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On almost bad Boolean bases

Vladimir Stetsenko
Department of Discrete Mathematics, Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119899, Russian Federation

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Abstract

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As known, there is a dependence of the formula complexity of explicit sequences of Boolean functions on the basis chosen. This paper is devoted to a description of bases for which one could hope to obtain nonlinear lower bounds on the formula complexity of such sequences most probably.

1. Introduction

It is well known that there are great difficulties with proving nonlinear lower bounds on the circuit complexity of explicit sequences of Boolean functions. We are currently able to prove only very weak lower bounds on circuit size except in very weak computational models. However it is not our purpose to go into a detailed discussion of this state of affairs. The reader is referred to [3, 6, 8, 13] for such a comprehensive discussion. One can only observe that the difficulty in proving that an explicit sequence has high circuit complexity seems to lie in the very nature of the circuit model of computation. One way to make some progress on this is to limit the capabilities of the circuit model. In this way it has been possible to achieve some interesting results. First of all, it is, of course, the lower bounds obtained by Razborov [9] and subsequently by Andreev [2] (also see [1]) for the monotone circuit model are almost exponential. But progress was mainly achieved for circuit models over incomplete bases, i.e., in reality for the computational models which are not universal computers. However hopes that such models can lead to a clear situation in the general case are not realized. It seems that there is a greater difference between a complete basis and any incomplete one

Correspondence to: V. Stetsenko, Department of Discrete Mathematics, Moscow State University, 119899 Moscow, Russian Federation. Email: stetsenko.matan@mpgu.msk.su

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than we now think. So it is important to study circuit models over complete bases in the first order. Today, in our opinion, a formula over a complete basis is one of the most promising models. Firstly, in this case we already have a number of examples of nonlinear lower bounds and so there is material for studying. Secondly, although among circuit models the formula over a complete basis has an especially simple definition and, hence, may be more amenable to combinatorial analysis, it is an universal computer, and so one could hope that in this case we will succeed in obtaining some new ideas which may lead the way to lower bounds for more powerful circuit models. It is this class of computational models that is considered in this paper.

Now, suppose we wish to obtain a nonlinear lower bound on the formula complexity of an explicit sequence of Boolean functions. It is known, however, that the formula complexity of such a sequence can depend essentially on the basis chosen (for example, see [12]), so the following natural question arises: "What are the bases are, for which we could hope to prove the most possible the nonlinear lower bounds?". Our purpose is to give an answer to this question.

2. Bad and almost bad Boolean bases

A function $f : \{0, 1\}^n \to \{0, 1\}$ is called a Boolean function (of n variables). Denote by P_2 the set of all Boolean functions and by P_2^n the set of all Boolean functions of n variables. Let B be a finite subset of P_2. *Formulas over* B are built as usual from variables and functions of B. Each formula over B represents a Boolean function in a natural way. We mean by a *basis* an arbitrary *finite* subset B of P_2 such that any Boolean function can be represented by a formula over B. For f in P_2 and a basis B, define *the complexity* of f in $BL_B(f)$ to be the number of occurrences of variables in the smallest formula over B representing f.

Analysis of all known methods of obtaining nonlinear lower bounds on the complexity of Boolean functions shows that here we achieve success more often in case of the de Morgan basis. The question arises whether this basis is special.

Let B_1 and B_2 be two bases.

B_1 *precedes* B_2 ($B_1 \preceq B_2$) if there is a constant $c > 0$ (depending only on B_1 and B_2) such that for any Boolean function f

$$L_{B_1}(f) \leqslant cL_{B_2}(f)\,.$$

B_1 and B_2 are *equivalent* ($B_1 \equiv B_2$) if $B_1 \preceq B_2$ and $B_2 \preceq B_1$, and *nonequivalent* otherwise.

Later we will not distinguish between equivalent bases.

B_1 *strictly precedes* $B_2 (B_1 \prec B_2)$ iff $B_1 \preceq B_2$ and $B_1 \not\equiv B_2$.

Denote by B_0 the de Morgan basis and by B_1 *the full binary* one (i.e., the basis consisting all of 16 Boolean funtions of two variables). As it is shown in [12] $B \preceq B_0$ for any basis B. In other words, formulas over the de Morgan basis are the most complicated. Also from [7] we know that

$$L_{B_0}(f) = O((L_{B_1}(f))^{\alpha}) \quad \text{for all } f \in P_2 \text{ where } \alpha = \log_3 10 = 2.095\{+\}.$$

On the other hand it is known that the function $x_1 \oplus \cdots \oplus x_n$ requires a formula of complexity at least n^2 over B_0 but at most n over B_1 [5]. ($\oplus$ denotes sum modulo 2.) This shows that although the basis B_1 differs from B_0, it differs very little from B_0 in the sense that it gives very little advantage over B_0 in representing Boolean functions by formulas. We know that in fact there are examples of nonlinear lower bounds in case of B_1 [4]. Now it seems natural to investigate bases which differ very little from the Morgan basis in order to prove nonlinear lower bounds in case of these bases. To be more precise, consider the definition:

a basis B is called *premaximal* if $B \prec B_0$ and there is no basis B' such that $B \prec B' \prec B_0$.

This definition is our variant of formalization of the property *"to differ very little from the de Morgan basis"*. In support of our way one can say that the full binary basis is premaximal [11]. At present, we do not know which bases are premaximal, besides this basis. However we know what kind of all premaximal bases must be and what bases the nearest neighborhood of the de Morgan one consists of. It turns out that these bases consist of so-called s-functions and Boolean functions represented by read-once formulas over the de Morgan one. Because of the peculiar role of the s-functions, it is desirable to obtain a better view of their discription, since using their peculiarities could give a posibility to obtain new methods of proving nonlinear lower bounds on the complexity of Boolean functions. Such a discription is given below. Since we use essentially some results from [12] we start with them.

For any Boolean function f, all functions obtained from f by replacing variables $x_{i_1}, \ldots, x_{i_m}$ by $\sigma_1, \ldots, \sigma_m \in \{0, 1\}$ are called its subfunctions and is denoted by ${}^{\sigma_1,\ldots,\sigma_m}f^{x_{i_1},\ldots,x_{i_m}}$.

It is also convenient to assume that any function itself is its subfunction. we will say that a subfunction of f is its proper one if it differs from f.

An essential variable x of f is called *a distinguished* one if both the subfunctions ${}^0f^x$ and ${}^1f^x$ depend essentially on all their variables except fictitious ones of f.

Lemma 2.1 (Subbotovskaya [12]). *For any basis B, $B \prec B_0$ iff there is a function in B such that at least one of its subfunctions depends esssentially on more than one variable and has a distinguished variable.*

We say that the formula F is *read-once* if none of its essential variables occurs more than once in F.

Lemma 2.2 (Lupanov). *For two bases B_1 and $B_2, B_1 \preceq B_2$ if each $f \in B_2$ can be represented by a read-once formula over B_1.*

Lemma 2.3 (Subbotovskaya [12]). *The Boolean function f can be represented by a read-once formula over B_0 iff none of its subfunctions which depends essentially on more than one variable has a distinguished variable.*

Now we introduce a basic conception of this paper.

A Boolean function of at least two variables f is called an *s-function* if

(i) f depends essentially on all its variables,

(ii) f has at least one distinguished variable, and

(iii) f has no proper subfunction depending essentially on more than one variable, which has a distinguished one.

By induction on the number of variables, we can easily prove the following lemma.

Lemma 2.4. *Any Boolean function of more than one variable, which depends essentially on all its variables and has a distinguished variable, has an s-function as its subfunction.*

A direct conclusion of Lemma 2.1 is the following result.

Lemma 2.5. *For any s-function g we have*

$$B_0 \cup \{g\} \prec B_0 .$$

Now using that conjunction, disjunction, negation and the constants 0,1 can be represented by read-once formulas over any basis we can easily show the following.

Lemma 2.6. *A basis B is equivalent to B_0 iff each function in B can be represented by a read-once formula over B_0.*

Likewise we can show that the following lemma is true.

Lemma 2.7. *For any basis $B \prec B_0$ there is an s-function g such that*

$$B \preceq B_0 \cup \{g\} \prec B_0 .$$

Now if we assume that the basis B in Lemma 2.7 is premaximal, we will easily obtain the next necessary condition for premaximal bases.

Lemma 2.8. *Each premaximal basis is equivalent to a basis of the type $B_0 \cup \{g\}$ where g is an s-function.*

The proof of next lemma is given in [11].

Lemma 2.9 (Stetsenko [11]). *If for a basis B and a finite set of s-functions B_s,*

$$B_0 \cup B_s \prec B \prec B_0$$

then B consists of s-functions among which at least one is not in B_s and Boolean functions represented by read-once formulas over B_0.

3. Main Theorem

As we have seen in the previous section, the s-functions enter, in an important manner, into the discription of the nearest neighborhood of the de Morgan basis. The following result gives a convenient discription of s-functions.

Theorem 3.1. *The following*[1] *are, up to renamings and negation of variables, all the possible s-functions:*

$$x_1x_2\cdots x_n \vee \bar{x}_1\bar{x}_2\cdots\bar{x}_n, \quad n \geqslant 2,$$

$$x_1(x_2 \vee x_2 \cdots \vee x_n) \vee x_2\cdots x_n, \quad n \geqslant 3,$$

$$x_1(x_2 \vee x_3 \cdots x_n) \vee x_2\bar{x}_3\cdots\bar{x}_n, \quad n \geqslant 3,$$

$$x_1(x_3x_4 \vee x_5) \vee x_2(x_3 \vee x_4x_5),$$

$$x_1(x_2 \vee x_3) \vee x_3x_4 .$$

To be more precise, define an equivalence relation on P_2^n in the following way.

Let Q_n be a group of all transformations of the type

$$f_1(x_1,\dots,x_n) \mapsto f_2^{\alpha_0}(x_{\pi(1)}^{\alpha_1},\dots,x_{\pi(n)}^{\alpha_n})$$

where $f_1, f_2 \in P_2^n, \alpha_0, \alpha_1, \dots, \alpha_n \in \{0, 1\}$ and π is a permutation of indices. For $f_1, f_2 \in P_2^n$ we will say that f_1 is *one-type* to f_2 and denote by $f_1 \equiv_1 f_2$ if there is $t \in Q_n$ such that $f_1 = t(f_2)$.

Obviously this relation retains the property "*to be an s-function*". Our final aim is to show that for each $n = 2, 3, \dots$ the list in Theorem 3.1 is a system of distinct representatives of s-functions w.r.t. $\equiv_1$.

4. Basic properties of read-once Boolean functions

In this section we consider properties of Boolean functions which can be represented by read-once formulas over B_0. For the sake of brevity, we will call such functions *read-once*.

First of all, let us note that each of subfunctions of a read-once function is itself read-once.

Let x_i and x_j be essential variables of $f \in P_2^n$, and let $\sigma \in \{0, 1\}$. We write $x_i \xrightarrow{\sigma} x_j(f)$ if x_j is a fictitious variable of ${}^{\sigma}f^{x_i}$.

Later we will often use the following simple lemma.

Lemma 4.1. *Let x_i, x_j be essential variables of Boolean functions f and ${}^{\tau}f^{x_k}$ (x_k differs from x_i and x_j). Then $x_i \xrightarrow{\sigma} x_j(f)$ implies $x_i \xrightarrow{\sigma} x_j({}^{\tau}f^{x_k})$ for any $\sigma \in \{0, 1\}$.*

Proof. If $x_i \xrightarrow{\sigma} x_j(f)$, then, by definition,

$${}^{\sigma 0}f^{x_ix_j} = {}^{\sigma 1}f^{x_ix_j},$$

so

$${}^{\sigma 0\tau}f^{x_ix_jx_k} = {}^{\sigma 1\tau}f^{x_ix_jx_k}.$$

[1] The result was announced in [10].

Obviously we can rewrite the second equality as

$$^{\sigma 0}({}^{\tau}f^{x_k})^{x_i x_j} = {}^{\sigma 1}({}^{\tau}f^{x_k})^{x_i x_j} ,$$

which, according to the definition, means $x_i \xrightarrow{\sigma} x_j({}^{\tau}f^{x_k})$. □

Let f be a Boolean function having at least two essential variables, and let x_i be one of them.

A constant $\sigma \in \{0, 1\}$ is called *a ramming value* of x_i in f if there is an essential variable x_j of f such that $x_i \xrightarrow{\sigma} x_j(f)$ and *non ramming* otherwise.

Obviously x_i is a distinguished variable of f iff x_i has no ramming value in f.

Let $X = \{x_1, \dots, x_n\}$. A collection of $\{\beta_1, \dots, \beta_m\}$ of non empty, disjoint subsets of X such that $X = \beta_1 \cup \cdots \cup \beta_m$ is called *a partition* of X.

Let $\pi = \{\beta_1, \dots, \beta_p\}$ and $\pi' = \{\beta'_1, \dots, \beta'_q\}$ be two partitions of X. It is said that π precedes $\pi'(\pi \trianglelefteq \pi')$ if each $\beta'_i, i \in \{1, \dots, q\}$ is contained in some β_j, $j \in \{1, \dots, p\}$. Also it is said that π and π' are *comparable* if $\pi \trianglelefteq \pi'$ or $\pi' \trianglelefteq \pi$ and *uncomparable* otherwise.

It is well known that $\trianglelefteq$ is a partial ordering on the set of all partitions of X. We will always write $X = A \mathring{\cup} B$ if $X = A \cup B$ and $A \cap B = \emptyset$.

The following almost obvious fact will be often used later on.

Lemma 4.2. $\pi \neg \trianglelefteq \pi'$ *iff there are*

$$\beta'_i, \beta_{i_1}, \dots, \beta_{i_k} (i \in \{1, \dots, q\}, i_1, \dots, i_k \in \{1, \dots, p\})$$

and disjoint sets

$$u_1, \dots, u_k, t_1, \dots, t_k \quad (2 \leqslant k \leqslant p)$$

such that

(i) $\beta'_i = u_1 \mathring{\cup} \cdots \mathring{\cup} u_k$ *and* $\beta_{i_1} = u_1 \mathring{\cup} t_1, \dots, \beta_{i_k} = u_k \mathring{\cup} t_k$,

(ii) $u_j \neq \emptyset$, $t_j \cap \beta'_r = \emptyset$ *and* $u_j \cap \beta'_r = \emptyset$ *for all* $j \in \{1, \dots, k\}$ *and* $r \in \{1, \dots, q\}$, $r \neq i$.

Remark. Note that if π and π' are uncomparable, then we have the equalities

$$\beta_i = u'_1 \mathring{\cup} \cdots \mathring{\cup} u'_s \quad \text{and} \quad \beta'_{i_1} = u'_1 \mathring{\cup} t'_1, \dots, \beta'_{i_s} = u'_s \mathring{\cup} t'_s \text{ as in Lemma 4.2}.$$

Besides at least one set t_i and at least one set t'_j is non empty. Later we will always assume that *all Boolean functions depend essentially on all their variables and differ from any constant.*

A Boolean function $f(\tilde{x})$ is called

(i) *a* $\vee$*-function* if $f(\tilde{x}) = f_1(\tilde{x}_1) \vee f_2(\tilde{x}_2)$, and

(ii) *a* $\wedge$*-function* if $f(\tilde{x}) = f_1(\tilde{x}_1) \wedge f_2(\tilde{x}_2)$

where $\{\{\tilde{x}_1\}, \{\tilde{x}_2\}\}$ is a partition of $\{\tilde{x}\}$.

Denote by $K_\vee$ and $K_\wedge$ the class of all $\vee$-functions and the class of all $\wedge$-functions, respectively. We will also denote by $D_c(f)$ the disjunction of all prime implicants of f. (All unexplained notions can be easily found, for example, in [13].)

Lemma 4.3. *The classes $K_\vee$ and $K_\wedge$ are disjoint.*

Proof. Suppose $K_\vee \cap K_\wedge \neq \emptyset$. Then there is $f(\tilde{x})$ such that

$$f(\tilde{x}) = f(\tilde{x}_1) \vee f(\tilde{x}_2) = f'(\tilde{y}_1) \wedge f'(\tilde{y}_2)$$

where $\{\tilde{x}\} = \{\tilde{x}_1\} \mathring{\cup} \{\tilde{x}_2\} = \{\tilde{y}_1\} \mathring{\cup} \{\tilde{y}_2\}$. Since the sets $\{\tilde{x}_1\}$, $\{\tilde{x}_2\}$ and the sets $\{\tilde{y}_1\}$, $\{\tilde{y}_2\}$ are disjoint, we have

$$D_c(f) = D_c(f_1) \vee D_c(f_2) = D_c(f'_1) \wedge D_c(f'_2).$$

Here $D_c(f'_1) \wedge D_c(f'_2)$ means an expression obtained after removing the parentheses.

Let K_i and K_j be two elementary conjunctions in $D_c(f_1)$ and $D_c(f_2)$ respectively. Then there are elementary conjunctions K'_i, K'_j in $D_c(f'_1)$ and K''_i, K''_j in $D_c(f'_2)$ such that $K_i = K'_i K'_j$ and $K_j = K'_i K''_j$. Consider the elementary conjunction $K'_i K''_j$ belonging to $D_c(f'_1) \wedge D_c(f'_2) = D_c(f)$. It is easy to see that $K'_i K''_j$ belongs to neither $D_c(f_1)$ nor $D_c(f_2)$, and so $K'_i K''_j$ cannot belong to $D_c(f_1) \vee D_c(f_2) = D_c(f)$. Thus we have a contradiction. $\square$

We now introduce the most important concept which is a convenient instrument in our research.

Suppose that $g(x_1, \ldots, x_n)$ and K_g denote either $x_1 \vee \cdots \vee x_n$ and $K_\vee$ or $x_1 \ldots x_n$ and $K_\wedge$ respectively.

Lemma 4.4. *Each function $f \in K_g$ can be uniquely represented up to permuting terms in the form*

$$f(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p))$$

where $R_1, \ldots, R_p$ are functions satisfying

(i) *each R_i depends essentially on all its variables, and differs from any constant,*

(ii) *no R_i belongs to K_g,*

(iii) *$\{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\}$ is a partition of $\{\tilde{x}\}$.*

The above representation is called a *g-representation* and we will always regard the above equality as a *g-representation of f*.

It is sometimes necessary to consider $K_\vee$ and $K_\wedge$ at the same time. In this case we will denote one of them by K_g and the other by K_φ, and will talk about g- and φ-representations respectively. As a matter of fact, it does not matter what kind of notation we use for these classes. It is only important to denote them by different symbols.

Proof of Lemma 4.4. The existence is obvious. For the uniqueness, suppose to the contrary that

$$f(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f(\tilde{x}) = g(R'_1(\tilde{w}_1), \ldots, R'_q(\tilde{w}_q)) \tag{1}$$

are two different g-representations of f.

There are the two possibilities:

$$\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\} \quad \text{and} \quad \pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$$

are different or

$$\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\} \quad \text{and} \quad \pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$$

are equal.

Let us consider each of them.

Case 1: $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\} \neq \pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$. Without loss of generality, one can assume $\pi_1 \neg\trianglelefteq \pi_2$. Then in the notation of Lemma 4.2

$$\{\tilde{w}_i\} = \{\tilde{u}_1\} \mathring{\cup} \cdots \mathring{\cup} \{\tilde{u}_k\} \quad \text{and} \quad \{\tilde{v}_i\} = \{\tilde{u}_1\} \mathring{\cup} \{\tilde{t}_1\}, \ldots, \{\tilde{v}_i\} = \{\tilde{u}_k\} \mathring{\cup} \{\tilde{t}_k\}$$

for some $i \in \{1, \ldots, q\}$. Since $R'_j(\tilde{w}_j)$ differs from any constant for all $j \in \{1, \ldots, q\}$, there are $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{i-1}, \tilde{\alpha}_{i+1}, \ldots, \tilde{\alpha}_q$ such that

$$g(R'_j(\tilde{\alpha}_j), x) = g(x, R'_j(\tilde{\alpha}_j)) = x$$

for all $j \in \{1, \ldots, i-1, i+1, \ldots, q\}$.

