Asymptotic behavior for a reaction-diffusion equation with inner absorption and boundary flux

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Abstract

This paper deals with a reaction-diffusion equation with inner absorption and boundary flux of exponential forms. The blow-up rate is determined with the blow-up set, and the blow-up profile near the blow-up time is obtained by the Giga–Kohn method. It is observed that the blow-up rate and profile are independent of the nonlinear absorption term.

Keywords: Blow-up rate; Blow-up profile; Asymptotic behavior; Reaction-diffusion

1. Introduction

Consider the following reaction-diffusion equation with inner absorption and boundary flux of exponential forms

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - ae^{pu}, \quad (x, t) \in \Omega \times (0, T), \\
\frac{\partial u}{\partial \eta} &= e^{qu}, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \); constants \( a, p \geq 0, q > 0 \); \( u_0(x) \geq 0 \) in \( \Omega \) and \( \frac{\partial u_0}{\partial \eta} = \exp\{qu_0\} \) on \( \partial \Omega \).

Eq. (1.1) can be used to describe, for example, heat propagations in solid media with nonlinear absorptions and nonlinear boundary flux [1–9]. The existence and uniqueness of local solutions to (1.1) is known by the standard theory [10].

The special case of (1.1) without inner absorption

\[
\begin{align*}
\frac{\partial u}{\partial t} &= u_{xx}, \quad (x, t) \in (0, 1) \times (0, T), \\
u_x(0, t) &= 0, \quad u_x(1, t) = e^{q(t)}, \quad t \in (0, T)
\end{align*}
\]

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was studied by Deng [11]. The blow-up rate of (1.2) was presented as
\[ c \leq (T - t)^{\frac{2}{p}} \exp[u(1, t)] \leq C, \] (1.3)
while the blow-up profile was given as
\[ (T - t)^{\frac{2}{p}} \exp[u(1 - y\sqrt{T - t}, t)] \rightarrow z_0(y) \text{ as } t \rightarrow T \] (1.4)
uniformly on any interval \([0, K]\), where \(z_0(y)\) is the unique positive bounded solution to the corresponding stationary problem of (1.2).

The similar reaction-diffusion equation problem
\[
\begin{align*}
\{ u_t &= u_{xx} - \lambda u^p, \quad (x, t) \in (0, 1) \times (0, T), \\
        u_x(0, t) = 0, u_x(1, t) &= u^q(1, t), \quad t \in (0, T)
\end{align*}
\] (1.5)
with \(p, q > 1\) and \(\lambda > 0\) has been well studied; see [12–14]. The blow-up criterion of (1.5) is \(2q - 1 > p\), or \(2q - 1 = p, \lambda < q, u_0 > v\) with \(v\) being any maximal stationary solution [12,13]. Recently, Rossi [14] obtained the blow-up rate for (1.5) under the additional condition \(u''_0(x) - \lambda u^p_0(x) \geq \delta_0 > 0\) in \((0, 1)\) that if \(p < 2q - 1\), or \(p = 2q - 1\) with \(\lambda < q\), there exist positive constants \(c, C\) such that
\[ c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{\frac{1}{2q-1}} \leq C \text{ as } t \rightarrow T. \]
The blow-up profile was obtained there as well for \(p < 2q - 1\).

In this paper, we will extend all these results to problem (1.1). We give the critical exponent for (1.1), and then the blow-up rate and blow-up set. In particular, we will determine the blow-up profile near the blow-up time. We will follow the methods of [11,15]. Due to the absorption term included in (1.1), some technical difficulties should be overcome.

2. Critical exponent

We begin with the critical exponent of (1.1). Let \(\varphi_0\) be the first eigenfunction of
\[ \Delta \varphi + \lambda \varphi = 0 \text{ in } \Omega; \quad \varphi = 0 \text{ on } \partial \Omega \]
with the first eigenvalue \(\lambda_0\), normalized by \(\|\varphi_0\|_\infty = 1\). It is well known that [16] \(\varphi_0 > 0\) in \(\Omega\), and there are positive constants \(c_i (i = 1, 2, 3, 4)\) and \(\varepsilon_0\) such that \(c_1 \leq -\frac{\partial \varphi_0}{\partial \eta} \|\varphi_0\| \leq c_2 \leq \max_\Omega |\nabla \varphi_0| = c_4, |\nabla \varphi_0| \geq \frac{1}{2} \) on \(\{x \in \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon_0\}\), and \(\varphi_0 \geq c_3\) on \(\{x \in \Omega : \text{dist}(x, \partial \Omega) \geq \varepsilon_0\}\).

