduce the well-known prediction that at the unification scale $m_b = m_\tau$. Here, the operator $\psi_2\psi_3\Omega'_{23}\Delta_{23}$ does not make these entries equal but gives them a ratio $(M_D)_{32}/(M_L)_{23} = a'/b'$, whose value depends on the parameters in the Higgs superpotential. (See Eq. (5).) (If, instead of Ω' in this operator there were Ω, then the contribution to M_L would vanish.)

In the same way, it is easy to see that the weak-scale VEV of Δ_{31} can generate contributions to the 31 element of M_D and the 13 element of M_L. If these too are comparable to the magnitudes of the 33 elements, then a so-called "doubly lopsided" model results [10]. As has been explained in the literature, such models can account for the so-called "bi-large" pattern of neutrino mixing in a very simple way, and have other attractive features as well.

One might expect the transposes of these lopsided mass matrix elements also to be induced by these quartic terms (e.g., the 23 element of M_D in addition to the 32 element, etc.). However, they are not. Nor are any off-diagonal elements of the up quark-mass matrices, induced until one takes into account terms higher-order than quartic. This may well be related to the stronger mass hierarchy observed among the up-type quarks. Indeed, in lopsided models, it is precisely the absence of large lopsided terms in M_U that is responsible for this. It is noteworthy that in the lopsided models published in the literature the placement of the lopsided entries (for example that they appear in the 32 elements but not the 12 elements, say) is to some extent contrived with an eye to reproducing the observed pattern of masses and mixings. Here, it is largely dictated by the $SO(10)^3 \times Z_3$ symmetry of the theory (and by the requirement that only one pair of Higgs doublets remains light).

The operators O_{Y1} can also play an important role. The term $\psi_1\psi_1 H_3\Omega'_{13}$ and the term $\psi_2\psi_2 H_3\Omega'_{23}$ give 11 and 22 elements respectively to all the quark and lepton mass matrices. Note that the two terms $\psi_1\psi_1 H_3\Omega'_{13}$ and $\psi_2\psi_2 H_3\Omega'_{23}$ are related to each other by S_3. However, if S_3 is broken spontaneously, either completely or down to a Z_3 subgroup, the equality of the 11 and 22 elements need not hold.

The following shows at what level various elements in the quark and lepton mass matrices arise. A "3" means that such an element can arise even if only terms cubic and lower exist in W;

a "4" means that such an element can arise only if quartic (or higher) terms are present, and a "5" means that such an element can arise only if quintic (or higher) terms are present.

$$M_U = \begin{pmatrix} 4 & 5 & 5 \\ 5 & 4 & 5 \\ 5 & 5 & 3 \end{pmatrix}, \qquad M_D = \begin{pmatrix} 4 & 5 & 5 \\ 5 & 4 & 5 \\ 4 & 4 & 3 \end{pmatrix},$$
$$M_L = \begin{pmatrix} 4 & 5 & 4 \\ 5 & 4 & 4 \\ 5 & 5 & 3 \end{pmatrix}. \tag{9}$$

The entries that are labelled "5" arise in a somewhat non-trivial way. Consider, for instance, the 12 and 21 elements of the mass matrices. The quintic term $\Delta_{12}\Delta_{13}\bar{\Delta}_{23}\Omega_{13}H_3$ induces a weak-scale VEV in the component $\Delta_{12}(\bar{5}^{-3}, 1^5, 1^0)$ through the product $(\bar{5}^{-3}, 1^5, 1^0) \cdot \langle(1^5, 1^0, 1^5)\rangle\langle(1^0, 1^{-5}1^{-5})\rangle\langle(5^{-2}, 1^0, \bar{5}^2)\rangle\langle(1^0, 1^0, 5^{-2})\rangle$. This VEV then gives $(M_D)_{12}$ and $(M_L)_{21}$ via the quartic Yukawa operator $\psi_1\psi_2\Omega_{12}\Delta_{12}$ as follows: $(10^1, 1^0, 1^0)(1^0, \bar{5}^{-3}, 1^0)\langle(\bar{5}^2, 5^{-2}, 1^0)\rangle\langle(\bar{5}^{-3}, 1^5, 1^0)\rangle$. The transposed elements $(M_D)_{21}$ and $(M_L)_{12}$ arise in a completely analogous way. (Just interchange the labels 1 and 2 everywhere in the preceding discussion.) The 12 and 21 elements of the up quark mass matrix M_U can come from a quintic Yukawa operator $\psi_1\psi_2\Omega_{12}\Omega_{12}\bar{\Delta}_{12}$ from the term $(10^1, 1^0, 1^0)(1^0, 10^1, 1^0)\langle(5^{-2}, \bar{5}^2, 1^0)\rangle\langle(5^{-2}, \bar{5}^2, 1^0)\rangle\langle(5^3, 1^{-5}, 1^0)\rangle$.