By replacing $\tilde{w}_1, \ldots, \tilde{w}_{i-1}, \tilde{w}_{i+1}, \ldots, \tilde{w}_q$ in (1) by $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{i-1}, \tilde{\alpha}_{i+1}, \ldots, \tilde{\alpha}_q$ we obtain

$$R'(\tilde{w}_i) = g(R_{i_1}(\tilde{u}_1, \tilde{\beta}_1), \ldots, R_{i_k}(\tilde{u}_k, \tilde{\beta}_k))$$

where $\tilde{\beta}_1, \ldots, \tilde{\beta}_k$ are parts of $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k$ respectively.

Therefore $R'_i \in K_g$ (because $\{\tilde{u}_j\} \neq \emptyset$ for all $j \in \{1, \ldots, k\}$ and $k \geqslant 2$) which contradicts (ii). Thus Case 1 is impossible.

Case 2: $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\} = \pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$. In this case we have

$$f(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f(\tilde{x}) = g(R'_1(\tilde{v}_1), \ldots, R'_p(\tilde{v}_p)). \tag{2}$$

Obviously

$$R_i(\tilde{v}_i) \neq R'_i(\tilde{v}_i) \tag{3}$$

at least for one $i \in \{1, \ldots, p\}$, since we have assumed that these g-representations are different.

It follows from (2) by associatively and commutativity of $g(x,y)$ that

$$g(R_i(\tilde{v}_i), S(\tilde{u})) = g(R'_i(\tilde{v}_i), T(\tilde{u}))$$

with some functions S and T different from any constant. This implies $R_i(\tilde{v}_i) = g(R_i(\tilde{v}_i), T(\tilde{\beta}))$ as above. Since R_i differs from any constant, $T(\tilde{\beta})$ must be equal to $a \in \{0,1\}$ such that $g(a,x) = g(x,a) = x$. Hence $R_i(\tilde{v}_i) = R'_i(\tilde{v}_i)$ which contradicts (3) so Case 2 is also impossible. □

Let us recall that any formula over B_0 is equivalent to a formula containing only $\wedge$, $\vee$, variables, and negation of variables. Because of this we can state the following fact.

Lemma 4.5. *Each read-once Boolean function depending essentially on at least two variables can belong only to either $K_\vee$ or $K_\wedge$.*

Let f be a Boolean function depending on at least two variables. For an essential variable x_i of f and $\sigma \in \{0,1\}$, let

$${}^{\sigma}Z_f^{x_j} = \{x_i \mid x_i \xrightarrow{\sigma} x_j(f)\}\,.$$

Where no confussion can occur, we omit σ in ${}^{\sigma}Z_f^{x_j}$.

For any finite set A, denote by $|A|$ the number of all elements of A.

Lemma 4.6. *Let $f(\tilde{x}) = g(R_1(\tilde{v}_1),\dots,R_p(\tilde{v}_p))$ and $|\{\tilde{v}_i\}| \geqslant 2$ for some $i \in \{1,\dots,p\}$. Then ${}^{\sigma}Z_f^{x_j} \subseteq \{\tilde{v}_i\}$ for all $x_j \in \{\tilde{v}_i\}$ and $\sigma \in \{0,1\}$.*

Proof. Now, assume $g = \vee$. Since $\{\tilde{v}_1\},\dots,\{\tilde{v}_p\}$ are disjoint, we have

$$D_c(f) = D_c(R_1) \vee \cdots \vee D_c(R_p)\,. \tag{4}$$

The case ${}^{\sigma}Z_f^{x_j} = \emptyset$ is trivial. Consider the case ${}^{\sigma}Z_f^{x_j} \neq \emptyset$. It follows immediately from $|\{\tilde{v}_i\}| \geqslant 2$ and $R_i \notin K_g$ that $D_c({}^{\sigma}R_i^{x_j}) \neq 1$. We easily see that ${}^{\sigma}f^{x_j}$ is represented by the disjunctive normal form obtained from $D_c(f)$ by replacing $D_c(R_i)$ by $D_c({}^{\sigma}R_i^{x_j})$. Since (1) contains different variables and $D_c({}^{\sigma}R_i^{x_j})(\neq 1)$ contains only variables from $D_c(R_i)$, none of the elementary conjunctions in $D_c({}^{\sigma}R_i^{x_j})$ absorbs elementary conjunction from $D_c(R_k)$ for all $k \neq i$. Therefore ${}^{\sigma}f^{x_i}$ depends essentially on $\tilde{v}_1,\dots,\tilde{v}_{i-1}$, $\tilde{v}_{i+1},\dots,\tilde{v}_p$ and so ${}^{\sigma}Z_f^{x_j} \subseteq \{\tilde{v}_i\}$. For the case $g = \wedge$ we obtain a proof of the lemma by dualizing our proof. □

Lemma 4.7. *Let $f(\tilde{x})$ be a read-once Boolean function and $|\{\tilde{x}\}| \geqslant 2$. Then each variable has only one ramming value in f.*

Proof. Lemma 2.3 implies the existence. We will give a proof of the uniqueness by induction on the number of variables of f. The basis of the induction is trivial. Suppose now that f is a read-once Boolean function depending on n $(n > 2)$ variables and the lemma holds for all such functions with the number of variables less than n. By Lemmas 4.5 and 4.4 f can be represented by its g-representation

$$f(\tilde{x}) = g(R_1(\tilde{v}_1),\ldots,R_p(\tilde{v}_p)).$$

Suppose to the contrary that there is $x_k \in \{\tilde{x}\}$ having two different ramming values σ_1 and σ_2 in f. In the other words, ${}^{\sigma_1}Z_f^{x_k} \neq \emptyset$ and ${}^{\sigma_2}Z_f^{x_k} \neq \emptyset$. Let $x_k \in \{\tilde{v}_i\}, i \in \{1,\ldots,p\}$. Obviously the case $R_i(\tilde{v}_i) = x_k^{x_k}$ is impossible. Suppose now $|\{\tilde{v}_i\}| \geqslant 2$. According to Lemma 4.6, ${}^{\sigma_1}Z_f^{x_k} \subseteq \{\tilde{v}_i\}$ and ${}^{\sigma_2}Z_f^{x_k} \subseteq \{\tilde{v}_i\}$ hence the function R_i with the number of variables less than n has two different ramming values which contradicts the induction hypothesis. □

5. Some facts about *s*-functions

Let us recall from the definition that any s-function depends essentially on at least two variables, has a distinguished variable, and has no proper subfunction depending essentially on at least two variables with a distinguished variable.

Lemma 5.1. *Let $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ be an s-function with a distinguished variable y. Then f_1 and f_2 are different read-once functions depending essentially on all their variables.*

Proof. We first prove that f_1 and f_2 depend essentially on all their variables. Suppose to the contrary that, for example, x_i is a fictitious variable of f_1. Then $y \xrightarrow{0} x_i(f)$ hence y is not a distinguished variable of f which contradicts an assumption of the lemma. Thus f_1 and f_2 depend essentially on all their variables. Likewise we can show that $f_1 \neq f_2$. We will now show that f_1 and f_2 are read-once functions. Suppose to the contrary that, for example, f_1 is not a read-once function. Then, by virtue of Lemma 4.2, there is a subfunction f_1' of f_1 depending essentially on at least two variables with a distinguished variable, and so f is not an s-function (because f_1' is a proper subfunction of f too). The obtained contradiction proves the statement. □

Although the converse is not true, a weaker statement is true.

Lemma 5.2. *If $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ where f_1 and f_2 are different functions depending essentially on all their variables, then y is a distinguished variable of f.*

The proof is left to the reader as an easy exercise.

Lemma 5.3. *Let $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ be an s-function where f_1 and f_2 are different functions depending essentially on all their variables, and let $|\{\tilde{x}\}| \geqslant 2$. Then the following two cases are the only possible.*

Case 1 : Each variable in $\{\tilde{x}\}$ has different ramming values in f_1 and f_2.

Case 2 : There are $x \in \{\tilde{x}\}$ and $\alpha \in \{0,1\}$ such that the functions ${}^{\alpha}f_1^{x}$ and ${}^{\alpha}f_1^{x}$ depend essentially on all their variables, and are equal.

Proof. Suppose to the contrary that there is a variable $x_i \in \{\tilde{x}\}$ having the same ramming value σ in f_1 and f_2. Then, by virtue of Lemma 4.7, $x_i = \bar{\sigma}$ is a ramming value neither in f_1 nor in f_2, and so the functions ${}^{\bar{\sigma}}f_1^{x_i}$ and ${}^{\bar{\sigma}}f_2^{x_i}$ depend essentially on all their variables. Assume now that ${}^{\bar{\sigma}}f_1^{x_i} \neq^{\bar{\sigma}} f_2^{x_i}$. Then, by virtue of Lemma 5.2, y is a distinguished variable of the function ${}^{\bar{\sigma}}f_1^{x_i} = \bar{y}^{\bar{\sigma}} f_1^{x_i} V y^{\bar{\sigma}} f_2^{x_i}$ which is a proper subfunction of f, hence f is not an s-function which contradicts an assumption of the lemma. Thus ${}^{\bar{\sigma}}f_1^{x_i} = {}^{\bar{\sigma}}f_2^{x_i}$. Therefore we can take x_i and $\bar{\sigma}$ for x and α respectively. □

6. One-type pairs of Boolean functions and a vector $\vec{V}(f_1, f_2, x)$

It is well known that each function $f \in P_2^n$ can be uniquely represented in the form

$$f(x_1,\ldots,x_n)$$
$$= \bar{x}_i f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) \vee x_i f(x_1,\ldots,x_{i-1},1,x_{i-1},\ldots,x_n) \tag{5}$$

for any $i \in \{1,\ldots,n\}$.

So, for each integer i with $1 \leqslant i \leqslant n$, we can define a bijection

$$p_i : P_2^n \to P_2^{n-1} \times P_2^{n-1}$$

by representation (5).

Let us consider the definition. Let $(f_1,f_2),(g_1,g_2) \in P_2^n \times P_2^n$.

$$(f_1,f_2) \equiv_2 (g_1,g_2)$$
$$\Leftrightarrow \exists t \in Q_n(f_1 = t(g_1) \wedge f_2 = t(g_2))$$
$$\vee \exists t' \in Q_n(f_1 = t'(g_2) \wedge f_2 = t'(g_1)).$$

We will call such pairs of Boolean functions *an one-type pairs,* and will also say that (f_1, f_2) is *one-type* to (g_1, g_2).

It is easy to see that $\equiv_2$ is an equivalence. We can obtain directly from the above definitions the following lemma.

Lemma 6.1. *If* $(f_1, f_2) \equiv_2 (g_1, g_2)$ *then* $p_i^{-1}(f_1, f_2) \equiv_1 p_i^{-1}(g_1,g_2)$ *for all* $i = 1,2,\ldots,n+1$.

In particular $p_i^{-1}(f_1,f_2) \equiv_1 p_i^{-1}(f_2,f_1)$.

Let $(f_1,f_2) \in P_2^n \times P_2^n$ with f_1 and f_2 satisfying the following conditions.

(i) each of them is a read-once function depending essentially on all its variables,

(ii) each of their variables has a different ramming value in these functions. Let us also assume that all variables of f_1 and f_2 are labelled by possitive integers as indices.

Given a pair (f_1, f_2) as above and a variable x_i of f_1 we now construct a vector $\vec{V}(f_1, f_2, x_i) = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ and an auxiliary vector of variables $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ in the following way.

(1) Take x_{i_1} as x_i and σ_1 as a ramming value of x_i in f_1. (Because of Lemma 4.7 there is only one ramming value for each variable of f_1 in f_1.)

(2) Take x_{i_2} as the variable in ${}^{\sigma_1}Z^{x_i}_{f_1}$ (consisting of fictitious variables ${}^{\sigma_1}f_1^{x_i}$) with the least index, and take σ_2 as a nonramming value of x_{i_2} in ${}^{\sigma_1}f_2^{x_i}$. (Since f_1 and f_2 have different ramming values for each of their variables, ${}^{\sigma_1}f_2^{x_i}$ depends essentially on all its variables. Besides, it is evident that it is a read-once function.)

(3) Take x_{i_3} as the variable in ${}^{\sigma_1}Z^{x_i}_{f_1}\backslash\{x_{i_2}\}$ with the least index and σ_3 as a nonramming value of x_{i_3} in ${}^{\sigma_2}({}^{\sigma_1}f_2^{x_i})^{x_{i_2}}$. (Obviously, by virtue of choosing σ_2, ${}^{\sigma_2}({}^{\sigma_1}f_2^{x_i})^{x_{i_2}}$ depends essentially on all its variables.)

(4) Continue the procedure until ${}^{\sigma_1}Z^{x_i}_{f_1}$ is reduced to the empty set.

Obviously, the length of $\vec{V}(f_1, f_2, x_i)$ is equal to $|{}^{\sigma_1}Z^{x_i}_{f_1}| + 1$. Later we will often use this construction.

For the sake of brevity, instead of "*each component of* $(x_{i_1}, \ldots, x_{i_m})$ *is replaced by each component of* $\vec{V}(f_1, f_2, x_i)$ *respectively*" we will say "${}^{\sigma_1}Z^{x_i}_f \cup \{x_i\}$ *is replaced by* $\vec{V}(f_1, f_2, x_i)$" (here we mean the above auxiliary vector of variables). Similarly, instead of "*each variable has different ramming values in* f_1 *and* f_2" *we say briefly* "f_1 *and* f_2 *have different ramming values*".

We can now state the lemma.

Lemma 6.2. *Let* (f_1, f_2), x_i *and* σ_1 *be as above, and let* g_1, g_2 *be functions obtained respectively from* f_1, f_2 *by replacing* ${}^{\sigma_1}Z^{x_i}_f \cup \{x_i\}$ *by* $\vec{V}(f_1, f_2, x_i)$. *Then* g_1 *and* g_2 *are read-once functions depending essentially on all their variables* (*i.e., the same variables as in* f_1 *and in* f_2 *except the variables from* ${}^{\sigma_1}Z^{x_i}_f \cup \{x_i\}$).

7. The proof of the main theorem

First we give a sketch of the proof.

Consider the bijections

$$p_i : P_2^{n+1} \to P_2^n \times P_2^n$$

discussed above.

Let $\{r_1, \ldots, r_q\}$ be a partition of $P_2^n \times P_2^n$ w.r.t $\equiv_2$, and let f be an arbitrary s-function of P_2^{n+1}. "To what classes of $r_1, \ldots, r_q$ can $p_i(f)$ belong ?" is the main question for us later on. Now, assume that we already know that $p_i(f)$ can belong only to $r_{j_1}, \ldots, r_{j_m}$. Then from Lemma 6.1 and that the relation $\equiv_1$ retains the property "to be an s-function" we can also see that a system of distinct representatives of s-functions can be chosen from $p_i^{-1}(r_{j_1}), \ldots, p_i^{-1}(r_{j_m})$. If their number is not very large such a choice can be really made. Thus, a new question arises: "In which way can we choose p_i such that the number of $p_i^{-1}(r_{j_1}),\dots,p_i^{-1}(r_{j_m})$ is not very great?" One of the answers is the following.

Let x_i be a distinguished variable of f. Let us represent f in the form

$$f(x_1,\dots,x_{n+1})$$
$$=\bar{x}_i f(x_1,\dots,x_{i-1},\,0,x_{i+1},\dots,x_{n+1}) \vee x_i f(x_1,\dots,x_{i-1},\,1,x_{i+1},\dots,x_{n+1})\,.$$

According to Lemma 5.1 ${}^0f^{x_i}$ and ${}^1f^{x_i}$ are different read-once functions depending essentially on all their variables. Also if they depend on at least two variables, then, by virtue of Lemma 5.3, there are only the two possibilities:

Case 1: ${}^0f^{x_i}$ and ${}^1f^{x_i}$ have different ramming values, or

Case 2: there are x_i of ${}^0f^{x_i}$ and ${}^1f^{x_i}$ and $\sigma\in\{0,1\}$ such that the functions ${}^\sigma({}^0f^{x_i})^{x_j}$ and ${}^\sigma({}^1f^{x_i})^{x_j}$ depend essentially on all their variables, and are equal.

Also, note that without loss of generality we can assume that one of the functions ${}^0f^{x_i}$ and ${}^1f^{x_i}$ is monotone in both the above cases. It is obvious that in case ${}^0f^{x_i}$ and ${}^1f^{x_i}$ depend on less than two variables we can easily check by hand whether f is an s-function. So we now assume that ${}^0f^{x_i}$ and ${}^1f^{x_i}$ depend on at least two variables. In this case, as we have already seen, it suffices to consider only the two above cases. Besides, since ${}^0f^{x_i}$ and ${}^1f^{x_i}$ are read-once functions, according to Lemma 4.5, each of them belongs only to one of the classes $K_\vee$ and $K_\wedge$ in both the above cases. Because of Lemma 6.1 we can easily see that only the two cases are possible:

Case 3: ${}^0f^{x_i}$ and ${}^1f^{x_i}$ belong to one and the same class (it is unimportant whether we mean $K_\vee$ or $K_\wedge$), or

Case 4: ${}^0f^{x_i}$ and ${}^1f^{x_i}$ belong to different classes.

Hence, by virtue of Lemma 4.4, we have in Case 3

$${}^0f^{x_i}=g(R_1(\tilde{v}_1),\dots,R_p(\tilde{v}_p))\,,$$
$${}^1f^{x_i}=g(R'_1(\tilde{w}_1),\dots,R'_q(\tilde{w}_q))\,,$$

and in Case 4

$${}^0f^{x_i}=g(R_1(\tilde{v}_1),\dots,R_p(\tilde{v}_p))\,,$$
$${}^1f^{x_i}=\varphi(R'_1(\tilde{w}_1),\dots,R'_q(\tilde{w}_q))\,,$$

where $\pi_1=\{\{\tilde{v}_i\}\mid i\in\{1,\dots,p\}\}$ and $\pi_2=\{\{\tilde{w}_i\}\mid i\in\{1,\dots,q\}\}$ are two partitions of $\{\tilde{x}\}$. Finally we get four cases by combining the above ones. But if ${}^0f^{x_i}$ and ${}^1f^{x_i}$ depend on more than two variables, then it is necessary to go into the depths of the g-representation of each of these functions that gives us two more cases depending on whether or not π_1 and π_2 are comparable. By combining them and the four above cases we get eight cases.

To prove the main theorem it is necessary to consider separately each of these cases.

We now give a complete proof of the main theorem, Theorem 3.1. The proof will be preceded by a number of short statements; nevertheless we try to stick to the above sketch. We start with the Case 1. Unless otherwise stated we assume that *any s-function is represented in the form*

$$f(y, \tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x}),$$

where y is a distinguished variable of f, f_1 and f_2 are different read-once functions depending essentially on all their variables with different ramming values. We also assume that f_1 is *a monotone function.*

Lemma 7.1. *Let $f(y,\tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function and $\emptyset \neq \{\tilde{x}_1\} \subset \{\tilde{x}\}$, and let f_1' and f_2' be functions obtained from f_1 and f_2 by replacing all variables in $\{\tilde{x}\} \backslash \{\tilde{x}_1\}$ by constants from $\{0,1\}$ in such a way that f_1' and f_2' depend essentially on all their variables. Then $f_1' = f_2'$.*

Proof. Assume that $f_1' \neq f_2'$. Consider the function

$$g(y, \tilde{x}_1) = \bar{y} f_1'(\tilde{x}_1) \vee y f_2'(\tilde{x}_1)$$

which is a proper subfunction of f. Since f_1' and f_2' differ from any constant (because $\{\tilde{x}_1\} \neq \emptyset$ and $f_1' \neq f_2'$), g depends essentially on at least two variables (y and $\tilde{x}_1$). Besides, y is a distinguished variable of g, since f_1' and f_2' depend essentially on all their variables. So f cannot be an s-function. The contradiction proves the lemma. □

Lemma 7.2. *Let $f(y,\tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function where $f_1 \in K_g$ and $f_2 \in K_\varphi$ are functions such that*

$$f_1(\tilde{x}) = \psi_1(P(\tilde{x}_1),\tilde{x}_2) \quad \text{and} \quad f_2(\tilde{x}) = \psi_2(P(\tilde{x}_1),\tilde{x}_2)$$

for some ψ_1, ψ_2 and P of P_2, and $\{\{\tilde{x}_1\},\{\tilde{x}_2\}\}$ is a partition of $\{\tilde{x}\}$. Then $|\{\tilde{x}_1\}| = 1$.