In [17], the authors studied the critical exponents for the coupled system corresponding to (1.1). By taking \(a_1 = a_2, \alpha_1 = \beta_1, \alpha_2 = \beta_2 = 0, p = q\) with \(u_0 = v_0\) in (1.1) of [17], we have the following conclusion on the critical exponent for (1.1) directly.

**Theorem 2.1.** (i) If \(2q < p\), the solutions of (1.1) are globally bounded.
(ii) If \(2q > p\), the solutions of (1.1) blow up in finite time for large initial data.
(iii) For \(2q = p\), if \(a \geq 2^p \left(\frac{\lambda_0^4 + 3c_1^2}{c_1^2}\right)\), the solutions are globally bounded; if \(a \leq \frac{p}{4c_2^2} \min \left\{\frac{c_2^2}{4}, \lambda_0 c_3^2\right\}\), the solutions blow up in finite time for large initial data. \(\square\)

3. Blow-up rate and set

In what follows, we deal with radial solutions of (1.1) with \(\Omega = B_R = \{|x| < R\} \subset \mathbb{R}^n, \partial B_R = S_R\). Firstly, consider the lower bound of the blow-up rate.

**Theorem 3.1.** Let \(u\) be a solution of (1.1) with blow-up time \(T\) and \(\Delta u_0(x) - \alpha e^{mu_0(x)} \geq 0\). If \(2q > p\), or \(2q = p\) with \(a \leq \frac{p}{4c_2^2} \min \left\{\frac{c_2^2}{4}, \lambda_0 c_3^2\right\}\), then there is a constant \(C_1\) such that
\[ \max_{\bar{B}_R} u(\cdot, t) \geq \log C_1(T - t)^{-\frac{1}{2q}}, \quad t \in (0, T). \] (3.1)
Proof. Define \( M(t) = \max_{B_R} u(\cdot, t) \) for \( t \in [0, T) \). Clearly, \( M(t) \) is increasing in \([0, T)\) due to the comparison principle with \( \Delta u_0(x) - ae^{pu_0(x)} \geq 0 \).

For \( 0 \leq z < t < T \), by Green’s identity,
\[
\begin{align*}
\int_B u(x, t) &= \int_B \Gamma(x, t, \xi, t) u(\xi, t) d\xi - \int_B \int_B \Gamma(x, t, \xi, \tau) u(\xi, \tau) e^{pu(\xi, \tau)} d\xi d\tau \\
&\quad \quad + \int_B \int_{S_R} \Gamma(x, t, \xi, \tau) e^{pu(\xi, \tau)} dS_{\xi} d\tau - \int_B \int_{S_R} u(\xi, \tau) \frac{\partial \Gamma}{\partial \eta_{\xi}}(x, t, \xi, \tau) dS_{\xi} d\tau,
\end{align*}
\]
where \( \Gamma \) is the fundamental solution of the heat equation. We employ \( C \) to denote positive constants independent of \( t \), which may change from line to line. It is known that [18, 19]
\[
\int_B \int_{S_R} \Gamma(x, t, \xi, \tau) dS_{\xi} d\tau \leq C \sqrt{t - z}, \quad 0 \leq z < t < T;
\]
\[
|\Gamma(x, t, \xi, \tau)| \leq \frac{C}{(t - \tau)^\beta} \frac{1}{|x - \xi|^{n - 2 + \beta}}, \quad 0 < \beta < 1;
\]
\[
|\frac{\partial \Gamma}{\partial \eta_{\xi}}(x, t, \xi, \tau)| \leq \frac{C}{(t - \tau)^\beta} \frac{1}{|x - \xi|^{n + 1 - 2\beta - \alpha}}, \quad \alpha \in (0, 1), \beta \in \left(1 - \frac{\alpha}{2}, 1\right).
\]
So, we have for (ii) and the second part of (iii) of Theorem 2.1,
\[
\begin{align*}
\int_B u(x, t) &\leq C e^{qM(t)} \sqrt{t - z} + CM(t) \sqrt{t - z} + M(z) \\
&\leq 2C e^{qM(t)} \sqrt{t - z} + M(z),
\end{align*}
\]
which means that \( M(t) \) \( \leq 2C e^{qM(t)} \sqrt{t - z} + M(z), \) \( 0 \leq z < t < T \). We can choose \( z < t < T \) such that \( M(t) - M(z) = C_0 > 0 \). Then
\[
C_0 \leq 2C \sqrt{T - z} \exp(qM(z) + qC_0),
\]
which implies (3.1). \( \square \)

Next, consider the upper bound of the blow-up rate for radial solutions, where \( \Omega = B_R \) and \( c_1 = c_2 \). Denote \( u(x, t) = u(r, t) \) with \( r = |x| \).