The same quintic term $\Delta_{12}\Delta_{13}\bar{\Delta}_{23}\Omega_{13}H_3$, induces a weak-scale VEV for $\Delta_{13}(\bar{5}^{-3}, 1^0, 1^5)$ through the product $\langle(1^5, 1^5, 1^0)\rangle \cdot (\bar{5}^{-3}, 1^0, 1^5) \cdot \langle(1^0, 1^{-5}, 1^{-5})\rangle\langle(5^{-2}, 1^0, \bar{5}^2)\rangle\langle(1^0, 1^0, 5^{-2})\rangle$. This VEV then induces $(M_D)_{13}$ and $(M_L)_{31}$ through the quartic term that was mentioned before as giving $(M_D)_{31}$ and $(M_L)_{13}$, namely $\psi_1\psi_3\Omega'_{13}\Delta_{13}$. In particular, this contains the product $(10^1, 1^0, 1^0)(1^0, 1^0, \bar{5}^{-3})\langle(\bar{5}^2, 1^0, 5^{-2})\rangle\langle(\bar{5}^{-3}, 1^0, 1^5)\rangle$. The entries $(M_D)_{23}$ and $(M_L)_{32}$ arise in a similar way. (Just interchange the indices 1 and 2 in the foregoing discussion.)

What remains is to show that the 13, 31, 23, and 32 elements of M_U can arise from quintic terms. The elements $(M_U)_{13}$ and $(M_U)_{31}$ arise from the quintic term $\psi_1\psi_3\Omega_{13}\Omega_{13}\bar{\Delta}_{13}$, which contains the product $(10^1, 1^0, 1^0)(1^0, 1^0, 10^1)\langle(\bar{5}^2, 1^0, 5^{-2})\rangle\langle(\bar{5}^2, 1^0, 5^{-2})\rangle\langle(1^{-5}, 1^0, 5^3)\rangle$. The elements $(M_U)_{12}$ and $(M_U)_{21}$ arise in a similar way.

One sees, then, that the requirements of $SO(10)^3 \times S_3$ symmetry imply that the quark and lepton mass matrices have a non-trivial structure that contains several promising features: (i) a hierarchy among the mass matrix elements, (ii) only one family obtaining mass at lowest order, (iii) a qualitative difference between the up quark mass matrix and the other mass matrices (in particular some of the elements of M_U arise at higher order than the corresponding elements of M_D and M_L, which is perhaps related to the stronger hierarchy observed among the up-type quarks); (iv) relatively large off-diagonal elements in the third row of M_D and third column of M_L, i.e., the "doubly lopsided" pattern that is known to explain in a simple way the bilarge pattern of neutrino mixing; (v) "Clebsches" in certain elements of M_D and M_L that may allow an explanation of the well-known Georgi–Jarlskog relations. Still, the construction of a complete model with fully realistic quark and lepton mass matrices has not been attempted here.

There are several issues that would have to be faced in constructing a fully realistic model based on $SO(10)^3 \times S_3$. The most difficult would be proton decay via the $d = 5$ operators that arise from the exchange of colored Higgsinos. The simple structure in Eq. (7) leads to no suppression of such decay amplitudes. It seems likely, however, that with more than two types of bifundamental Higgs fields adequate suppression may be achieved. Another issue is the existence of Landau poles above the unification scale (i.e., the $SO(10)^3 \times S_3$ scale) due to the large number of fields in the bispinor and bifundamental Higgs multiplets. These issues require further study.

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