Proof. Suppose that $|\{\tilde{x}_1\}| > 1$. Then $P(\tilde{x}_1)$ depends essentially on at least two variables. Since we have assumed above $f_1(\tilde{x})$ as a monotone function, $P(\tilde{x}_1)$ is a monotone function as well, and so we can obtain from $P(\tilde{x}_1)$ a variable x_i by replacing some variables in $\tilde{x}_1$ by some $\sigma_1, \ldots, \sigma_k \in \{0,1\}$. Having done analogous replacement in f_1 and f_2 we obtain functions $f_1'(x_i,\tilde{x}_2)$ and $f_2'(x_i,\tilde{x}_2)$ belonging to the different classes K_g and K_φ. By Lemma 4.3 $f_1'(x_i,\tilde{x}_2) \neq f_2'(x_i,\tilde{x}_2)$ which contradicts Lemma 7.1. Thus the lemma is proved. □

Lemma 7.3. *Let $f(y,\tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function and*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{w}_1), \ldots, R_q'(\tilde{w}_q)),$$

and let $\{\tilde{v}_i\} \subseteq \{\tilde{w}_j\}$ for some i, j ($i \in \{1,\ldots,p\}, j \in \{1,\ldots,q\}$) and $|\{\tilde{v}_i\}| \geqslant 2$. Then $f_1(\tilde{x}) = g(R(\tilde{v}_i), x_i)$ and $f_2(\tilde{x}) = \varphi(R'(\tilde{v}_i),x_i)$.

Proof. Without loss of generality, we can assume that $\{\tilde{v}_1\} \subseteq \{\tilde{w}_1\}, |\{\tilde{v}_1\}| \geqslant 2$, and $x_1 \in \{\tilde{v}_1\}$. First we show that $p = q = 2$. Suppose to the contrary $p > 2$. (The case $q > 2$ can be considered by analogy.) According to an assumption of the lemma we have

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = \varphi(R'_1(\tilde{w}_1), \ldots, R'_q(\tilde{w}_q)). \tag{6}$$

By Lemma 4.6 $Z^{x_1}_{f_1} \cup \{x_1\} \subseteq \{\tilde{v}_1\}$, hence by replacing $Z^{x_1}_{f_1} \cup \{x_1\}$ by $\vec{V}(f_1, f_2, x_1)$ in f_1 and f_2 we can obtain the functions

$$f'_1 = g(P(\tilde{v}_{11}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f'_2 = \varphi(Q(\tilde{w}_{11}), R'_2(\tilde{w}_2), \ldots, R'_q(\tilde{w}_q)) \tag{7}$$

with $\{\tilde{v}_{11}\} = \{\tilde{v}_1\} \backslash (Z^{x_1}_{f_1} \cup \{x_1\}) \subseteq \{\tilde{w}_{11}\} = \{\tilde{w}_1\} \backslash (Z^{x_1}_{f_1} \cup \{x_1\})$.

By Lemma 7.1 $f'_1 = f'_2$. We now show that $\{\tilde{w}_{11}\} = \emptyset$ and $q = 2$. Assume that at least one of the two equalities is not satisfied. Then $f'_2 \in K_\varphi$. On the other hand, since $p > 2$, $f^1_1 \in K_g$. Hence, by virtue of Lemma 4.3, $f'_1 \neq f'_2$. Thus we have obtained a contradiction and so $\{\tilde{w}_{11}\} = \emptyset$ and $q = 2$. From (7) we easily see that under these equalities:

$$R'_2(\tilde{w}_2) = g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)).$$

By replacing $R'_2(\tilde{w}_2)$ in (6) by its g-representation obtained above we obtain

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p))),$$

$$f_2(\tilde{x}) = \varphi(R'_1(\tilde{w}_1), g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)))$$

with $\{\tilde{v}_1\} = \{\tilde{w}_1\}$.

Since $p > 2$ and $\{\tilde{v}_i\} \neq \emptyset$ for all $i \in \{1, \ldots, p\}$, the function

$$g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p))$$

depends on at least two variables which contradicts Lemma 7.2. Thus $p = q = 2$.

We will now show that $\{\tilde{v}_1\} = \{\tilde{w}_1\}$ and $\{\tilde{v}_2\} = \{\tilde{w}_2\}$. Using the equalities $p = q = 2$ we can rewrite (6) as

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), R_2(\tilde{v}_2)),$$

$$f_2(\tilde{x}) = \varphi(R'_1(\tilde{w}_1), R'_2(\tilde{w}_2)). \tag{8}$$

Having done the the same replacement as above we have obtained the functions

$$f^2_1 = g(P(\tilde{v}_{11}), R_2(\tilde{v}_2)),$$

$$f^2_2 = \varphi(Q(\tilde{w}_{11}), R'_2(\tilde{w}_2)),$$

which are, by virtue of Lemma 7.1, equal to each other. It is also clear that

$$\{\tilde{v}_{11}\} = \{\tilde{v}_1\}\backslash(Z_{f_1}^{x_1} \cup \{x_1\}) \subseteq \{\tilde{w}_{11}\} = \{\tilde{w}_1\}\backslash(Z_{f_1}^{x_1} \cup \{x_1\}).$$

We will now prove $\{\tilde{w}_{11}\} = \emptyset$. Suppose that $\{\tilde{w}_{11}\} \neq \emptyset$. Then we can prove $\{\tilde{v}_{11}\} = \emptyset$ as above. From this equality and the equality of f_1^2 and f_2^2 we see that

$$R_2(\tilde{v}_2) = \psi(Q(\tilde{w}_{11}), R_2'(\tilde{w}_2)).$$

By replacing $R_2(\tilde{v}_2)$ in (8) by the above experession of $R_2(\tilde{v}_2)$ we obtain the equalities

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \psi(Q(\tilde{w}_{11}), R_2'(\tilde{w}_2)),$$
$$f_2(\tilde{x}) = \psi(R_1'(\tilde{w}_1), R_2'(\tilde{w}_2)). \tag{9}$$

There are only two possibilities:

$$|\{\tilde{w}_2\}| > 1 \quad \text{or} \quad |\{\tilde{w}_2\}| = 1.$$

It is necessary to consider separately each of them.

(i) Suppose that $|\{\tilde{w}_2\}| > 1$. Then the function $R_2'(\tilde{w}_2)$ depends on at least two variables which contradicts Lemma 7.2. Thus (i) is impossible.

(ii) Now suppose that $|\{\tilde{w}_2\}| = 1$. In this case we can rewrite (9) as

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \psi(Q(\tilde{w}_{11}), x)),$$
$$f_2(\tilde{x}) = \psi(R_1'(\tilde{w}_1), \bar{x}) \tag{10}$$

(here f_2 contains $\bar{x}$, since we have already assumed above that f_1 is monotone, and f_1 and f_2 have different ramming values.)

We again replace $Z_{f_1}^{x_1} \cup \{x_1\}$ in (10) by $\vec{V}(f_1, f_2, x_1)$ and as a result we obtain the functions:

$$f_1^3(\tilde{x}') = \psi(Q(\tilde{w}_{11}), x) \quad \text{and} \quad f_2^3(\tilde{x}') = \psi(Q(\tilde{w}_{11}), \bar{x})$$

with $\{\tilde{w}_{11}\} = \{\tilde{w}_1\}\backslash(Z_{f_1}^{x_1} \cup \{x_1\})$. (Recall that $\{\tilde{v}_{11}\} = \emptyset$.) It is easy to see that $f_1^3 \neq f_2^3$ which contradicts Lemma 7.1. Thus $\{\tilde{w}_{11}\} = \emptyset$.

We can now see from

$$\{\tilde{v}_{11}\} = \{\tilde{v}_1\}\backslash(Z_{f_1}^{x_1} \cup \{x_1\}) \subseteq \{\tilde{w}_{11}\} = \{\tilde{w}_1\}\backslash(Z_{f_1}^{x_1} \cup \{x_1\})$$

that $\{\tilde{v}_1\} = \{\tilde{w}_1\}$, and so $\{\tilde{v}_2\} = \{\tilde{w}_2\}$ as well, since $\{\tilde{x}\} = \{v_1\}\mathring{\cup}\{\tilde{v}_2\} = \{\tilde{w}_1\}\mathring{\cup}\{\tilde{w}_2\}$. We now show that $|\{\tilde{v}_2\}| = 1$. Suppose $|\{\tilde{v}_2\}| > 1$. First we rewrite (1) as

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), R_2(\tilde{v}_2)),$$
$$f_2(\tilde{x}) = \psi(R_1'(\tilde{v}_1), R_2'(\tilde{v}_2)). \tag{11}$$

By replacing $Z_{f_1}^{x_1} \cup \{x_1\}$ in (11) by $\vec{V}(f_1, f_2, x_1)$ we can easily show that $R_2(\tilde{v}_2) = R_2'(\tilde{v}_2)$ in the same way as before. But this contradicts Lemma 7.2. Thus $|\{\tilde{v}_2\}| = 1$. Obviously we can now write

$$f_1(\tilde{x}) = g(R(\tilde{v}), x),$$

$$f_2(\tilde{x}) = \psi(R'(\tilde{v}), x). \qquad \square$$

Lemma 7.4. *Let $f(y, \tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function with $f_1, f_2 \in P_2^n$, $n \geqslant 2$, such that*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{w}_1), \ldots, R_q'(\tilde{w}_q)).$$

If $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\}$ and $\pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$ are comparable partitions of $\{\tilde{x}\}$, then (f_1, f_2) may be one-type to only one of the following pairs:

$$(x_1 \vee x_2 \cdots x_n, x_1(x_2 \vee \cdots \vee x_n)), \quad n \geqslant 3,$$

$$(x_1 \vee \cdots \vee x_n, x_1 \cdots x_n), \quad n \geqslant 2,$$

$$(x_1 \vee \cdots \vee x_n, x_1(\bar{x}_2 \vee \cdots \vee \bar{x}_n)), \quad n \geqslant 3.$$

Proof. Without loss of generality we can assume $\pi_2 \trianglelefteq \pi_1$. According to the definition, $\pi_2 \trianglelefteq \pi_1$ iff each $\{\tilde{v}_i\} (i \in \{1, \ldots, p\})$ is contained in some $\{\tilde{w}_j\}$ $(j \in \{1, \ldots, q\})$. There are only two possibilities:

- there is $\{\tilde{v}_i\}$ $(i \in \{1, \ldots, p\})$ such that $|\{\tilde{v}_i\}| \geqslant 2$, or
- $|\{\tilde{v}_i\}| = 1$ for all $i \in \{1, \ldots, p\}$.

We will now consider separately each of them.

Case 1: Suppose that there is $\{\tilde{v}_i\}(i \in \{1, \ldots, p\})$ such that $|\{\tilde{v}_i\}| \geqslant 2$. Then, by Lemma 7.3, we have

$$f_1(\tilde{x}) = g(R(\tilde{v}), x),$$

$$f_2(\tilde{x}) = \varphi(R'(\tilde{v}), x), \tag{12}$$

with $|\{\tilde{v}\}| \geqslant 2$.

Since $R \notin K_g$ and $R' \notin K_\varphi$ (see the definition of the g-representation), and R, R' are read-once functions depending on at least two variables, by virtue of Lemmas 4.3 and 4.4, we can represent R and R' in the form:

$$R(\tilde{v}) = \varphi(R_{11}(\tilde{v}_{11}), \ldots, R_{1l})),$$

$$R'(\tilde{v}) = g(R_{11}'(\tilde{w}_{11}), \ldots, R_{1r}'(\tilde{w}_{1r})).$$

We first show that $|\{\tilde{v}_{1i}\}| = 1$ for all $i \in \{1, \ldots, l\}$. Suppose to the contrary that there is $\{\tilde{v}_{1i}\}$ such that $|\{\tilde{v}_{1i}\}| \geqslant 2$. Let $|\{\tilde{v}_{11}\}| \geqslant 2$ and $x_1 \in \{\tilde{v}_{11}\}$. Since f_1 and f_2 have different ramming values, R and R' have different ramming values as well. By Lemma 4.6 $Z_R^{x_1} \cup \{x_1\} \subseteq \{\tilde{v}_{11}\}$, also $l \geqslant 2$, hence by replacing $Z_R^{x_1} \cup \{x_1\}$ by $\vec{V}(R, R', x_1)$ in f_1 and f_2 respectively we can obtain the functions:

$$f_1^1(\tilde{x'}) = g(P(\tilde{u}), x) \quad \text{and} \quad f_2^1(\tilde{x'}) = \varphi(Q(\tilde{u}), x) \quad \text{(see (12))}.$$

Since $Z_R^{x_1} \cup \{x_1\} \subseteq \{\tilde{v}_{11}\} \subset \{\tilde{v}\}$, x is an essential variable of these functions and, by virtue of Lemma 6.2, the functions P and Q depend essentially on at least one variable, $f_1^1 \in K_g$ and $f_2^1 \in K_\varphi$. Hence, by Lemma 4.3, $f_1^1 \neq f_2^1$ which contradicts Lemma 7.1. Thus $|\{\tilde{v}_{1i}\}| = 1$ for all $i \in \{1, \ldots, l\}$.

Similarly, we can prove $|\{\tilde{w}_{1j}\}| = 1$ for all $j \in \{1, \ldots, r\}$. Seeing that f_1 and f_2 have different ramming values, and f_1 is a monotone function, we can now write

$$f_1(\tilde{x}) = g(x_1, \varphi(x_2, \ldots, x_n)),$$

$$f_2(\tilde{x}) = \varphi(x_1, g(x_2, \ldots, x_n)), \quad n \geqslant 3 \tag{13}$$

Case 2: Now suppose that $|\{\tilde{v}_i\}| = 1$ for all $i \in \{1, \ldots, p\}$. Then we clearly have

$$f_1(\tilde{x}) = g(x_1, \ldots, x_n),$$

$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{w}), \ldots, R_q'(\tilde{w}_q)).$$

Here there are two more possibilities. Namely,

$$|\{\tilde{w}_i\}| = 1 \text{ for all } \quad i \in \{1, \ldots, q\} \quad \text{or} \quad |\{\tilde{w}_i\}| > 1 \text{ for some } j \in \{1, \ldots, q\}.$$

We will again consider saparately each of them.

Case 2.1: Let $|\{\tilde{w}_i\}| = 1$ for all $i \in \{1, \ldots, q\}$. Obviously we can now write

$$f_1(\tilde{x}) = g(x_1, \ldots, x_n),$$

$$f_2(\tilde{x}) = \varphi(x_1, \ldots, x_n), \quad n \geqslant 2 \tag{14}$$

Case 2.2: Let $|\{\tilde{w}_1\}| > 1$ and $x_1 \in \{\tilde{w}_1\}$. By Lemma 4.6, $Z_{f_2}^{x_1} \cup \{x_1\} \subseteq \{\tilde{w}_1\}$, and so by replacing $Z_{f_2}^{x_1} \cup \{x_1\}$ by $\vec{V}(f_2, f_1, x_1)$ in f_1 and f_2, we can obtain the functions:

$$f_1^2 = g(x_{i_1}, \ldots, x_{i_k}) \quad \text{and} \quad f_2^2 = \varphi(P(\tilde{w}_{11}), R_2'(\tilde{w}_2), \ldots, R_q'(\tilde{w}_q))$$

where $\{\tilde{w}_{11}\} = \{\tilde{w}_1\} \backslash (Z_{f_2}^{x_1} \cup \{x_1\})$.

Hence, by Lemma 7.1 $f_1^2 = f_2^2, \{\tilde{w}_{11}\} = \emptyset$ and $q = 2$ (because $f_1^2 \in K_g$ and $f_2^2 \in K_\varphi$, so $f_1^2 \neq f_2^2$ otherwise). Thus $R_2'(\tilde{w}_2) = g(x_{i_1}, \ldots, x_{i_k})$. Clearly we can now write

$$f_1(\tilde{x}) = g(g(i_1, \ldots, x_{i_k}), x_{j_1}, \ldots, x_{j_r}),$$

$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{w}_1), g(x_{i_1}, \ldots, x_{i_k})).$$

By Lemma 7.2 $k = 1$. Therefore

$$f_1(\tilde{x}) = g(x_1, \ldots, x_n),$$

$$f_2(\tilde{x}) = \varphi(x_1, g(\bar{x}_2, \ldots, \bar{x}_n)), \quad n \geqslant 3 \tag{15}$$

The φ-representation of f_2 contains the negation of some variables, since the function f_1 is monotone, and f_1 and f_2 have different ramming values.

Recalling the preceding notation one can rewrite (13) as

$$(x_1 \vee x_2 \cdots x_n, x_1(x_2 \vee \cdots \vee x_n)), \quad n \geqslant 3,$$

$$(x_1 \vee \cdots \vee x_n, x_1 \cdots x_n), \quad n \geqslant 2,$$

$$(x_1 \vee \cdots \vee x_n, x_1(\bar{x}_2 \vee \cdots \vee \bar{x}_n)), \quad n \geqslant 3,$$

Lemma 7.5. *Let $f(y,\tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function and*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{w}_1), \ldots, R'_q(\tilde{w}_q)).$$

Then none of the functions R_i ($i \in \{1, \ldots, p\}$) is equal to a function R'_j ($j \in \{1, \ldots, q\}$).

Proof. Suppose to the contrary that $R_i = R'_j$ for some $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Since, according to the definition of the g-representation, R_i and R'_j differ from any constant, there is $\tilde{\sigma}$ such that by replacing the variables $\tilde{v}_i$ and $\tilde{w}_j$ by $\tilde{\sigma}$ in R_i and R'_j, respectively, we can obtain a constant a such that $g(x,a) = g(a,x) = x$.

It is easy to see that

$$f_1(\tilde{x}) = g(g(R_1(\tilde{v}_1), \ldots, R_{i-1}(\tilde{v}_{i-1}), R_{i+1}(\tilde{v}_{i+1}), \ldots, R_p(\tilde{v}_p)), R_i(\tilde{v}_i)),$$

$$f_2(\tilde{x}) = g(g(R'_1(\tilde{w}_1), \ldots, R'_{j-1}(\tilde{w}_{j-1}), R'_{j+1}(\tilde{w}_{j+1}), \ldots, R'_q(\tilde{w}_q)), R'_j(\tilde{w}_j)). \quad (16)$$

Hence, by replacing $\tilde{v}_i$ and $\tilde{w}_j$ by $\tilde{\sigma}$, we can obtain from (16) the following functions:

$$f_1^1 = g(R_1(\tilde{v}_1), \ldots, R_{i-1}(\tilde{v}_{i-1}), R_{i+1}(\tilde{v}_{i+1}), \ldots, R_p(\tilde{v}_p)),$$

$$f_2^1 = g(R'_1(\tilde{w}_1), \ldots, R'_{j-1}(\tilde{w}_{j-1}), R'_{j+1}(\tilde{w}_{j+1}), \ldots, R'_q(\tilde{w}_q)).$$

By Lemma 7.1 $f_1^1 = f_2^1$, and so $f_1 = f_2$, since $R_i = R_j$. Therefore y is a fictitious variable of f which contradicts that y is a distinguished variable of f. The contradiction proves the lemma. $\square$

Lemma 7.6. *Let $f(y,\tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function and*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{w}_1), \ldots, R'_q(\tilde{w}_q)).$$

If $\{\tilde{v}_i\} \subseteq \{\tilde{w}_j\}$ for some $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$, then $|\{\tilde{v}_i\}| = 1$.