Theorem 3.2. Let \( u(r, t) \) be a solution of (1.1) with blow-up time \( T \) and
\[
u''(r) + \frac{n - 1}{r} u'_{00}(r) - ae^{pu_0(r)} \geq \delta_0 > 0, \quad u_0'(r) \geq 0, \quad r \in [0, R).
\]

If \( 2q > p \), or \( 2q = p \) with \( a \leq \frac{q}{4} \min\left\{\frac{1}{4}, \lambda_0 c_3^2/c_2^2\right\} \), then there is a constant \( C_2 > 0 \) such that
\[
u(R, t) \leq \log C_2(T - t)^{-\frac{3}{2q}}, \quad t \in [0, T).
\]

Proof. We know \( u_t > 0 \) and \( u_r \geq 0 \) for \( (r, t) \in [0, R) \times [0, T) \) by the comparison principle [19] with (3.2). Set \( J(x, t) = \sqrt{u_t} - \epsilon u_r, (x, t) \in B_R \times [0, T) \). Let \( \epsilon > 0 \) be small such that
\[
J(x, 0) = \sqrt{u_t(0, r) - \epsilon u_r(0, r)} \geq 0, \quad x \in \bar{B}_R,
\]
\[
J_t - \Delta J + \frac{1}{2} ae^{pu} J = \frac{1}{4} u_t^{\frac{3}{2}} u_r^2 + \frac{1}{2} \epsilon ae^{pu} u_r + \epsilon \frac{n - 1}{r^2} u_r \geq 0.
\]

By the comparison principle (see Theorem 2.1 of [19, p. 145]), we have \( J \geq 0 \), and hence
\[
 u_t(R, t) \geq \varepsilon^2 u_r^2(R, t) = e^{2qu(R, t)}, \quad t \in [0, T).
\] (3.4)

Integrating (3.4) from \( t \) to \( T \), we get (3.3) immediately. \( \square \)

Now we deal with the blow-up set of (1.1).

**Theorem 3.3.** Under the conditions of Theorem 3.2, the blow-up set of (1.1) consists of \( S_R = \{ |x| = R \} \). Moreover, there exist suitable positive constants \( A, B \) such that
\[
 u(x, t) \leq \log[A(R^2 - |x|^2)^2 + B(T - t)]^{-\frac{1}{4q}}, \quad (x, t) \in B_R \times [0, T).
\]

**Proof.** Set \( w(x, t) = w(r, t) = \log[A(R^2 - r^2)^2 + B(T - t)]^{-\frac{1}{4q}}, r = |x| \) with
\[
 B \leq \min \left\{ C_2^{-2q}, \frac{4(n + 1)}{R^2 + 4(n + 1)T} \exp[-2q\|u_0\|_\infty] \right\}, \quad 4(n + 1)R^2 A \leq B.
\]
A simple computation shows
\[
 w_t - w_{rr} - \frac{n - 1}{r}w_r \geq 0, \quad -ae^{pu} = u_t - u_{rr} - \frac{n - 1}{r}u_r, \quad (r, t) \in [0, R] \times (0, T),
\]
\[
 w|_{r=R} = \log B^{-\frac{1}{2q}}(T - t)^{-\frac{1}{2q}} \geq u|_{r=R}, \quad t \in [0, T),
\]
\[
 w|_{r=0} \geq \log(A R^4 + B T)^{-\frac{1}{2q}} \geq u_0(r), \quad r \in [0, R).
\]
By the comparison principle, we have \( w \geq u \) in \( B_R \times (0, T) \). \( \square \)

4. Blow-up profile

Throughout this section, we always assume that \( 2q > p \) and \( \Omega = (0, 1) \). Introduce the similarity variables,
\[
 w(y, s) = (T - t)^{\frac{1}{2q}} e^{s(y, t)}, \quad y = \frac{1 - x}{\sqrt{T - t}}, \quad s = \frac{x}{\sqrt{T - t}}.
\] (4.1)

Then \( w \) solves the following system:
\[
 \begin{cases}
 w_y = w_{yy} - \frac{1}{w} w_y^2 - \frac{y}{2} w_y - \frac{1}{2q} w - ae^{-ks} w^{p+1}, \\
 w_y(0, s) = -w^{q+1}(0, s), \quad w_y(e^{\frac{s}{2}}, s) = 0, \\
 w(y, -\log T) = T^{\frac{1}{2q}} e^{u_0(1 - \sqrt{T}) y}.
 \end{cases}
\] (4.2)

in \( \{(y, s) | 0 < y < e^{s/2}, \quad s > -\log T \} \) with \( k = 1 - \frac{p}{2q} > 0 \).