Proof. Suppose the contrary. Without loss of generality one can assume that $\{\tilde{v}_1\} \subseteq \{\tilde{w}_1\}, |\{\tilde{v}_1\}| \geqslant 2$, and $x_1 \in \{\tilde{v}_1\}$. By Lemma 4.6 $Z^{x_1}_{f_1} \cup \{x_1\} \subseteq \{\tilde{v}_1\}$. Hence, by replacing $Z^{x_1}_{f_1} \cup \{x_1\}$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_1)$, we can obtain the functions:

$$f_1^1 = g(P(\tilde{v}_{11}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f_2^1 = g(Q(\tilde{w}_{11}), R'_2(\tilde{w}_2), \ldots, R'_q(\tilde{w}_q)) \tag{17}$$

where $\{\tilde{v}_{11}\} = \{\tilde{v}_1\} \backslash (Z^{x_1}_{f_1} \cup \{x_1\}) \subseteq \{\tilde{w}_{11}\} = \{\tilde{w}_1\} \backslash (Z^{x_1}_{f_1} \cup \{x_1\})$.

By Lemma 7.1 $f_1^1 = f_2^1$. There are the two possibilities:

$$\{\tilde{w}_{11}\} = \emptyset \text{ or } \{\tilde{w}_{11}\} \neq \emptyset .$$

We will now consider separately each of them.

Case 1: Suppose that $\{\tilde{w}_{11}\} = \emptyset$. Since $\{\tilde{v}_{11}\} \subseteq \{\tilde{w}_{11}\}$ $\{\tilde{v}_{11}\} = \emptyset$ in this case as well. Then the equality $f_1^1 = f_2^1$ and the uniqueness of the g-representation imply the equalities $p = q$ and $R_2 = R'_{i_2}, \ldots, R_p = R'_{i_p}$ which contradicts Lemma 7.5. Thus this case is impossible.

Case 2: Now suppose that $\{\tilde{w}_{11}\} \neq \emptyset$. Here there are also the two possibilities:

$$\{\tilde{v}_{11}\} = \emptyset \text{ or } \{\tilde{v}_{11}\} \neq \emptyset .$$

We will again consider separately each of them.

Case 2.1: Suppose that $\{\tilde{v}_{11}\} = \emptyset$. Let $Q(\tilde{w}_{11}) = g(R'_{11}(\tilde{u}_1), \ldots, R'_{1k}(\tilde{u}_k))$. We will now do the following: if $Q \in K_g$, then we replace $Q(\tilde{w}_{11})$ in (17) by its g-representation. As a result we have obtained

$$f_1^1 = g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f_2^1 = g(R'_{11}(\tilde{u}_1), \ldots, R'_{1k}(\tilde{u}_k), R'_2(\tilde{w}_2), \ldots, R'_q(\tilde{w}_q)) .$$

Then the equality $f_1^1 = f_2^1$ and the uniqueness of the g-representation imply the equality $R'_2 = R_i$ for some $i \in \{2, \ldots, p\}$ which contradicts Lemma 7.1. (Note that, according to the definition of the g-representation, $q \geqslant 2$.) Thus the Case 2.1 is also impossible.

Case 2.2: Now suppose that $\{\tilde{v}_{11}\} \neq \emptyset$.

Let $P(\tilde{v}_{11}) = g(R_{11}(\tilde{z}_1), \ldots, R_{1l}(\tilde{z}_l))$. Here we first transform

$$g(P(\tilde{v}_{11}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)) \quad \text{and} \quad g(Q(\tilde{w}_{11}), R'_2(\tilde{w}_2), \ldots, R'_q(\tilde{w}_q),$$

in the same way as above. It is comparatively easy to check that under the transformation we obtain f_1^1 and f_2^1 in the form

$$f_1^1 = g(R_{11}(\tilde{z}_1), \ldots, R_{1l}(\tilde{z}_l), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f_2^1 = g(R'_{11}(\tilde{u}_1), \ldots, R'_{1k}(\tilde{u}_k), R'_2(\tilde{w}_2), \ldots, R'_q(\tilde{w}_q)) .$$

Since $\{\tilde{v}_{11}\} \subseteq \{\tilde{w}_{11}\}$, none of R_{1i} ($i \in \{1, \ldots, l\}$) is equal to R'_j ($j \in \{2, \ldots, q\}$) hence $R'_2 = R_i$ for some $i \in \{2, \ldots, p\}$ which contradicts Lemma 7.1. Thus this case is also impossible. □

Lemma 7.7. *Let* $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ *be an s-function with* $f_1, f_2 \in P_2^n, n \geqslant 2$ *such that*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = g(R_1'(\tilde{w}_1), \ldots, R_q'(\tilde{w}_q)).$$

If $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\}$ *and* $\pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$ *are comparable partitions of* $\{\tilde{x}\}$, *then* (f_1, f_2) *may be one-type only to the following pair:*

$$(x_1 \vee \cdots \vee x_n, \bar{x}_1 \vee \cdots \vee \bar{x}_n), \quad n \geqslant 2.$$

Proof. Without loss of generality we can assume $\pi_2 \trianglelefteq \pi_1$. According to the definition, $\pi_2 \trianglelefteq \pi_1$ iff each $\{\tilde{v}_i\}(i \in \{1, \ldots, p\})$ is contained in some $\{\tilde{w}_j\}$ $(j \in \{1, \ldots, q\})$. By Lemma 7.6 $|\{\tilde{v}_i\}| = 1$ for all $i \in \{1, \ldots, p\}$ hence

$$f_1(\tilde{x}) = g(x_1, \ldots, x_n)$$

(recall that f_1 is a monotone function).

We will now prove that $|\{\tilde{w}_j\}| = 1$ for all $j \in \{1, \ldots, q\}$. Suppose to the contrary that, for example, $|\{\tilde{w}_1\}| \geqslant 2$. Let $x_1 \in \{\tilde{w}_1\}$.

By Lemma 4.6 $Z_{f_2}^{x_1} \cup \{x_1\} \subseteq \{\tilde{w}_1\}$ and so by replacing $Z_{f_2}^{x_1} \cup \{x_1\}$ in f_1 and f_2 by $\vec{V}(f_2, f_1, x_1)$, we have obtained the functions:

$$f_1^1 = g(x_{i_1}, \ldots, x_{i_m}),$$

$$f_2^1 = g(Q(\tilde{w}_{11}), R_2'(\tilde{w}_2), \ldots, R_q'(\tilde{w}_q)),$$

where $\{\tilde{w}_{11}\} = \{\tilde{w}_1\} \backslash (Z_{f_2}^{x_1} \cup \{x_1\})$.

By Lemma 7.1 $f_1^1 = f_2^1$. In this case, as we easily see, there are only two possibilities: either

$$q > 2 \text{ or } \{\tilde{w}_{11}\} \neq \emptyset,$$

or

$$q = 2 \text{ and } \{\tilde{w}_{11}\} = \emptyset.$$

We will now consider separately each of them.

Case 1: Suppose that either $q > 2$ or $\{\tilde{w}_{11}\} \neq \emptyset$. In this case we transform the functions f_1^1 and f_2^1 in the same way as in the Case 2.1 of Lemma 7.6. As above, we can show that $R_2' = x_{i_j}$ for some $j \in \{1, \ldots, m\}$. However this contradicts Lemma 7.5. Thus Case 1 is impossible.

Case 2: Now suppose that both $q = 2$ and $\{\tilde{w}_{11}\} = \emptyset$. Obviously, this assumption implies $R_2'(\tilde{w}_2) = g(x_{i_1}, \ldots, x_{i_m}$. Since $R_2' \notin K_g, m = 1$ because $R_2' \in K_g$ otherwise. Therefore $R_2' = x_{i_j}$ for some $j \in \{1, \ldots, m\}$ which contradicts Lemma 7.5 again, and so Case 2 is also impossible.

Thus we have shown that $|\{\tilde{w}_j\}| = 1$ for all $j \in \{1, \ldots, q\}$. From this equality and the fact of that f_1 and f_2 have different ramming values we easily see that

$$f_2(\tilde{x}) = g(\bar{x}_1, \ldots, \bar{x}_n).$$

Finally we can write

$$f_1(\tilde{x}) = g(x_1, \ldots, x_n),$$

$$f_2(\tilde{x}) = g(\bar{x}_1, \ldots, \bar{x}_n).$$

Since g denotes either conjunction or disjunction, the lemma has been proved. □

Lemma 7.8. *Let $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ be an s-function with $\{\tilde{x}\} = \{x_1, \ldots, x_n\}$ $(n \geqslant 3)$ and*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{w}_1), \ldots, R'_q(\tilde{w}_q)).$$

If $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\}$ and $\pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$ are uncomparable partitions, then

$$\pi_1 = \{\{x_1, \ldots, x_{n-2}, x_{n-1}\}, \{x_n\}\} \quad \textit{and} \quad \pi_2 = \{\{x_1, \ldots x_{n-2}, x_n\}\{x_{n-1}\}\}$$

up to permuting indices.

Proof. According to Lemma 4.2 we can assume, for example, $\{\tilde{v}_1\} = \{\tilde{u}_1\} \mathring{\cup} \ldots \mathring{\cup} \{\tilde{u}_k\}$ for some nonempty, disjoint sets $\{\tilde{u}_i\}$ and $\{\tilde{w}_i\} = \{\tilde{u}_i\} \mathring{\cup} \{\tilde{t}_i\}$ for some sets $\{\tilde{t}_i\}, i = 1, 2, \ldots, k \geqslant 2$. Also, at least one of $\{\tilde{t}_i\}$ is nonempty and $\{\tilde{t}_i\} \cap \{\tilde{u}_j\} = \emptyset$ for all $i, j \in \{1, \ldots, k\}$. Taking this into account, we can write

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1, \ldots, \tilde{u}_k), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{u}_1, \tilde{t}_1), \ldots, R'_k(\tilde{u}_k, \tilde{t}_k), R'_{k+1}(\tilde{w}_{k+1}), \ldots, R'_q(\tilde{w}_q)).$$

We first show that $q = k$. Assume $q > k$. Let $x_1 \in \{\tilde{u}_1, \ldots, \tilde{u}_k\}$. Since $k \geqslant 2$, by virtue of Lemma 4.6, $Z^{x_1}_{f_1} \cup \{x_1\} \subseteq \{\tilde{u}_1, \ldots, \tilde{u}_k\}$. Then by replacing $Z^{x_1}_{f_1} \cup \{x_1\}$ by $\vec{V}(f_1, f_2, x_1)$ in f_1 and f_2 respectively, we obtain the functions:

$$f^1_1 = g(Q(\tilde{z}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f^1_2 = g(P_1(\tilde{z}_1, \tilde{t}_1), \ldots, P_k(\tilde{z}_k, \tilde{t}_k), R'_{k+1}(\tilde{w}_{k+1}), \ldots, R'_q(\tilde{w}_q)),$$

where $\{\tilde{z}\} = \{\tilde{u}_1, \ldots, \tilde{u}_k\} \backslash (Z^{x_1}_{f_1} \cup \{x_1\})$ and $\{\tilde{z}_i\} = \{\tilde{u}_i\} \backslash (Z^{x_1}_{f_1} \cup \{x_1\})$, $\quad i = 1, 2, \ldots, k$.

We will now transform f_1 and f_2 in the following way: first, in case $Q \in K_g$, replace $Q(\tilde{z})$ in f^1_1 by its g- representation then, replace each $P_i(\tilde{z}_i, \tilde{t}_i)$ in f^1_2 by its g-representation as well. As a result, we will clearly have obtained the following g-representations of f^1_1 and f^1_2:

$$f^1_1 = g(R_{11}(\tilde{z}_{11}), \ldots, R_{1l}(\tilde{z}_{1l}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$

$$f^1_2 = g(R'_{11}(\tilde{w}_{11}), \ldots, R'_{1k}(\tilde{w}_{1k}), R'_{k+1}(\tilde{w}_{k+1}), \ldots, R'_q(\tilde{w}_q)),$$

where $\{\tilde{z}\} = \{\tilde{z}_{11}\}, \mathring{\cup} \ldots, \mathring{\cup} \{\tilde{z}_{1l}\}$.

Since $\{\tilde{z}\} \cap \{\tilde{w}_i\} = \emptyset$ for all $i \in \{1,\dots,k\}$, none of R_{1i}, $i \in \{1,\dots,l\}$ is equal to R'_q, hence, by virtue of the uniqueness of the g-representation, $R'_q = R_i$ for some $i \in \{2,\dots,p\}$ which contradicts Lemma 7.5. Thus $q = k$, and so we have

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1,\dots,\tilde{u}_k), R_2(\tilde{v}_2),\dots,R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{u}_1,\tilde{t}_1),\dots,R'_k(\tilde{u}_k,\tilde{t}_k)).$$

We now prove that $|\{\tilde{v}_i\}| = 1$ for all $i \in \{2,\dots,p\}$. Since $\{\tilde{u}_i\} \cap \{\tilde{v}_j\} = \emptyset$ for all $i \geqslant 1, j \geqslant 2$, we can show that each $\{\tilde{v}_i\}$ $(i \in \{2,\dots,p\})$ is contained in some $\{\tilde{t}_j\}(j \in \{1,\dots,k\})$ by analogy with the preceding case. Hence, by virtue of Lemma 7.6, $|\{\tilde{v}_i\}| = 1$ for all $i \in \{2,\dots,p\}$. Thus

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1,\dots,\tilde{u}_k), x_{i_2},\dots,x_{i_p}),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{u}_1,\tilde{t}_1),\dots,R'_k(\tilde{u}_k,\tilde{t}_k)).$$

We now prove that $p = 2$. Suppose the contrary. There are two possibilities:
– there are at least two nonempty sets among $\{\tilde{t}_i\}$ $(i \in \{1,\dots,k\})$, or
– there is the only nonempty set among $\{\tilde{t}_i\}$ $(i \in \{1,\dots,k\})$.
We will consider separately each of them.

Case 1: For the sake of definiteness, we assume that $\{\tilde{t}_1\}$ and $\{\tilde{t}_2\}$ be nonempty sets, $x_i \in \{\tilde{t}_1\}$ and $x_j \in \{\tilde{t}_2\}$. Since $\{\tilde{u}_1\} \neq \emptyset$, $|\{\tilde{u}_1,\tilde{t}_1\}| \geqslant 2$ hence, by virtue of Lemma 4.6, $Z^{x_i}_{f_2} \cup \{x_i\} \subseteq \{\tilde{u}_1,\tilde{t}_1\}$ and so by replacing $Z^{x_i}_{f_2} \cup \{x_i\}$ by $\vec{V}(f_2, f_1, x_i)$ in f_1 and f_2, we obtain the functions:

$$f_1^2 = g(P(\tilde{u}_{11},\tilde{u}_2,\dots,\tilde{u}_k), x_j, x_{j_1}, \dots, x_{j_1}),$$

$$f_2^2 = g(Q(\tilde{u}_{11},\tilde{t}_{11}), R'_2(\tilde{u}_2,\tilde{t}_2),\dots,R'_k(\tilde{u}_k,\tilde{t}_k)),$$

where $\{\tilde{u}_{11}\} = \{\tilde{u}_1\} \backslash (Z^{x_i}_{f_2} \cup \{x_i\})$ and $\{\tilde{t}_{11}\} = \{\tilde{t}_1\} \backslash (Z^{x_i}_{f_2} \cup \{x_i\})$.

According to Lemma 4.1, x_j has the same ramming value in f_1 as in f_1^2. On the other hand, since $x_j \in \{\tilde{t}_2\}$ and $\{\tilde{u}_2\} \neq \emptyset$, then – as follows from Lemma 4.6 – $Z^{x_j}_{f_2} \cup \{x_j\} \subseteq \{\tilde{u}_2,\tilde{t}_2\}$. So according to Lemma 4.1, x_j has the same ramming value in f_2 as in f_2^2. Therefore $f_1^2 \neq f_2^2$ since f_1 and f_2 have different ramming values. Also f_1^2 and f_2^2 depend essentially on at least two variables because $\{\tilde{u}_2\} \neq \emptyset$ and $\{\tilde{t}_2\} \neq \emptyset$. Thus we have obtained a contradiction to Lemma 7.1, and so this case is impossible.

Case 2: Now, for the sake of definiteness, we assume that $\{\tilde{t}_1\}$ is the single nonempty set among $\{\tilde{t}_i\}$ $(i \in \{1,\dots,k\})$. Obviously we can now write

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1,\dots,\tilde{u}_k), x_{i_2},\dots,x_{i_p}),$$

$$f_2(\tilde{x}) = g(R'_1(\tilde{u}_1, x_{i_2},\dots,x_{i_p}), R'_2(\tilde{u}_2),\dots,R'_k(\tilde{u}_k)).$$

Let $x_1 \in \{\tilde{u}_2\}$, $x_2 \in \{x_{i_2},\dots,x_{i_p}\}$ and $x_3 \in \{\tilde{u}_1\}$. (Recall that $k \geqslant 2$.) We will now consider the two possibilities:

$$x_3 \in Z^{x_2}_{f_2} \quad \text{and} \quad x_3 \notin Z^{x_2}_{f_2}.$$

Case 2.1: Suppose that $x_3 \in Z_{f_2}^{x_2}$. Since $k \geqslant 2$ and $\{\tilde{u}_i\} \neq \emptyset$ for all $i \in \{1,\dots,k\}$, then, by virtue of Lemma 4.6, $Z_{f_1}^{x_1} \cup \{x_1\} \subseteq \{\tilde{u}_1,\dots,\tilde{u}_k\}$. Hence, by replacing $Z_{f_1}^{x_1} \cup \{x_1\}$ by $\vec{V}(f_1,f_2,x_1)$ in f_1 and f_2, we obtain the functions:

$$f_1^3 = g(P(\tilde{u}),x_{i_2},\dots,x_{i_p}),$$

$$f_2^3 = Q(\tilde{u},x_{i_2},\dots,x_{i_p}),$$

where $\{\tilde{u}\} = \{\tilde{u}_1,\dots,\tilde{u}_k\} \backslash (Z_{f_1}^{x_1} \cup \{x_1\})$.

By Lemma 4.1 and $p \geqslant 2$, x_2 has the same ramming values in f_1 and f_2 as in f_1^3 and f_2^3 respectively. Hence $f_1^3 \neq f_2^3$ since f_1 and f_2 have different ramming values. Also it is easy to see that f_1^3 and f_2^3 depend essentially on at least two variables. Thus we have again obtained a contradiction to Lemma 7.1, and so the Case 2.1 is impossible.

Case 2.2: Now suppose that $x_3 \notin Z_{f_2}^{x_2}$. Here there are also the two possibilities:

$$x_2 \in Z_{f_2}^{x_3} \quad \text{or} \quad x_2 \notin Z_{f_2}^{x_3}.$$

We will consider separately each of them.

Case 2.2.1: Suppose that $x_2 \in Z_{f_2}^{x_3}$. It is easy to see that this case is an analogy to the Case 2.1, and so it is also impossible.

Case 2.2.2: Now suppose that $x_2 \notin Z_{f_2}^{x_3}$. In this case we can write

$$R_1'(\tilde{u}_1, \tilde{t}_1) = \varphi(\dots,R_{1i}(\tilde{u}_{1i}, x_2),\dots,R_{1j}(\tilde{u}_{1j},x_3),\dots),$$

where $\{\tilde{u}_{1i}\} \neq \emptyset$ and $\{\tilde{u}_{1j}\} \neq \emptyset$.

By Lemma 4.6 $Z_{f_2}^{x_3} \cup \{x_3\} \subseteq \{\tilde{u}_{1j},x_3\}$. Hence, by replacing $Z_{f_2}^{x_3} \cup \{x_3\}$ by $\vec{V}(f_2,f_1,x_3)$ in f_1 and f_2, we obtain the functions:

$$f_1^4 = g(\dots,P(\tilde{u}_{1i}),\dots,x_2,\dots),$$

$$f_2^4 = Q(\dots,\tilde{u}_{1i},\dots,x_2,\dots).$$

In exactly the same way as in the Case 2.1 we can show that $f_1^4 \neq f_2^4$ and these functions depend essentially on at least two variables (one can only observe that $Z_{f_2}^{x_2} \cup \{x_2\} \subseteq \{\tilde{u}_{1i},x_2\}$). Therefore the Case 2.2 is impossible.

Thus we have shown $p = 2$. Now, since $|\{\tilde{v}_i\}| = 1$ for all $i \in \{2,\dots,p\}$ and because of the monotony of f_1 we can write

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1),x_2).$$

Similarly, we can show that

$$f_2(\tilde{x}) = g(R_1'(\tilde{w}_1), x_i^\sigma).$$

To complete the proof it suffices to observe that f_1 and f_2 are different. □

For the subsequent discussion we need more concepts.

Let us recall that here *all Boolean functions depend essentially on all their variables, and differ from any constant.*

Let f be a read-once Boolean function depending on at least two variables. As shown in Lemmas 4.4 and 4.5, for such a function there is only one g-representation (in fact, of course, up to permuting terms). Using this representation, we can inductively define the depth of f.

- If $f(\tilde{x}) = g(x_1^{\sigma_1}, \ldots, x_n^{\sigma_n})$ then the depth of f is equal to 1.
- Assume that we already know what functions have depth equal to k, $k \leqslant n$.
- Let $f(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p))$. f is said to have depth equal to $n+1$ iff all functions R_i, $i \in \{1, \ldots, p\}$ have depth not greater than n, and there is at least one R_i having depth equal to n.

We will denote the depth of f by $d(f)$. Let us observe that the definition of the depth does not depend on an ordering of terms of the g-representation of a function so all equal functions have the same depth.

The following statements follow directly from the above definition.

Lemma 7.9. *If $d(f) = k$, then the function f depends essentially on at least $k+1$ variables.*

Lemma 7.10. *If the functions f and ${}^{\sigma}f^{x_i}$ depend on at least two variables and $d(f) = k$, then $d({}^{\sigma}f^{x_i}) \leqslant k$.*

For the sake of convenience we will now prefer to use a geometrical language. (We assume in this paper that the reader has some familiarity with basic concepts of Graph Theory.) Without going into details we observe only that one can establish a correspondence between a read-once Boolean function f of the depth k and a rooted tree D_f of height k in which each leaf is labelled by a literal from the set $\{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$ and each internal node is labelled by either $\wedge$ or $\vee$ in such a way that no adjacent nodes in D_f can be labelled by the same symbol. (Note that in general, $\wedge$ and $\vee$ denote here many-place conjunction and disjunction.)

We will say that a node v of D_f is a node of level k iff the distance between v and a root of D_f is equal to k.

Many statements get trivial by using the representation of read-once functions in the form trees.

Lemma 7.11. *Let $(f) = k$, $k \geqslant 2$, and let x_i, x_j be variables of f with the levels k_i, k_j respectively. If $k_i < k_j \leqslant k$ then $x_i \notin Z_f^{x_j}$.*

The proof of the two above lemmas is trivial.

Lemma 7.12. *Let $f \in K_g$, $d(f) = k$, $k \geqslant 3$, and let x_i be a variable f with the level k and f' be obtained from f by replacing x_i by its ramming value in f, and $Z_f^{x_i}$ by constants from $\{0,1\}$. Then $f' \in K_g$ and $d(f') \geqslant k-2$.*

The lemma gets trivial if we represent f in the form a tree D_f (see Fig. 1) (it is necessary to observe only that in D_f any two adjacent nodes are labelled by different symbols).

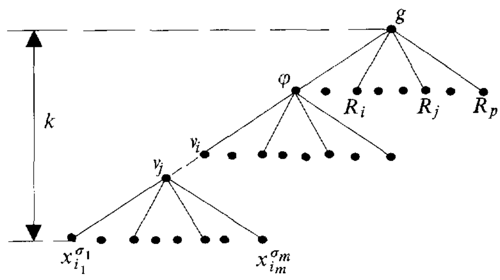

Fig. 1.

Remark. Let $\beta_1 \in \{0,1\}$ be a ramming value of x_{i_1} in f. Then as it is easy to see from Fig. 1 we obtain the variables $x_{i_2},\dots,x_{i_m}$ as fictitious ones of f by replacing x_{i_1} in f by β_1. Also k is reduced by one unit if there is at least two nodes of the level $k-1$, which differ from v_j and are adjacent to v_i, otherwise k is reduced by two units. (Recall that two nodes in D_f are adjacent iff they are joined by an edge.)

Lemma 7.13. *Let $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ be an s-function. If*

$$f_1(\tilde{x}) = g(R(x_1,\dots,x_{n-2},x_{n-1}),x_n)\,,$$
$$f_2(\tilde{x}) = g(R'(x_1,\dots,x_{n-2},x_n),x_{n-1}^{\sigma})\,, \tag{18}$$

or

$$f_1(\tilde{x}) = g(R(x_1,\dots,x_{n-2},x_{n-1}),x_n)\,,$$
$$f_2(\tilde{x}) = \varphi(R'(x_1,\dots,x_{n-2},x_n),x_{n-1}^{\sigma}) \tag{19}$$

then $d(R) \leqslant 2$ and $d(R') \leqslant 2$.

Proof. Where the proof in case of (18) is just like the proof in case of (19) we limit ourselves to the consideration the case of (18), otherwise we will consider separately each of the two cases.

Suppose the contrary. For example, let $d(R) = k$, $k \geqslant 3$. (Note that in this case $n \geqslant 3$.) We will now show that D_{f_1} is a tree of the same kind as in Fig. 2. Since $d(R) \geqslant 3$, φ-representation of R has the following form:

$$R = \varphi(g(R_{11}(\tilde{v}_{11}),\dots,R_{1r}(\tilde{v}_{1r})),R_{22}(\tilde{v}_{22}),\dots,R_{2p}(\tilde{v}_{2p}))\,,$$

where at least one set among $\{\tilde{v}_{1i}\}$, $i \in \{1,\dots,r\}$ contains no less than two elements. For the sake of definiteness, let $|\{\tilde{v}_{11}\}| \geqslant 2$ and $x_1 \in \{\tilde{v}_{11}\}$. We first show that $x_{n-1} \in \{\tilde{v}_{11}\}$. Assume $x_{n-1} \notin \{\tilde{v}_{11}\}$. Since $|\{\tilde{v}_{11}\}| \geqslant 2$, by virtue of Lemma 4.6, $Z_{f_1}^{x_1} \cup \{x_1\} \subseteq \{\tilde{v}_{11}\}$. Hence, by replacing $Z_{f_1}^{x_1} \cup \{x_1\}$ by $\vec{V}(f_1,f_2,x_1)$ in f_1 and f_2 we obtain the functions:

$$f_1^1 = g(R_1(\dots,x_1,\dots,x_{n-1}),x_n), \quad R_1 \in K_\varphi\,,$$
$$f_2^1 = g(Q_1(\dots,x_1,\dots,x_n),x_{n-1}^{\sigma})\,.$$

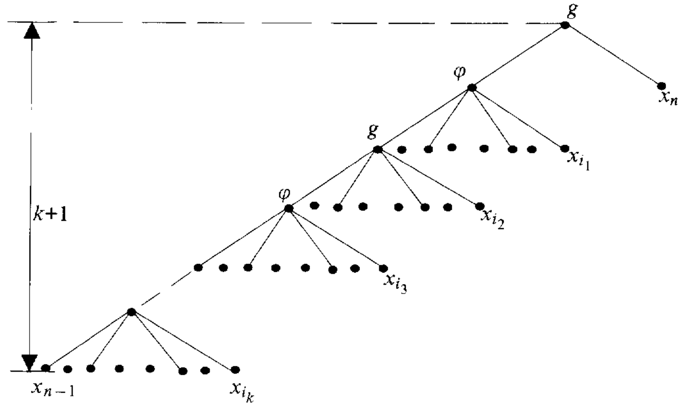

Fig. 2.

As we easily see $f_1^1 \neq f_2^1$ which contradicts Lemma 7.1. Thus $x_{n-1} \in \{\tilde{v}_{11}\}$. Similarly, we can prove that $|\{\tilde{v}_{1i}\}| \geqslant 2$ implies $x_{n-1} \in \{\tilde{v}_{1i}\}$ for $i \in \{2,\ldots,r\}$. On the other hand, we have already shown that $x_{n-1} \in \{\tilde{v}_{11}\}$ hence $|\{\tilde{v}_{1i}\}| = 1$ for all $i \in \{2,\ldots,r\}$, since $\{\tilde{v}_{11}\} \cap \{\tilde{v}_{1i}\} = \emptyset$ for $i \neq 1$.

Now we show that $|\{\tilde{v}_{2j}\}| = 1$ for all $j \in \{2,\ldots,p\}$. Suppose the contrary. Without loss of generality, we can assume that, for example, $|\{\tilde{v}_{22}\}| \geqslant 2$ and $x_2 \in \{\tilde{v}_{22}\}$. Now we can obtain different functions depending essentially on at least one variable by replacing $Z_{f_1}^{x_2} \cup \{x_2\}$ by $\vec{V}(f_1,f_2,x_2)$ in f_1 and f_2 as above.

The contradiction to Lemma 7.1 proves that $|\{\tilde{v}_{2j}\}| = 1$ for all $j \in \{2,\ldots,p\}$. This way we have proved that

$$f_1(\tilde{x}) = g(\varphi(g(R_{11}(\tilde{v}_{11}),x_{i_2},\ldots,x_{i_r}),x_{j_2},\ldots,x_{j_p}),x_n)$$

with $x_{n-1} \in \{\tilde{v}_{11}\}$.

With the aid of this equality, we can easily prove by induction on $d(f_1)$ that D_{f_1} is a tree of the same kind as in Fig. 2.

We will now show that $d(R') \leqslant 2$. Suppose the contrary.

Let $d(R') = k'$, $k' \geqslant 3$. (Recall that we have already assumed $d(R) \geqslant 3$.) In exactly the same way as for D_{f_1} we can now show that in case of (1) D_{f_1} is a tree of the same kind as in Fig. 3 and in case of (19) also as in Fig. 3 with the only exception that the root is labelled by φ, the internal node adjacent to the root is labelled by g, etc.

We will now prove that all variables of the greatest level of D_{f_1} and D_{f_2} are alike except x_{n-1} and x_n. Suppose to the contrary that there is a variable x_i different from x_{n-1} and x_n which belongs to the $(k+1)$th level of D_{f_1} and to the k_1th level of D_{f_2}, $k_1 < k'+1$. Then as we easily see from Figs. 2 and 3 $x_{n-1} \to x_i(f_1)$ and $x_{n-1} \to x_i(f_2)$. Since x_n belongs to the largest level of D_{f_2}, by virtue of Lemma 7.9, we obtain functions f_1^2 and f_2^2 depending essentially on x_{n-1} and x_i by replacing $Z_{f_2}^{x_n}$ by $\vec{V}(f_2,f_1,x_n)$ in f_1 and f_2. According to Lemma 4.1, $x_{n-1} \to x_i(f_1^2)$ and $x_{n-1} \to x_i(f_2^2)$

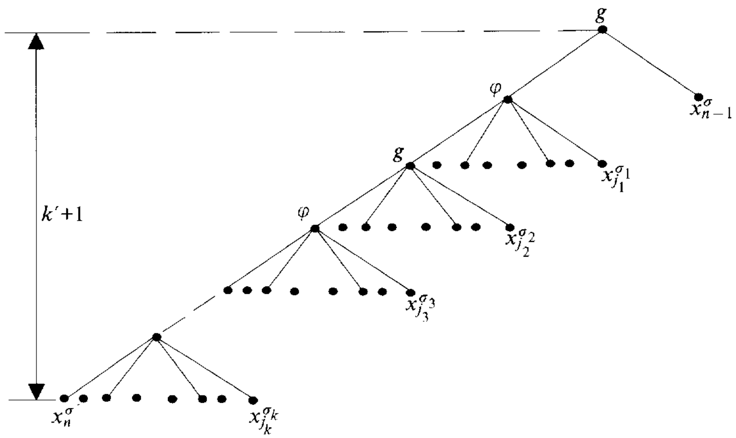

Fig. 3.

hence x_{n-1} has the same ramming values in f_1 as in f_1^2 and in f_2 as in f_2^2. So x_{n-1} has different ramming values in f_1^2 and f_2^2, since the functions f_1 and f_2 have different ramming values. Hence $f_1^2 \neq f_2^2$ which contradicts Lemma 7.1. Thus each variable of the $(k+1)$th level of D_{f_1} different from x_{n-1} belongs to the $(k'+1)$th level of D_{f_2}. Likewise we can prove that each variable of the $(k'+1)$th level of D_{f_2} different from x_n and x_{n-1} belongs to the $(k+1)$th level of D_{f_1}.

Let us now obtain functions f_1^3 and f_2^3 by replacing $Z_{f_1}^{x_{n-1}} \cup \{x_{n-1}\}$ by $\vec{V}(f_1, f_2, x_{n-1})$ in f_1 and f_2. We can easily show that both the functions depend essentially on x_n in the same manner as above. Also it follows from $d(f_1) \geqslant 4$ and Lemmas 7.7 and 7.10 that each of the functions also depend essentially on at least other variables x_i and x_j. Note that after the above replacement no more than one variable can change its ramming value to the opposite one in f_1 and f_2 (x_n in f_2, and a variable of the kth level of D_{f_1} in f_1 if it is the only variable of the kth level). Thus at least one variable has the same ramming value in f_1 as in f_1^3 and in f_2^3. But since f_1 and f_2 have different ramming values, this variable also has different ramming values in f_1^3 and in f_2^3. Hence $f_1^3 \neq f_2^3$ which contradicts Lemma 7.1. The contradiction proves that $d(R') \leqslant 2$.

Starting at this point, we consider (18) and (19) separately.

(1) Since $d(R) \geqslant 3$ and $d(R') \leqslant 2, f_1$ and f_2 can be represented in the form:

$$f_1(\tilde{x}) = g(\varphi(g(R(x_{n-1}, \tilde{v}_1), \tilde{v}_2), \tilde{v}_3), x_n),$$

$$f_2(\tilde{x}) = g(\varphi(R'_1(\tilde{w}_1), \dots, R'_q(\tilde{w}_q)), x_{n-1}^{\sigma}),$$

where $\{\tilde{v}_i\} \neq \emptyset$ for $i = 1,2,3$ and R'_j is either a variable or g for all $j \in \{1, \dots, q\}$.

Now we easily see that by replacing $Z_{f_1}^{x_{n-1}} \cup \{x_{n-1}\}$ by $\vec{V}(f_1, f_2, x_{n-1})$ in f_1 and f_2, we obtain the function

$$f_1^4 = g(R_1(\tilde{v}), x_n) \quad \text{with} \quad |\{\tilde{v}\}| \geqslant 2, \quad R_1 \in K_{\varphi}$$

and a function

$$f_2^4 \text{ such that either } f_2^4 \in K_\varphi \text{ or } f_2^4 = g\,.$$

In either of the cases $f_1^4 \neq f_2^4$, which contradicts Lemma 7.1, so $d(R) \leqslant 2$.

(2) This case needs a longer proof.

First, we show that $d(R') \geqslant k-1$. Let us replace $Z_{f_1}^{x_{n-1}} \cup \{x_{n-1}\}$ by $\vec{V}(f_1, f_2, x_{n-1})$ in f_1 and f_2. One readily checks that in this way we obtain functions f_1^5 and f_2^5 depending essentially on at least three variables. Since x_{n-1} belongs to the $(k+1)$th level of D_{f_1}, by virtue of Lemma 7.10, $d(f_1^5) \geqslant k-1$. But according to Lemma 7.1 $f_1^5 = f_2^5$, since f is an s-function. Hence $d(f_2^5) = d(f_1^5) \geqslant k-1$. Moreover it is easy to see that f_2^5 is a subfunction of R' hence, by virtue of Lemma 7.8, $d(R') \geqslant k-1$.

We will now show that $d(R) < 4$. Assume $d(R) \geqslant 4$. Then $d(R') \geqslant 3$, since $d(R') \geqslant k-1$. (Recall that $d(R) = k, k \geqslant 3$.) On the other hand, we have already shown that $d(R') \leqslant 2$. The contradiction proves that $d(R) = k$, $k < 4$. Thus we have shown that this is the only possibility:

$$d(R) = 3 \quad \text{and} \quad d(R') = 2\,.$$

Taking this into account, we can write

$$f_1(\tilde{x}) = g(\varphi(g(\varphi(x_{n-1}, \tilde{v}_1), \tilde{v}_2), \tilde{v}_3), x_n)\,,$$

$$f_2(\tilde{x}) = g(\varphi(R_1'(\tilde{w}_1), \ldots, R_q'(\tilde{w}_q)), \bar{x}_{n-1})\,,$$

where $\{\tilde{v}_i\} \neq \emptyset$ for $i = 1,2,3$ and $q \geqslant 2$.

(Here the last equality contains $\bar{x}_{n-1}$ because f_1 and f_2 have different ramming values and f_1 is monotone.)

Let $x_1 \in \{\tilde{w}_1\} \mathring{\cup} \cdots \mathring{\cup} \{\tilde{w}_q\}$. Then we obtain different functions depending essentially on x_{n-1} from f_1 and f_2 by replacing $Z_{f_1}^{x_{n-1}} \cup \{x_{n-1}\}$ by $\vec{V}(f_1, f_2, x_{n-1})$ in f_1 and f_2. The contradiction to Lemma 7.1 proves that $d(R') \leqslant 2$. □

Lemma 7.14. *Let $f(y, \tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function with $\{\tilde{x}\} = \{x_1, \ldots, x_n\}$ $(n \geqslant 3)$ and*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p))\,,$$

$$f_2(\tilde{x}) = g(R_1'(\tilde{w}_1), \ldots, R_q'(\tilde{w}_q))\,.$$

If $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\}$ and $\pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$ are uncomparable partitions of $\{\tilde{x}\}$, then (f_1, f_2) may be one-type to only one of the pairs:

$$(x_1 \ldots x_{n-2} x_{n-1} \vee x_n, \bar{x}_1 \ldots \bar{x}_{n-2} x_n \vee x_{n-1}), \quad n \geqslant 3\,,$$

$$(x_1 \ldots x_{n-2} x_{n-1} \vee x_n, (x_1 \vee \cdots \vee x_{n-2})\, x_n \vee x_{n-1}), \quad n \geqslant 4\,,$$

$$(x_1 \vee \cdots \vee x_{n-2})\, x_{n-1} \vee x_n, (\bar{x}_1 \vee \cdots \vee \bar{x}_{n-2})\, x_n \vee x_{n-1}), \quad n \geqslant 4\,,$$

for all n given above.