In order to discuss the asymptotic behavior of the solution near the blow-up time, we need four lemmas. First, consider the corresponding stationary solution of (4.2), which solves
\[
 v'' - \frac{1}{v} v'^2 - \frac{y}{2} v' - \frac{1}{2q} v = 0, \quad y > 0 \quad \text{and} \quad v'(0) = -v^{q+1}(0).
\] (4.3)

**Lemma 4.1.** There is a unique positive bounded solution of (4.3),
\[
 V(y) = \exp \left\{ \frac{1}{2q} \int_0^y e^{\xi^2} \left[ \int_0^\xi e^{-\eta^2} d\eta - \sqrt{\pi} \right] d\xi + \frac{1}{q} \log \left( \frac{\sqrt{\pi}}{2q} \right) \right\}
\] (4.4)
satisfying \( |V_y| < CV \), where \( C \) is a positive constant.
Proof. Let \( h(y) = \log v(y) \). Then \( h \) satisfies
\[
h'' - \frac{y}{2} h' - \frac{1}{2q} = 0, \quad y > 0 \quad \text{and} \quad h'(0) = -e^{h(0)}.
\]
If \(|v'| < C \), where \( C \) is a positive constant, then \(|h'| \leq C \). Thus, we have
\[
h'(y) = \frac{1}{2q} e^{\frac{y^2}{4}} \left[ \int_0^y e^{-\frac{\xi^2}{4}} d\xi - \sqrt{\pi} \right] \quad \text{and} \quad h'(0) = -\frac{1}{2q} \sqrt{\pi},
\]
and hence (4.4) follows. \( \square \)

Second, we give some estimates on \( w \).

Lemma 4.2. If \( w \) is defined by (4.2), and (3.2) holds, then there exists some constant \( C > 0 \) such that \( w, |w_y|/w, |w_{yy}| \leq C, |w_x| \leq C(1 + y) \).

Proof. From Theorem 3.2, we know that \( w \) is positive and bounded. By (3.2) and the comparison principle, \( u_x, u_{xx} \geq 0 \), and hence
\[
-w_y = e^{-\frac{y^2}{4}} u_x(x, t) w \leq e^{-\frac{y^2}{4}} u_x(1, t) w \leq C_y w,
\]
that is \(|w_y|/w \leq C\). Similarly to the proof for Proposition 1 of [15], we get \(|w_{yy}| \leq C\). Therefore,
\[
|w_x| = \left| w_{yy} - \frac{1}{w} w^2 y - \frac{1}{2q} w - ae^{-ks} w^{p+1} \right| \leq C(1 + y).
\]

Lemma 4.3. Assume \( w \) is defined by (4.2), and (3.2) holds, \( \{s_j\} \) is an increasing sequence such that \( s_j \to +\infty \) and \( s_{j+1} - s_j \to +\infty \) as \( j \to +\infty \). Then \( w(y, s) = w(y, s + s_j) \) converges to a limit \( w_\infty(y, s) \) uniformly on compact subsets of \([0, +\infty) \times (-\infty, +\infty) \), and for any integer \( m \), \( (w_j)_y(y, m) \to (w_\infty)_y(y, m) \) a.e. in \([0, +\infty) \). Also either \( w_\infty(y, s) \equiv 0 \), or \( w_\infty(y, s) > 0 \) on \([0, +\infty) \times (-\infty, +\infty) \).

Proof. Since \( w, |w_y| \) and \(|w_x| \) are bounded on compact subsets by Lemma 4.2, there is a subsequence of \( \{w_j\} \) converging to some \( w_\infty \) uniformly on compact subsets. Since \(|w_y| \) is also bounded, a diagonal argument yields a subsequence (still denoted by \( \{s_j\} \)) such that \( w_j(y, m + s_j) \to (w_\infty)_y(y, m) \) a.e. for each integer \( m \).