Proof. According to Lemma 7.8, we have

$$\pi_1 = \{\{x_1,\dots,x_{n-2},x_{n-1}\},\{x_n\}\} \quad \text{and} \quad \pi_2 = \{\{x_1,\dots,x_{n-2},x_n\},\{x_{n-1}\}\}.$$

(To be more precise, note that π_1 and π_2 can differ from them in permuting indices. But without loss of generality we can assume that π_1 and π_2 are the same as above.) This implies that

$$f_1(\tilde{x}) = g(R(x_1,\dots,x_{n-2},x_{n-1}),x_n),$$

$$f_2(\tilde{x}) = g(R'(x_1,\dots,x_{n-2},x_n),x_{n-1}^{\sigma}). \tag{20}$$

On the other hand, according to Lemma 7.13, $d(R) \leqslant 2$ and $d(R') \leqslant 2$. So, in order to prove the lemma it suffices to consider only the three possibilities:

$$d(R) = d(R') = 1; \quad d(R) = 1, \quad d(R') = 2 \quad \text{and} \quad d(R) = d(R') = 2.$$

We will now consider each of them taken separately.

Case 1: Suppose that $d(R) = d(R') = 1$. Because of (20) we have

$$f_1(\tilde{x}) = g(\varphi(x_1,\dots,x_{n-2},x_{n-1}),x_n),$$

$$f_2(\tilde{x}) = g(\varphi(\bar{x}_1,\dots,\bar{x}_{n-2},x_n),x_{n-1}), \quad n \geqslant 3.$$

(Recall that f_1 and f_2 have different ramming values and f_1 is monotone, so f_1 contains $x_1,\dots,x_{n-2}$ and f_2 contains $\bar{x}_1,\dots,\bar{x}_{n-2}$).

Case 2: Now suppose that $d(R) = 1$, $d(R') = 2$. Here, by (20), there are two more possibilities. First

$$f_1(\tilde{x}) = g(\varphi(\tilde{z}_1,\tilde{z}_2,x_{n-1}),x_n),$$

$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z}_1,\bar{x}_n),\psi_1(\tilde{z}_2)),x_{n-1}), \tag{21}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{x_{n-1},x_n\}$ and $|\{\tilde{z}_1,\tilde{z}_2,x_{n-1}\}| \geqslant 2$.

As it is easy to see we can obtain the functions: x_n and $\bar{x}_n$ by replacing $\tilde{z}_1,\tilde{z}_2,x_{n-1}$ in f_1 and f_2 by $\vec{V}(f_1,f_2,x_{n-1})$. Since the functions are different, we have a contradiction to Lemma 7.1 which shows that this case is impossible. And then

$$f_1(\tilde{x}) = g(\varphi(\tilde{z}_1,\tilde{z}_2,x_{n-1}),x_n),$$

$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z}_1,\psi_2(\tilde{z}_2,x_n)),x_{n-1}), \tag{22}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{x_{n-1}\} \mathring{\cup} \{x_n\}$ and $|\{\tilde{z}_1\}| \geqslant 2$.

We will now show that $\{\tilde{z}_2\} = \emptyset$. For the sake of definiteness, suppose $x_1 \in \{\tilde{z}_1\}$. One can easily check that the functions:

$$f_1^1 = g(\varphi(\tilde{z}_2,x_{n-1}),x_n),$$

$$f_2^1 = g(\psi_2(\tilde{z}_2,x_n),x_{n-1})$$

will be obtained from f_1 and f_2 by replacing $\tilde{z}_1$ in these functions by $\vec{V}(f_2,f_1,x_1)$.

If $\{\tilde{z}_2\} \neq \emptyset$ then $f_1^1 \neq f_2^1$ which contradicts Lemma 7.1, so $\{\tilde{z}_2\} = \emptyset$. Thus

$$f_1(\tilde{x}) = g(\varphi(\tilde{z}_1, x_{n-1}), x_n),$$
$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z}_1), x_n), x_{n-1}) \quad \text{with} \mid \{\tilde{z}_1\} \mid \geqslant 2.$$

Case 3: Now suppose that $d(R) = d(R') = 2$. For $\tilde{x}_i = (x_{i_1}, \ldots, x_{i_k})$, denote by $\tilde{x}_i'$ the vector $(\bar{x}_{i_1}, \ldots, \bar{x}_{i_k})$. By (1) there are three more possibilities.

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2), x_{n-1}), \psi_{11}(\tilde{z}_3, \tilde{z}_4)), x_n),$$
$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z'}_1, \tilde{z}_3, \bar{x}_n), \psi_{12}(\tilde{z}_2, \tilde{z}_4)), \bar{x}_{n-1}), \tag{23}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{\tilde{z}_3\} \mathring{\cup} \{\tilde{z}_4\} \mathring{\cup} \{x_{n-1}, x_n\}$ and $\mid \{\tilde{z}_1, \tilde{z}_2, x_{n-1}\} \mid \geqslant 2$.

Clearly we can obtain the functions:

$$f_1^2 = g(\psi_{11}(\tilde{z}_3, \tilde{z}_4), x_n),$$
$$f_2^2 = \varphi(g(\tilde{z}_3, \bar{x}_n), \psi'_{12}(\tilde{z}_4))$$

from f_1 and f_2 by replacing $\tilde{z}_1$, $\tilde{z}_2$, x_{n-1} by $\vec{V}(f_1, f_2, x_{n-1})$. As it is easy to see $f_1^2 \neq f_2^2$. The contradiction to Lemma 7.1 shows that this case is impossible.

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2), \psi_{21}(\tilde{z}_3, \tilde{z}_4, x_{n-1})), x_n),$$
$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z'}_1, \tilde{z}_3, \bar{x}_n), \psi_{22}(\tilde{z}_2, \tilde{z}_4)), x_{n-1}), \tag{24}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{\tilde{z}_3\} \mathring{\cup} \{\tilde{z}_4\} \mathring{\cup} \{x_{n-1}, x_n\}$ and $\mid \{\tilde{z}_1, \tilde{z}_2,\} \mid \geqslant 2$. Let $x_1 \in \{\tilde{z}_1, \tilde{z}_2\}$. Let us replace $\tilde{z}_1, \tilde{z}_2$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_1)$. As a result, we obtain the following functions:

$$f_1^3 = g(\psi_{21}(\tilde{z}_3, \tilde{z}_4, x_{n-1}), x_n),$$
$$f_2^3 = g(\varphi(\tilde{z}_3, \bar{x}_n), \psi_{22}'(\tilde{z}_4)), x_{n-1}).$$

Since one of f_1^3, f_2^3 is a monotone function, but the other is not monotone, as it is easy to see, $f_1^3 \neq f_2^3$. The contradiction to Lemma 7.1 shows that this case is also impossible.

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2), \psi_{31}(\tilde{z}_3, \tilde{z}_4, x_{n-1})), x_n),$$
$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z'}_1, \tilde{z}_3,), \psi_{32}(\tilde{z}_2, \tilde{z}_4, x_n)), x_{n-1}), \tag{25}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{\tilde{z}_3\} \mathring{\cup} \{\tilde{z}_4\} \mathring{\cup} \{x_{n-1}, x_n\}$, $\mid \{\tilde{z}_1, \tilde{z}_2,\} \mid \geqslant 2$, and $\mid \{\tilde{z}_1, \tilde{z}_3,\} \mid \geqslant 2$.

First, we show that $\{\tilde{z}_3\} = \emptyset$. Assume $\{\tilde{z}_3\} \neq \emptyset$. We have already assumed above that $x_1 \in \{\tilde{z}_1, \tilde{z}_2\}$. In this case, by replacing $\tilde{z}_1, \tilde{z}_2$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_1)$, we clearly obtain the following functions:

$$f_1^4 = g(\psi_{31}(\tilde{z}_3, \tilde{z}_4, x_{n-1}), x_n),$$
$$f_2^4 = g(\varphi(g(\tilde{z}_3,), \psi'_{32}(\tilde{z}_4, x_n)), x_{n-1}).$$

Since, according to the above assumption, $\{\tilde{z}_3\} \neq \emptyset, f_1^4 \neq f_2^4$ which contradicts Lemma 7.1. Therefore $\{\tilde{z}_3\} = \emptyset$. We will now show that $\{\tilde{z}_2\} = \emptyset$ as well. Assume $\{\tilde{z}_2\} \neq \emptyset$. Let $x_2 \in \{\tilde{z}_1\}$. Since $\{\tilde{z}_3\} = \emptyset$, we obtain the functions:

$$f_1^5 = g(\varphi(g(\tilde{z}_2), \psi'_{31}(\tilde{z}_4, x_{n-1})), x_n),$$

$$f_2^5 = g(\psi_{32}(\tilde{z}_2, \tilde{z}_4, x_n)), x_{n-1})$$

by replacing $\tilde{z}_1$ in f_1 and f_2 by $\vec{V}(f_2, f_1, x_2)$. Since $\{\tilde{z}_2\} \neq \emptyset$, $f_1^5 \neq f_2^5$ which contradicts Lemma 7.1, so $\{\tilde{z}_2\} = \emptyset$. Since $\{\tilde{z}_2\} = \{\tilde{z}_3\} = \emptyset$, we can write

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1), \psi_{31}(\tilde{z}_4, x_{n-1})), x_n),$$

$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z'}_1), \psi_{32}(\tilde{z}_4, x_n)), x_{n-1}) \quad \text{with} \, |\{\tilde{z}_1\}| \geqslant 2.$$

We will now show that $\{\tilde{z}_4\} = \emptyset$. Assume $\{\tilde{z}_4\} \neq \emptyset$. Then, by replacing $\tilde{z}_1$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_2)$, we obtain the functions:

$$f_1^6 = g(\psi_{31}(\tilde{z}_4, x_{n-1}), x_n),$$

$$f_2^6 = g(\psi_{32}(\tilde{z}_4, x_n), x_{n-1}).$$

Since f is an s-function, by virtue of Lemma 7.1, $f_1^6 = f_2^6$ but this equality is possible only if

$$\psi_{31}(\tilde{z}_4, x_{n-1}) = g(P(\tilde{z}_4), x_{n-1}) \quad \text{and} \quad \psi_{32}(\tilde{z}_4, x_n) = g(P(\tilde{z}_4), x_n).$$

Hence both x_{n-1} and x_n have the same ramming value in f_1 and f_2, but, according to the above assumption, f_1 and f_2 have different ramming values. The contradiction proves that $\{\tilde{z}_4\} = \emptyset$. This way we have shown that

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1), x_{n-1}), x_n),$$

$$f_2(\tilde{x}) = g(\varphi(g(\tilde{z'}_1), x_n), x_{n-1}) \quad \text{with} \, |\{\tilde{z}_1\}| \geqslant 2. \qquad \square$$

Lemma 7.15. *Let* $f(y, \tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ *be an s-function with* $\{\tilde{x}\} = \{x_1, \ldots, x_n\}$ $(n \geqslant 3)$ *and*

$$f_1(\tilde{x}) = g(R(x_1, \ldots, x_{n-2}, x_{n-1}), x_n),$$

$$f_2(\tilde{x}) = \varphi(R'(x_1, \ldots, x_{n-2}, x_n), x_{n-1}^{\sigma}).$$

Then (f_1, f_2) *may be one-type only to one of the pairs*

$$(x_1 \cdots x_{n-2} x_{n-1} \vee x_n, (x_1 \vee \cdots \vee x_{n-2} \vee \bar{x}_n)\bar{x}_{n-1}), \quad n \geqslant 3,$$

$$((x_1 \vee \cdots \vee x_{n-3} \vee x_{n-1}) x_{n-2} \vee x_n, (x_1 \cdots x_{n-3} x_n \vee x_{n-2}) x_{n-1}), \quad n \geqslant 4.$$

Proof. According to Lemma 7.13, $d(R) \leqslant 2$ and $d(R') \leqslant 2$. There are the four possibilities:

$$d(R) = d(R') = 1; \quad d(R) = 1, \quad d(R') = 2; \text{or}$$

$$d(R) = 2, \quad d(R') = 1; \quad d(R) = d(R') = 2.$$

We will consider each of them taken separately.

Case 1: Suppose that $d(R) = d(R') = 1$. Having recalled that f_1 and f_2 have different ramming values we can write

$$f_1(\tilde{x}) = g(\varphi(x_1, \ldots, x_{n-2}, x_{n-1}), x_n),$$

$$f_2(\tilde{x}) = \varphi(g(x_1, \ldots, x_{n-2}, \bar{x}_n), \bar{x}_{n-1}).$$

Case 2: Suppose that $d(R) = 1, d(R') = 2$. In this case we must have

$$f_1(\tilde{x}) = g(\varphi(\tilde{z}, x_{n-1}), x_n),$$

$$f_2(\tilde{x}) = \varphi(R'(\tilde{z}, x_n), \bar{x}_{n-1}),$$

where $\{\tilde{x}\} = \{\tilde{z}\} \mathring{\cup} \{x_{n-1}, x_n\}$.

Since $d(R') = 2$, $|\{\tilde{z}, x_n\}| \geqslant 2$ and, as follows from Lemma 4.6, $Z_{f_2}^{x_n} \cup \{x_n\} \subseteq \{\tilde{z}, x_n\}$, so, by replacing $Z_{f_2}^{x_n} \cup \{x_n\}$ in f_1 and f_2 by $\vec{V}(f_2, f_1, x_n)$, we obtain the functions:

$$f_1^1 = \varphi(\tilde{z}_1, x_{n-1}) \quad \text{and} \quad f_2^1 = \varphi(P(\tilde{z}_1), \bar{x}_{n-1}),$$

where $\{\tilde{z}_1\} = \{\tilde{z}\} \backslash (Z_{f_2}^{x_n} \cup \{x_n\})$.

It is easy to see that $f_1^1 \neq f_2^1$ which contradicts Lemma 7.1. Thus this case is impossible.

We can show that the case $d(R) = 2, d(R') = 1$ is also impossible in the same way as in the preceding case.

Case 4: Now suppose that $d(R) = d(R') = 2$. Here we must consider separately four more possibilities.

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2, x_{n-1}), \psi_{11}(\tilde{z}_3, \tilde{z}_4)), x_n),$$

$$f_2(\tilde{x}) = \varphi(g(\varphi(\tilde{z}_1, \tilde{z}_3, x_n), \psi_{12}(\tilde{z}_2, \tilde{z}_4)), x_{n-1}), \tag{26}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{\tilde{z}_3\} \mathring{\cup} \{\tilde{z}_4\} \mathring{\cup} \{x_{n-1}, x_n\}$, $|\{\tilde{z}_1, \tilde{z}_2, x_{n-1}\}| \geqslant 2$, and $|\{\tilde{z}_1, \tilde{z}_3, x_n\}| \geqslant 2$.

First we will show that $\{\tilde{z}_2\} = \emptyset$. Let us replace $\tilde{z}_1, \tilde{z}_3, x_n$ in f_1 and f_2 by $\vec{V}(f_2, f_1, x_n)$. As a result, we have obtained the functions:

$$f_1^2 = \varphi(g(\tilde{z}_2, x_{n-1}), \psi'_{11}(\tilde{z}_4)),$$

$$f_2^2 = \varphi(\psi_{12}(\tilde{z}_2, \tilde{z}_4)), x_{n-1}).$$

Obviously if $\{\tilde{z}_2\} \neq \emptyset$ then $f_1^2 \neq f_2^2$ which contradicts Lemma 7.1, hence $\{\tilde{z}_2\} = \emptyset$.

Likewise, by replacing $\tilde{z}_1, \tilde{z}_2, x_{n-1}$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_{n-1})$, we can show that $\{\tilde{z}_3\} = \emptyset$. It follows from $\{\tilde{z}_3\} = \emptyset$ that $\{\tilde{z}_4\} \neq \emptyset$, otherwise $\{\tilde{z}_3, \tilde{z}_4\} = \emptyset$, and so $d(R) = 1$, while $d(R) = 2$ here.

We will now show that $|\{\tilde{z}_4\}| = 1$. Since $\{\tilde{z}_2\} = \{\tilde{z}_3\} = \emptyset$, we can obtain the functions:

$$f_1^3 = g(\psi_{11}(\tilde{z}_4), x_n) \quad \text{and} \quad f_2^3 = g(\psi_{12}(\tilde{z}_4), x_n)$$

by replacing $\tilde{z}_1, x_{n-1}$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_{n-1})$. Since f is an s-function, by virtue of Lemma 7.1, $f_1^3 = f_2^3$. Hence $\psi_{11}(\tilde{z}_4) = \psi_{12}(\tilde{z}_4)$ and so, as follows from Lemma 7.2, $|\{\tilde{z}_4\}| = 1$. Thus we have shown that

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, x_{n-1}), x_{n-2}), x_n),$$

$$f_2(\tilde{x}) = \varphi(g(\varphi(\tilde{z}_1, x_n), x_{n-2}), x_{n-1}) \quad \text{with} \{\tilde{z}_1\} \neq \emptyset .$$

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2, x_{n-1}), \psi_{21}(\tilde{z}_3, \tilde{z}_4)), x_n),$$

$$f_2(\tilde{x}) = \varphi(g(\varphi(\tilde{z}_1, \tilde{z}_3), \psi_{22}(\tilde{z}_2, \tilde{z}_4, x_n)), x_{n-1}), \tag{27}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{\tilde{z}_3\} \mathring{\cup} \{\tilde{z}_4\} \mathring{\cup} \{x_{n-1}, x_n\}$ and $|\{\tilde{z}_1, \tilde{z}_3\}| \geqslant 2$.

Let $x_1 \in \{\tilde{z}_1, \tilde{z}_3\}$. Since $|\{\tilde{z}_1, \tilde{z}_3\}| \geqslant 2$, by replacing $\tilde{z}_1, \tilde{z}_3$, in f_1 and f_2 by $\vec{V}(f_2, f_1, x_1)$, we obtain the functions:

$$f_1^4 = g(\varphi(g(\tilde{z}_2, x_{n-1}), \psi'_{21}(\tilde{z}_4)), x_n),$$

$$f_2^4 = \varphi(g(\psi_{22}(\tilde{z}_2, \tilde{z}_4, x_n)), x_{n-1}).$$

By Lemma 4.3 $f_1^4 \neq f_2^4$ which conradicts Lemma 7.1. Therefore this case is impossible.

We have two possibilities left:

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2), \psi_{31}(\tilde{z}_3, \tilde{z}_4, x_{n-1})), x_n),$$

$$f_2(\tilde{x}) = \varphi(g(\varphi(\tilde{z}_1, \tilde{z}_3), \psi_{32}(\tilde{z}_2, \tilde{z}_4, x_n)), x_{n-1}), \tag{28}$$

and

$$f_1(\tilde{x}) = g(\varphi(g(\tilde{z}_1, \tilde{z}_2), \psi_{41}(\tilde{z}_3, \tilde{z}_4, x_{n-1})), x_n),$$

$$f_2(\tilde{x}) = \varphi(g(\varphi(\tilde{z}_1, \tilde{z}_3, x_n), \psi_{42}(\tilde{z}_2, \tilde{z}_4)), x_{n-1}), \tag{29}$$

where $\{\tilde{x}\} = \{\tilde{z}_1\} \mathring{\cup} \{\tilde{z}_2\} \mathring{\cup} \{\tilde{z}_3\} \mathring{\cup} \{\tilde{z}_4\} \mathring{\cup} \{x_{n-1}, x_n\}$ and $|\{\tilde{z}_1, \tilde{z}_2\}| \geqslant 2$.

We can show that the last two cases are impossible in the same way as in (2).

Thus the lemma is proved. □

Lemma 7.16. *Let $f(y, \tilde{x}) = \bar{y} f_1(\tilde{x}) \vee y f_2(\tilde{x})$ be an s-function with $\{\tilde{x}\} = \{x_1, \ldots, x_n\}$ $(n \geqslant 2)$ and*

$$f_1(\tilde{x}) = g(R_1(\tilde{v}_1), \ldots, R_p(\tilde{v}_p)),$$

$$f_2(\tilde{x}) = \varphi(R'_1(\tilde{w}_1), \ldots, R'_q(\tilde{w}_q)).$$

If $\pi_1 = \{\{\tilde{v}_i\} \mid i \in \{1, \ldots, p\}\}$ and $\pi_2 = \{\{\tilde{w}_i\} \mid i \in \{1, \ldots, q\}\}$ are uncomparable partitions of $\{\tilde{x}\}$, then (f_1, f_2) may be one-type to only one of the pairs:

$$(x_1 \cdots x_{n-2} x_{n-1} \vee x_n, (x_1 \cdots x_{n-2} \vee \bar{x}_n) \bar{x}_{n-1}), \quad n \geqslant 3,$$

$$((x_1 \vee \cdots \vee x_{n-3} \vee x_{n-1}) x_{n-2} \vee x_n, (x_1 \cdots x_{n-3} x_n \vee x_{n-2}) x_{n-1}), \quad n \geqslant 4,$$

$$(x_1 x_2 \vee x_3 x_4, (x_1 \vee x_3)(x_2 \vee x_4)).$$

Proof. According to Lemma 4.2, we can assume, for example, that

$$\{\tilde{v}_1\} = \{\tilde{u}_1\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{u}_k\},$$
$$\{\tilde{w}_i\} = \{\tilde{u}_i\}\mathring{\cup}\{\tilde{t}_i\} \tag{30}$$

for some nonempty, disjoint sets $\{\tilde{u}_i\}$ and some sets $\{\tilde{t}_i\}$, $i = 1,\ldots,k \geqslant 2$, also at least one of $\{\tilde{t}_i\}$ is nonempty and $\{\tilde{t}_i\}\cap\{\tilde{u}_j\} = \emptyset$ for all $i,j \in \{1,\ldots,k\}$.

Taking this into account, we can write

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1,\ldots,\tilde{u}_k), R_2(\tilde{v}_2)\ldots,R_p(\tilde{v}_p)),$$
$$f_2(\tilde{x}) = \varphi(R'_1(\tilde{u}_1,\tilde{t}_1),\ldots,R'_k(\tilde{u}_k,\tilde{t}_k), R'_{k+1}(\tilde{w}_{k+1}),\ldots,R'_q(\tilde{w}_q)).$$

Without loss of generality, one can assume that $\{\tilde{t}_1\} \neq \emptyset$. First we will show that $k = q$. Assume $q > k$. We now show that in this case

$$\{\tilde{t}_1\} \neq \{\tilde{v}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{v}_p\}.$$

Consider the equality:

$$\{\tilde{u}_1\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{u}_k\}\mathring{\cup}\{\tilde{t}_1\}\mathring{\cup}\{\tilde{t}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{t}_k\}\mathring{\cup}\{\tilde{w}_{k+1}\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{w}_q\}$$
$$= \{\tilde{u}_1\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{u}_k\}\mathring{\cup}\{\tilde{v}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{v}_p\}$$

in which both the expressions on either side of = are a partition of $\{\tilde{x}\}$.

If $\{\tilde{t}_1\} = \{\tilde{v}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{v}_p\}$. Then, by removing equal sets from both members of the equality, we obtain the equality

$$\{\tilde{t}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{t}_k\}\mathring{\cup}\{\tilde{w}_{k+1}\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{w}_q\} = \emptyset$$

which yields $\{\tilde{w}_i\} = \emptyset$ for all $i \in \{k+1,\ldots,q\}$. However, according to the definition, $\{\tilde{w}_i\} \neq \emptyset$ for all $i \in \{1,\ldots,q\}$. The contradiction proves that

$$\{\tilde{t}_1\} \neq \{\tilde{v}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{v}_p\}.$$

Let $x_1 \in \{\tilde{u}_1,\tilde{t}_1\}$. Since $\{\tilde{u}_1\} \neq \emptyset$ and $\{\tilde{t}_1\} \neq \emptyset$, $|\{\tilde{u}_1,\tilde{t}_1\}| \geqslant 2$ hence, by virtue of Lemma 4.6, $Z^{x_1}_{f_2}\cup\{x_1\} \subseteq \{\tilde{u}_1,\tilde{t}_1\}$ so, by replacing $Z^{x_1}_{f_2}\cup\{x_1\}$ in f_1 and f_2 by $\vec{V}(f_2,f_1,x_1)$, we obtain the functions

$$f_1^1 = g(R_1(\tilde{u}_{11},\tilde{u}_2,\ldots,\tilde{u}_k), P_2(\tilde{v}_{21})\ldots,P_p(\tilde{v}_{p1})),$$

and

$$f_2^1 = \varphi(Q(\tilde{u}_{11},\tilde{t}_{11}),R'_2(\tilde{u}_2,\tilde{t}_2),\ldots,R'_k(\tilde{u}_k,\tilde{t}_k), R'_{k+1}(\tilde{w}_{k+1}),\ldots,R'_q(\tilde{w}_q)).$$

where $\{\tilde{u}_{11}\} = \{\tilde{u}_1\}\backslash(Z^{x_1}_{f_2}\cup\{x_1\})$, $\{\tilde{t}_{11}\} = \{\tilde{t}_1\}\backslash(Z^{x_1}_{f_2}\cup\{x_1\})$, and $\{\tilde{v}_{i1}\} = \{\tilde{v}_i\}\backslash(Z^{x_1}_{f_2}\cup\{x_1\})$, $i = 2,\ldots,p$.

It follows from $\{\tilde{t}_1\}\cap\{\tilde{u}_i\} = \emptyset$ for all $i \in \{1,\ldots k\}$, $\{\tilde{t}_1\} \neq \{\tilde{v}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{v}_p\}$, and $\{\tilde{u}_1\}\cap\{\tilde{u}_i\} = \emptyset$ for $i \in \{2,\ldots k\}$ that $f_1^1 \neq f_2^1$ which contradicts Lemma 7.1. Therefore $k = q$.

Thus we can write that

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1,\dots,\tilde{u}_k), R_2(\tilde{v}_2)\dots, R_p(\tilde{v}_p)),$$
$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{u}_1,\tilde{t}_1),\dots,R_k'(\tilde{u}_k,\tilde{t}_k)).$$

There are only two possibilities:
- the set $\{\tilde{t}_1\}$ is the only nonempty set among $\{\tilde{t}_i\}$, $i \in \{1,\dots,k\}$, or
- there is another nonempty set among $\{\tilde{t}_i\}$ different from $\{\tilde{t}_1\}$.

We will consider separately each of these cases.

Case 1: Suppose that the set $\{\tilde{t}_1\}$ is the only nonempty set among $\{\tilde{t}_i\}, i \in \{1,\dots,k\}$. In this case we have

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1,\dots,\tilde{u}_k), R_2(\tilde{v}_2)\dots, R_p(\tilde{v}_p)),$$
$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{u}_1,\tilde{t}_1), R_2'(\tilde{u}_2),\dots,R_k'(\tilde{u}_k)).$$

Since $\{\tilde{v}_i\} \cap \{\tilde{u}_j\} = \emptyset$ for all $i \in \{2,\dots p\}$ and $j \in \{1,\dots,k\}$

$$\{\tilde{t}_1\} = \{\tilde{v}_2\} \mathring{\cup} \cdots \mathring{\cup} \{\tilde{v}_p\}.$$

We now show that $|\{\tilde{v}_i\}| = 1$ for all $i \in \{2,\dots p\}$. Suppose, for some $j \in \{2,\dots p\}$, $|\{\tilde{v}_j\}| \geqslant 2$. Then it follows from $\{\tilde{v}_j\} \subseteq \{\tilde{t}_1\}$ that, by virtue of Lemma 7.3, $R_1(\tilde{u}_1,\dots,\tilde{u}_k) = x_i$, i.e., $k = 1$ which contradicts Lemma 4.2, since π_1 and π_2 are uncomparable. Therefore $|\{\tilde{v}_i\}| = 1$ for all $i \in \{2,\dots,p\}$.

We can also show that $|\{\tilde{u}_j\}| = 1$ for all $j \in \{2,\dots,k\}$ in the same manner as above.

Thus we have proved that

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1, x_{i_2}\dots, x_{i_k}), x_{j_2},\dots,x_{j_p}),$$
$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{u}_1, x_{j_2},\dots,x_{j_p}), x_{i_2}^{\sigma_2}\dots,x_{i_k}^{\sigma_k}).$$

We will now show that $p = 2$. Assume $p > 2$. Here we must consider separately the two possibilities:

$$x_{j_2} \in Z_{f_2}^{x_{j_3}} \quad \text{and} \quad x_{j_2} \notin Z_{f_2}^{x_{j_3}}.$$

Case 1.1: Suppose that $x_{j_2} \in Z_{f_2}^{x_{j_3}}$. Since $\{\tilde{u}_1\} \neq \emptyset$ and $k > 2$, by Lemma 4.6 we have $Z_{f_1}^{x_{i_2}} \cup \{x_{i_2}\} \subseteq \{\tilde{u}_1, x_{i_2},\dots,x_{i_k}\}$ so, by replacing $Z_{f_1}^{x_{i_2}} \cup \{x_{i_2}\}$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_{i_2})$, we obtain functions: f_1^2 and f_2^2 depending essentially on $x_{j_2},\dots,x_{j_p}$. It is easy to see that $x_{j_2} \in Z_{f_1}^{x_{j_3}}$. So, by virtue of Lemma 4.1, x_{j_3} has the same ramming value in f_1 as in f_1^2. On the other hand, since $x_{j_2} \in Z_{f_2}^{x_{j_3}}, x_{j_3}$ has the same ramming value in f_2 as in f_2^2. Hence x_{j_3} has different ramming values in f_1^2 and f_2^2 because f_1 and f_2 have different ramming values. Therefore $f_1^2 \neq f_2^2$ which contradicts Lemma 7.1. Thus we have shown that $p = 2$ in this case.

Case 1.2: Now suppose that $x_{j_2} \notin Z_{f_2}^{x_{j_3}}$. Then by replacing $Z_{f_2}^{x_{j_3}} \cup \{x_{j_3}\}$ in f_1 and f_2 by $\vec{V}(f_2, f_1, x_{j_3})$, we can obtain the functions:

$$f_1^3 = g(\dots, x_{i_2},\dots,x_{j_2},\dots) \quad \text{and} \quad f_2^3 = \varphi(\dots,x_{i_2},\dots,x_{j_2},\dots)$$

which, by Lemma 4.3, are different. The contradiction to Lemma 7.1 shows that $p = 2$ in this case as well.

Similarly, one can show that $k = 2$. Thus we can write that

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1, x_{i_2}), x_{j_2}),$$

$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{u}_1, x_{j_2}), x_{i_2}^{\sigma_2}).$$

Then, by virtue of Lemma 7.15, (f_1, f_2) may be one-type to only one of the pairs:

$$(x_1 \dots x_{n-2}x_{n-1} \vee x_n, (x_1 \vee \dots \vee x_{n-2} \vee \bar{x}_n)\bar{x}_{n-1}), \quad n \geqslant 3,$$

$$((x_1 \vee \dots \vee x_{n-3} \vee x_{n-1})x_{n-2} \vee x_n, (x_1 \dots x_{n-3}x_n \vee x_{n-2})x_{n-1}), \quad n \geqslant 4.$$

Case 2: Suppose that there is another nonempty set among $\{\tilde{t}_i\}$ different from $\{\tilde{t}_1\}$. (Note that, according to our assumption $\{\tilde{t}_1\} \neq \emptyset$.) For the sake of definiteness, let $\{\tilde{t}_2\} \neq \emptyset$. This implies

$$\{\tilde{t}_1\} \neq \{\tilde{v}_2\} \mathring{\cup} \cdots \mathring{\cup} \{\tilde{v}_p\}.$$

We now show that $p = 2$. Assume $p > 2$. Let $x_2 \in \{\tilde{u}_1, \dots, \tilde{u}_k\}$. Since $\{\tilde{u}_1\} \neq \emptyset$, $\dots, \{\tilde{u}_k\} \neq \emptyset$, and $k \geqslant 2$, by virtue of Lemma 4.6, $Z_{f_1}^{x_2} \cup \{x_2\} \subseteq \{\tilde{u}_1, \dots, \tilde{u}_k\}$. Hence, by replacing $Z_{f_1}^{x_2} \cup \{x_2\}$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_2)$, we obtain the functions:

$$f_1^4 = g(P(\tilde{u}), R_2(\tilde{v}_2) \dots, R_p(\tilde{v}_p)),$$

$$f_2^4 = \varphi(P_1(\tilde{u}_{11}, \tilde{t}_1), \dots, P_k(\tilde{u}_{1k}, \tilde{t}_k)),$$

where $\{\tilde{u}\} = \{\tilde{u}_1, \dots, \tilde{u}_k\}$ and $\{\tilde{u}_{1i}\} = \{\tilde{u}_i\} \backslash (Z_{f_1}^{x_2} \cup \{x_2\}), i \in \{1, \dots, k\}$. Since $\{\tilde{t}_1\} \neq \emptyset, \{\tilde{t}_2\} \neq \emptyset, \{\tilde{v}_i\} \neq \emptyset$ for all $i \in \{1, \dots, p\}$ and $p \geqslant 2, f_1^4 \in K_g$ and $f_2^4 \in K_\varphi$; hence, by virtue of Lemma 4.3, $f_1^4 \neq f_2^4$ which contradicts Lemma 7.1. Therefore $p = 2$.

Similarly, by replacing $Z_{f_2}^{x_1} \cup \{x_1\}$ in f_1 and f_2 by $\vec{V}(f_2, f_1, x_1)$, we can show that $k = 2$ as well.

Thus we can now write that

$$f_1(\tilde{x}) = g(R_1(\tilde{u}_1, \tilde{u}_2), R_2(\tilde{t}_1, \tilde{t}_2)),$$

$$f_2(\tilde{x}) = \varphi(R_1'(\tilde{u}_1, \tilde{t}_1), R_2'(\tilde{u}_2, \tilde{t}_2)).$$

We will now show that R_1 and R_2 are functions of the type $\varphi(\tilde{z})$. Suppose to the contrary that, for example,

$$R_1(\tilde{u}_1, \tilde{u}_2) = \varphi(R_{11}(v_{11}), \dots, R_{1m}(\tilde{v}_{1m}))$$

and at least one of the set among $\{\tilde{v}_{1i}\}, i \in \{1, \dots, m\}$ contains no less that two elements. Without loss of generality one can assume that $|\{\tilde{v}_{11}\}| \geqslant 2$ and $x_i \in \{\tilde{v}_{11}\}$. By Lemma 4.6 $Z_{f_1}^{x_i} \cup \{x_i\} \subseteq \{\tilde{v}_{11}\}$. Then, by replacing $Z_{f_1}^{x_i} \cup \{x_i\}$ in f_1 and f_2 by $\vec{V}(f_1, f_2, x_i)$, we obtain the functions:

$$f_1^5 = g(\varphi(P(\tilde{z}), R_{12}(v_{12}), \dots, R_{1m}(\tilde{v}_{1m})), R_2(\tilde{t}_1, \tilde{t}_2)),$$

$$f_2^5 = \varphi(P_1(\tilde{u}_{11}, \tilde{t}_1), R_2'(\tilde{u}_2, \tilde{t}_2)),$$

where $\{\tilde{z}\} = \{\tilde{v}_{11}\} \backslash (Z_{f_1}^{x_i} \cup \{x_i\})$ and $\{\tilde{u}_{i1}\} = \{\tilde{u}_i\} \backslash (Z_{f_1}^{x_i} \cup \{x_i\})$, $i = 1, 2$.

Since $\{\tilde{t}_1\} \neq \emptyset, \{\tilde{t}_2\} \neq \emptyset, \{\tilde{v}_{1i}\} \neq \emptyset$ for all $i \in \{1,\dots,m\}$ and $m \geqslant 2$, then, by Lemma 4.3, $f_1^5 \neq f_2^5$ which contradicts Lemma 7.1. Therefore R_1 is a function of the type $\varphi(\tilde{z})$ (perhaps, with other variables). In the same manner one can prove that R_2 is also a function of the type $\varphi(\tilde{z})$ and R_1', R_2' are functions of the type $g(\tilde{z})$.

Thus we can write that

$$f_1(\tilde{x}) = g(\varphi(\tilde{u}_1,\tilde{u}_2), \varphi(\tilde{t}_1,\tilde{t}_2)),$$

$$f_2(\tilde{x}) = \varphi(g(\tilde{u}_1,\tilde{t}_1), g(\tilde{u}_2,\tilde{t}_2)).$$

Finally, we want to show that each of the sets $\{\tilde{u}_1\}, \{\tilde{u}_2\}, \{\tilde{t}_1\}$ and $\{\tilde{t}_2\}$ contains only one element. This is shown in the same way as above.

Let us show, for example, that either of $\{\tilde{u}_1\}, \{\tilde{u}_2\}$ contains only one element. Suppose to the contrary that at least one of them contains no less than two elements. Let $x_i \in \{\tilde{t}_1,\tilde{t}_2\}$. Since are $\{\tilde{t}_1\}$, and $\{\tilde{t}_2\}$ are nonempty, by Lemma 4.6, $Z_{f_1}^{x_j} \cup \{x_j\} \subseteq \{\tilde{t}_1,\tilde{t}_2\}$. Hence, by replacing $Z_{f_1}^{x_j} \cup \{x_j\}$ in f_1 and f_2 by $\vec{V}(f_1,f_2,x_j)$, we obtain the functions

$$f_1^6 = \varphi(\tilde{u}_1,\tilde{u}_2) \quad \text{and} \quad f_2^6 = \varphi(g(\tilde{u}_1), g(\tilde{u}_2)).$$

Since, acoording to our assumption, at least one of $\{\tilde{u}_1\}, \{\tilde{u}_2\}$ contains no less than two elements, then $f_1^6 \neq f_2^6$ which contradicts Lemma 7.1. Therefore $|\{\tilde{u}_1\}| = |\{\tilde{u}_2\}| = 1$.

Hence we have proved that

$$f_1(\tilde{x}) = g(\varphi(x_1,x_2), \varphi(x_3,x_4)),$$

$$f_2(\tilde{x}) = \varphi(g(x_1,x_3), g(x_2,x_4)).$$

Thus the proof is completed. □

The following statement which is an immediate consequence of Lemmas 7.4, 7.7, 7.14, and 7.16 completes the Case 1 of Lemma 5.3.

Lemma 7.17. *If $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ is an s-function with different functions f_1 and f_2 having different ramming values, then the pair (f_1,f_2) may be one-type only to one of the following pairs:*

$$(x_1 \vee x_2 \cdots x_n, x_1(x_2 \vee \cdots \vee x_n)), \quad n \geqslant 3,$$

$$(x_1 \vee \cdots \vee x_n, x_1 \cdots x_n), \quad n \geqslant 2,$$

$$(x_1 \vee \cdots \vee x_n, x_1(\bar{x}_2 \vee \cdots \vee \bar{x}_n)), \quad n \geqslant 3,$$

$$(x_1 \vee \cdots \vee x_n, \bar{x}_1 \vee \cdots \vee \bar{x}_n), \quad n \geqslant 2,$$

$$(x_1 \cdots x_{n-2}x_{n-1} \vee x_n, \bar{x}_1 \cdots \bar{x}_{n-2}x_n \vee x_{n-1}), \quad n \geqslant 3,$$

$$(x_1 \cdots x_{n-2}x_{n-1} \vee x_n, (x_1 \vee \cdots \vee x_{n-2})x_n \vee x_{n-1}), \quad n \geqslant 4,$$

$$(x_1 \vee \cdots \vee x_{n-2})x_{n-1} \vee x_n, (\bar{x}_1 \vee \cdots \vee \bar{x}_{n-2})x_n \vee x_{n-1}), \quad n \geqslant 4,$$

$$(x_1x_2 \vee x_3x_4, (x_1 \vee x_3)(x_2 \vee x_4))$$

$$(x_1 \cdots x_{n-2}x_{n-1} \vee x_n, (x_1 \cdots x_{n-2} \vee \bar{x}_n)\bar{x}_{n-1}), \quad n \geqslant 3,$$

$$((x_1 \vee \cdots \vee x_{n-3} \vee x_{n-1})x_{n-2} \vee x_n, (x_1 \cdots x_{n-3}x_n \vee x_{n-2})x_{n-1}), \quad n \geqslant 4.$$

Before considering the Case 2 of Lemma 5.3, we must introduce some auxiliary notions which will be convenient later.