It follows from (4.5) that \(-w_j(y, s) / w(y, s) \leq C\). Integrating with respect to \( y \), we have
\[
w(y_1, s) \leq w(y_2, s) e^{C(y_2 - y_1)} \quad \text{for} \quad y_1 < y_2,
\]
and hence \( w_\infty(y_1, s) \leq w_\infty(y_2, s) e^{C(y_2 - y_1)} \). Then \( w_\infty(y, s) \equiv 0 \), or \( w_\infty(y, s) > 0 \). \( \square \)

Introduce the energy function of \( w \) on \([0, R] \) at time \( s \):
\[
E_R[w](s) = \frac{1}{2} \int_0^R \rho w^2 w_y dy + \frac{1}{2q} \int_0^R \rho \log w dy - \frac{1}{q} \int_0^R w^q(0, s) + \frac{a}{p} e^{-ks} \int_0^R w^p dy,
\]
where weight \( \rho = \exp(-\frac{1}{4} y^2) \) and \( k = 1 - \frac{p}{2q} > 0 \).

Lemma 4.4. The limit function \( w_\infty \) in Lemma 4.3 is independent of \( s \), and \( E[w_\infty] \), \( w_\infty \) are independent of the choice of the sequence \( \{s_j\} \).

Proof. We rewrite the equation in (4.2) as
\[
\rho w_x = (\rho w_y)_y - \frac{1}{w} w_y - \frac{1}{2q} \rho w - a \rho e^{-ks} w^{p+1}.
\]
Multiplying both sides of (4.7) by \( w_x / w^2 \) and integrating with respect to \( y \) from 0 to \( s \),
\[
\int_0^s \rho \frac{w^2_x}{w^2} dy = - \frac{d}{ds} E_s[w](s) + G(s),
\]
where $E$ is defined in (4.6) and
\[ G(s) = \rho(s) \frac{w'_y(s,s)w_s(s,s)}{w^2(s,s)} + \frac{1}{2} \rho(s) \frac{w'_y(\alpha)}{w^2(\alpha)} + \frac{1}{2q} \rho(s) \log w(s,s) - \frac{ak}{p} e^{-ks} \int_0^s \rho w^p \,dy + \frac{a}{p} e^{-ks} \rho(s) w^p(s,s). \]

By the upper bound of the blow-up rate and (4.1), we have
\[ \frac{1}{C_2} \leq \frac{1}{w} \leq e^{\frac{w}{C_2}} \quad \text{and} \quad -\frac{s}{2q} \leq \log w \leq \log C_2. \]

It follows that $|G(s)| \leq C(1 + s)e^{-\frac{s}{2q} - \frac{1}{4}s^2} + C e^{-ks}$, and thus
\[ \int_{s_0}^{+\infty} |G(s)| \,ds < +\infty. \tag{4.9} \]

Integrating (4.8), we have
\[ \int_0^\beta \int_0^s \rho \frac{w'^2}{w_1} \,dy \,ds = E_a[w](\alpha) - E_\beta[w](\beta) + \int_0^\beta G(s) \,ds \tag{4.10} \]
for any $\alpha < \beta$. Similarly to the proof for Propositions 4 and 5 of [15], together with (4.9) and (4.10), we can prove that $w_\infty$ is independent of $s$, $E[w_\infty]$ is independent of the choice of the sequence $\{s_j\}$, and so is $w_\infty$, provided $E(0) = -\infty$. \hfill \Box

Now we give the main result of this section.

**Theorem 4.1.** Let $u$ be a solution of (1.1) with blow-up time $T$, $\Omega = (0, 1)$, $2q > p$ and (3.2). Then
\[ (T - t)^{-\frac{1}{2q}} \exp \left[ u(1 - y\sqrt{T - t}, t) \right] \rightharpoonup V(y) \quad \text{as} \quad t \to T \]
uniformly on each set $|y| \leq C$ for any constant $C > 0$, where $V$ is defined in (4.4).

**Remark.** Let us compare the results of this paper with those without absorptions. If $a = 0$, $q = 1$ and $\Omega = (0, 1)$, then (1.1) becomes (1.2), and the blow-up rate (3.1) and (3.3) is coincident with (1.3) of problem (1.2) obtained in [11]. The blow-up set of (1.1) is the same as that of (1.2). Similarly, (4.11) with $q = 1$ becomes (1.4) related to (1.2). Due to key role of the nonlinear boundary flux in the blow-up process, it is reasonable that the limit $V(y)$ in Theorem 4.1 is not a constant (cf. [11,14]). In this paper, it is interesting to observe that the absorption term in (1.1) does not affect the asymptotic behavior of the solutions at all near the blow-up time. For example, $V = z_0$ (defined in (1.4)) for $q = 1$ and $p < 2$.

**References**