Let f_1, f_2 be functions of P_2^n depending essentially on $\tilde{x}$. We will say that a variable $x \in \{\tilde{x}\}$ and a constant $\sigma \in \{0,1\}$ *form a special pair into* (f_1, f_2) if the functions ${}^{\sigma}f_1^x$ and ${}^{\sigma}f_2^x$ depend essentially on all their variables and ${}^{\sigma}f_1^x = {}^{\sigma}f_2^x$.

A pair $(f_1, f_2) \in P_2^n \times P_2^n$ is called a *B*-pair if the following conditions hold:

(i) the functions f_1 and f_2 are read-once, and depend essentially on all their variables,

(ii) $f_1 \neq f_2$,

(iii) there is a variable and a constant forming a special pair into (f_1, f_2). A proof of the following statement is left to the reader.

Lemma 7.18. *Let (f_1, f_2) and (g_1, g_2) be one-type pairs. Then (f_1, f_2) is a B-pair iff (g_1, g_2) is also B-pair.*

Note that for the functions $g(x, y)$ and $\varphi(x.y)$ (i.e., $\vee$ and $\wedge$) there are constants a and b in $\{0,1\}$ such that $g(a,x) = g(x,a) = x$ and $\varphi(b,x) = \varphi(x,b) = x$. We will denote these constants by c_g and by c_φ respectively.

Lemma 7.19. *Let f_1 and f_2 be different functions depending essentially on $\tilde{x}$, and let*

$$f_1(\tilde{x}) = g(P(\tilde{v}), Q(\tilde{u})),$$

$$f_2(\tilde{x}) = g(P'(\tilde{v}), Q(\tilde{u})) \quad \textit{with } \{\tilde{x}\} = \{\tilde{v}\} \mathring{\cup} \{\tilde{u}\}.$$

Then, by replacing $\tilde{u}$ in f_1 and f_2 by constants $\tilde{\sigma}$ such that $Q(\tilde{\sigma}) = c_g$, we obtain different functions.

A direct conclusion of the preceding lemma is the result.

Lemma 7.20. *Assume that we are in the notations of the preceding lemma and all assumptions of the lemma hold. Also let (f_1, f_2) be a B-pair and let a variable $x \in \{\tilde{x}\}$ and a constant $\sigma \in \{0,1\}$ form a special pair into (f_1, f_2). Then we can obtain different functions in exactly the same way as in the preceding lemma which form a B-pair with (x, σ) as a special pair into it.*

Lemma 7.21. *Let $(f_1, f_2) \in P_2^n \times P_2^n$. Then*

(i) *for $n = 1$ there is no B-pair;*

(ii) *for $n = 2$ there is the only equivalence class*

$$(x_1\bar{x}_2, x_1 \vee x_2);$$

(iii) *for $n = 3$ there are only the two equivalence classes*

$$(x_1 \vee x_2 \vee x_3, \bar{x}_1(x_2 \vee x_3)) \quad \text{and} \quad (x_1 \vee x_2 x_3, (x_1 \vee x_2)x_3)).$$

To prove the last lemma it is necessary to search consecutively all pairs of Boolean functions of one, two, and three variables. Of course verification is tedious, we can, however, easily do it.

Lemma 7.22. *From each B-pair $(f_1, f_2) \in P_2^n \times P_2^n$, $n \geqslant 4$ one can obtain a B-pair $(g_1, g_2) \in P_2^m \times P_2^m$ with $m < n$ by replacing the same variables in f_1 and f_2 by the same constants of $\{0, 1\}$.*

Proof. To prove the lemma we will consider a number of cases and in each of them we will give an effective way of constructing the pair (g_1, g_2) with a smaller number of variables than (f_1, f_2). Although the cases look alike we will examine them separately because in fact they differ.

Clearly, there are the following two possibilities:
- f_1 and f_2 belong to one and the same class of $K_\vee$ and $K_\wedge$ or
- f_1 and f_2 belong to different classes.

We will now consider separately each of them. Let a variable x and a constant $\sigma \in \{0, 1\}$ form a special pair into (f_1, f_2).

Case 1: Suppose that f_1 and f_2 belong to one and the same class of $K_\vee$ and $K_\wedge$. Denote the class containing f_1 and f_2 by K_g. Let

$$f_1(\tilde{x}) = g(R_1(x, \tilde{v}_1), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$
$$f_2(\tilde{x}) = g(R'_1(x, \tilde{w}_1), R'_2(\tilde{w}_2) \ldots, R'_q(\tilde{w}_q)).$$

where $|\{\tilde{x}\}| \geqslant 4$ and $\{\tilde{v}_1\}, \{\tilde{w}_1\}$ may be empty.

It is easy to see that here it suffices to consider only the following three possibilities:

$$\{\tilde{v}_1\} = \{\tilde{w}_1\} = \emptyset, \quad \{\tilde{v}_1\} = \emptyset, \{\tilde{w}_1\} \neq \emptyset, \quad \text{or} \quad \{\tilde{v}_1\} \neq \emptyset, \quad \{\tilde{w}_1\} \neq \emptyset.$$

Case 1.1: Let $\{\tilde{v}_1\} = \{\tilde{w}_1\} = \emptyset$. We will now show that this case is impossible. Since $\{\tilde{v}_1\} = \{\tilde{w}_1\} = \emptyset$, $R_1(x, \tilde{v}_1) = x^{\sigma_1}$ and $R'_1(x, \tilde{w}_1) = x^{\sigma_2}$. First, we show that $\sigma_1 \neq \sigma_2$. Indeed, since x and σ form a special pair into $(f_1, f_2), \sigma$ is a ramming value of x neither in f_1 nor in f_2 hence, by virtue of Lemma 4.7, $\bar{\sigma}$ is a ramming value of x in f_1 and in f_2. However, if $\sigma_1 \neq \sigma_2$, then either $R_1(\sigma, \tilde{v}_1)$ or $R'_1(\sigma, \tilde{w}_1)$ is equal to $\bar{\sigma}$, i.e., σ is a ramming value of x either in f_1 or in f_2, which contradicts the above-mentioned.

Now, assume $\sigma_1 = \sigma_2$. We then have

$$f_1(\tilde{x}) = g(x^\sigma, g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)),$$
$$f_2(\tilde{x}) = g(x^\sigma, g(R'_2(\tilde{w}_2) \ldots, R'_q(\tilde{w}_q)).$$

According to the definition, ${}^{\sigma}f_1^x = {}^{\sigma}f_2^x$, i.e.,

$$g(R_2(\tilde{v}_2),\ldots,R_p(\tilde{v}_p)) = g(R_2'(\tilde{w}_2),\ldots,R_q'(\tilde{w}_q))$$

which, in its turn, implies $f_1 = f_2$. Thus, we have obtained a contradiction again.

Before considering other cases we give a basic idea of their examination. First, represent ${}^{\sigma}f_1^x$ and ${}^{\sigma}f_2^x$ in the form of their g-representations and then, use the equality ${}^{\sigma}f_1^x = {}^{\sigma}f_2^x$ and the uniqueness of g-representation.

Case 1.2: Let $\{\tilde{v}_1\} = \emptyset$ and $\{\tilde{w}_1\} \neq \emptyset$. Clearly, we here have

$$\begin{aligned}{}^{\sigma}f_1^x &= g(R_2(\tilde{v}_2),\ldots,R_p(\tilde{v}_p)),\\ {}^{\sigma}f_2^x &= g(R_1'(\sigma,\tilde{w}_1),R_2'(\tilde{w}_2),\ldots,R_q'(\tilde{w}_q)).\end{aligned}$$

Of course, it may be that the second expression is not the g-representation. So we first make the transformation: in the case $R_1'(\sigma,\tilde{w}_1) \in K_g$, replace $R_1'(\sigma,\tilde{w}_1)$ by its g-representation in the expression, otherwise, keep $R_1'(\sigma,\tilde{w}_1)$ without any change. Obviously, under this transformation we obtain the g-representation of ${}^{\sigma}f_2^x$. Since ${}^{\sigma}f_1^x ={}^{\sigma} f_2^x$, by virtue of the uniqueness of g-representation, there is $i \in \{2,\ldots,p\}$ such that, for example, $R_2'(\tilde{w}_2) = R_i(\tilde{v}_i)$. Hence, by virtue of Lemma 7.20, the lemma is true in this case.

Case 1.3: Now suppose that $\{\tilde{v}_1\} \neq \emptyset$ and $\{\tilde{w}_1\} \neq \emptyset$. This case is more tedious. Clearly, in its turn, here there are the following three possibilities:
- neither $R_1(\sigma,\tilde{v}_1)$ nor $R_1'(\sigma,\tilde{w}_1)$ belong to K_g,
- only one of them belongs to K_g, or
- either of them belongs to K_g.

We will now consider separately each of them.

Case 1.3.1: Suppose that neither $R_1(\sigma,\tilde{v}_1)$ nor $R_1'(\sigma,\tilde{w}_1)$ belong to K_g.
Then it is clear that

$$\begin{aligned}{}^{\sigma}f_1^x &= g(R_1(\sigma,\tilde{v}_1),R_2(\tilde{v}_2),\ \ldots,\ R_p(\tilde{v}_p)),\text{and}\\ {}^{\sigma}f_2^x &= g(R_1'(\sigma,\tilde{w}_1),R_2'(\tilde{w}_2)\ldots,R_q'(\tilde{w}_q)).\end{aligned}$$

are already g-representations, hence ${}^{\sigma}f_1^x = {}^{\sigma}f_2^x$ and the uniqueness of g-representation imply $p = q$ and the two possibilities:
- $R_1(\sigma,\tilde{v}_1) = R_1'(\sigma,\tilde{w}_1)$ or
- $R_1(\sigma,\tilde{v}_1) = R_i'(\tilde{w}_i)$ and $R_1'(\sigma,\tilde{w}_1) = R_j(\tilde{v}_j)$ for some $i \in \{2,\ \ldots,\ q\}$ and $j \in \{2,\ \ldots,\ p\}$.

We will also consider separately each of them.

Case 1.3.1.1: Let $R_1(\sigma,\tilde{v}_1) = R_1'(\sigma,\tilde{w}_1)$. Then it is easy to see that in this case

$$g(R_2(\tilde{v}_2),\ \ldots,\ R_p(\tilde{v}_p)) = g(R_2'(\tilde{w}_2)\ldots,R_q'(\tilde{w}_q)).$$

On the other hand,

$$f_1(\tilde{x}) = g(R_1(x,\tilde{v}_1), g(R_2(\tilde{v}_2), \dots, R_p(\tilde{v}_p))),$$

$$f_2(\tilde{x}) = g(R'_1(x,\tilde{w}_1), g(R'_2(\tilde{w}_2) \dots, R'_q(\tilde{w}_q))).$$

By Lemma 7.20, the pair (R_1, R'_1) depending on the number of variables less than (f_1, f_2) is a B-pair. This proves the lemma in this case.

Case 1.3.1.2: Let $R_1(\sigma, \tilde{v}_1) = R'_i(\tilde{w}_i)$ for some $i \in \{2, \dots, q\}$. Then two more cases depending on the quantity of $p = q$ are possible.

$$p = q = 2 \quad \text{and} \quad p = q > 2.$$

Case 1.3.1.2.1: Let $p = q = 2$. Clearly, here we have

$$f_1(\tilde{x}) = g(R_1(x,\tilde{v}_1), R_2(\tilde{v}_2)),$$

$$f_2(\tilde{x}) = g(R'_1(x,\tilde{v}_2), R'_2(\tilde{v}_1)), \tag{31}$$

and

$$R_1(\sigma,\tilde{v}_1) = R'_2(\tilde{v}_1),$$

$$R'_1(\sigma, \tilde{v}_2) = R_2(\tilde{v}_2). \tag{32}$$

Since $\{\tilde{v}_1\} \neq \emptyset$ and $\{\tilde{w}_1\} = \{\tilde{v}_2\} \neq \emptyset$, either of R_1 and R'_1 belongs to K_φ. Let

$$R_1(x,\tilde{v}_1) = \varphi(R_{11}(x, \tilde{v}_{11}), R_{12}(\tilde{v}_{12}), \dots, R_{1k}(\tilde{v}_{1k})),$$

$$R'_1(x,\tilde{v}_2) = \varphi(R'_{21}(x, \tilde{w}_{21}), R'_{22}(\tilde{w}_{22}), \dots, R'_{1l}(\tilde{w}_{1l})),$$

where $\{\tilde{v}_{11}\}$ and $\{\tilde{w}_{21}\}$ may be empty.

Then, using (32), we can rewrite (31) in the form

$$f_1(\tilde{x}) = g(\varphi(R_{11}(x,\tilde{v}_{11}), R_{12}(\tilde{v}_{12}), \dots, R_{1k}(\tilde{v}_{1k})), \varphi(R'_{21}(\sigma,\tilde{w}_{21}), R'_{22}(\tilde{w}_{22}), \dots, R'_{1l}(\tilde{w}_{1l})),$$

$$f_2(\tilde{x}) = g(\varphi(R'_{21}(x,\tilde{w}_{21}), R'_{22}(\tilde{w}_{22}), \dots, R'_{1l}(\tilde{w}_{1l})), \varphi(R_{11}(\sigma,\tilde{v}_{11}), R_{12}(\tilde{v}_{12}), \dots, R_{1l}(\tilde{v}_{1l})).$$

Since the sets $\{\tilde{v}_{11}\}$ and $\{\tilde{w}_{21}\}$ may be empty, there are the two possibilities: either of these sets is empty, or at least one of them is nonempty and they will be consider separately.

Case 1.3.1.2.1.1: Let either of the sets $\{\tilde{v}_{11}\}$ and $\{\tilde{w}_{21}\}$ be empty. Then since, according to our assumption, $n \geqslant 4$, i.e., $|\{\tilde{x}\}| \geqslant 4$, at least one of the two possibilities must be satisfied:

- $k \geqslant 3$ or $l \geqslant 3$, or
- at least one of the sets $\{\tilde{v}_{12}\}, \{\tilde{w}_{22}\}$ contains no less than two elements.

Case 1.3.1.2.1.1.1: For the sake of definiteness, let $k \geqslant 3$. Since R_{12} differs from a constant, there are constants $\tilde{\beta}$ such that $R_{12}(\tilde{\beta}) = c_\varphi$. It is easy to check that we obtain a B-pair of the number of variables less than (f_1, f_2) by replacing $\tilde{v}_{12}$ in f_1 and f_2 by $\tilde{\beta}$. In case $l \geqslant 3$ we can show the same one by replacing $\tilde{v}_{22}$ in f_1 and f_2 by analogous constants for R_{22}.

Case 1.3.1.2.1.1.2: For the sake of definiteness, suppose that $|\{\tilde{v}_{12}\}| \geqslant 2$. (The case $|\{\tilde{w}_{22}\}| \geqslant 2$ can be easily considered by analogy.) Clearly, there are constants $\tilde{\gamma}$ such that by replacing some variables of R_{12} by these constants in R_{12}, we can obtain $x_i^{\sigma_i}$. It is easy to check that we obtain a B-pair of the number of variables less than (f_1, f_2) by replacing the same variables by $\tilde{\gamma}$ in f_1 and f_2.

Case 1.3.1.2.1.2: Now, suppose that at least one of $\{\tilde{v}_{11}\}$ and $\{\tilde{v}_{21}\}$ is nonempty. Here we act in the same way as in Case 1.3.1.2.1.1.1. It is easy to check that we obtain a B-pair of the number of variables less than (f_1, f_2) as a result again.

Case 1.3.1.2.2: Let $p = q > 2$. By the uniqueness of g-representation there is $i \in \{3, \ldots, q\}$ such that, for example, $R_3(\tilde{v}_3) = R_i'(\tilde{w}_i)$. In this case, by Lemma 7.20, we obtain a B-pair of the number of variables less than (f_1, f_2) by replacing $\tilde{v}_3$ in f_1 and f_2 by such constants $\tilde{\beta}$ that $R_3(\tilde{\beta}) = R_i'(\tilde{\beta}) = c_g$.

Case 1.3.2: Let only one of $R_1(\sigma, \tilde{v}_1)$ and $R_1'(\sigma, \tilde{w}_1)$ belong to K_g. Without loss of generality one can assume that $R_1(\sigma, \tilde{v}_1) \in K_g$ and $R_1'(\sigma, \tilde{w}_1) \notin K_g$. Suppose, also, that

$$R_1(\sigma, \tilde{v}_1) = g(R_{11}(\tilde{v}_{11}), \ldots, R_{1k}(\tilde{v}_{1k}))\,.$$

Then

$$R_1(x, \tilde{v}_1) = \varphi(x^\sigma, g(R_{11}(\tilde{v}_{11}), \ldots, R_{1k}(\tilde{v}_{1k})))\,.$$

The number of terms of the above φ-representation of R_1 is equal to 2, otherwise ${}^\sigma R_1^x \in K_\varphi$ which contradicts the assumption ${}^\sigma R_1^x \in K_g$. The first term is x^σ by the same reason. By replacing $R_1(\sigma, \tilde{v}_1)$ by its g-representation in

$${}^\sigma f_1^x = g(R_1(\sigma, \tilde{v}_1), g(R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p)))\,,$$

we obtain

$${}^\sigma f_1^x = g(R_{11}(\tilde{v}_{11}), \ldots, R_{1k}(\tilde{v}_{1k}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p))\,.$$

Since

$${}^\sigma f_1^x = g(R_{11}(\tilde{v}_{11}), \ldots, R_{1k}(\tilde{v}_{1k}), R_2(\tilde{v}_2), \ldots, R_p(\tilde{v}_p))\,,$$

$${}^\sigma f_2^x = g(R_1'(\sigma, \tilde{w}_1), R_2'(\tilde{w}_2) \ldots, R_q'(\tilde{w}_q))\,,$$

then due to the uniqueness of g-representation we have two possibilities:

- $R_i(\tilde{v}_i) = R_j'(\tilde{w}_j)$ for some $i \in \{2, \ldots, p\}$ and $j \in \{2, \ldots, q\}$, or
- $R_{1s}(\tilde{v}_{1s}) = R_i'(\tilde{w}_i)$ and $R_{1t}(\tilde{v}_{1t}) = R_j'(\tilde{w}_j)$ for some $s, t \in \{1, \ldots, k\}, s \neq t$ and $i, j \in \{2, \ldots, q\}$, $i \neq j$.

Let us consider separately each of them.

Case 1.3.2.1: Let

$$R_i(\tilde{v}_i) = R'_j(\tilde{w}_j) \quad \text{for some} i \in \{2,\ldots,p\} \text{and} j \in \{2,\ldots,q\}$$

In this case we replace $\tilde{v}_i$ in f_1 and f_2 by constants $\tilde{\sigma}$ such that

$$R_i(\tilde{\sigma}) = R'_j(\tilde{\sigma}) = c_g\,.$$

Case 1.3.2.2: Let

$$R_{1s}(\tilde{v}_{1s}) = R'_i(\tilde{w}_i)\,,$$

$$R_{1t}(\tilde{v}_{1t}) = R'_j(\tilde{w}_j)$$

for some $s,t \in \{1,\ldots,k\}, s \neq t$ and $i,j \in \{2,\ldots,q\}, i \neq j$.

In this case we replace $\tilde{v}_{1s}$ in f_1 and f_2 by constants $\tilde{\sigma}$' such that

$$R_{1s}(\tilde{\sigma}') = R'_i(\tilde{\sigma}') = c_g\,.$$

In both of the above cases we obtain a B-pair with a number of variables less that (f_1, f_2). The verification is left to the reader.

Case 1.3.3: Suppose that each of $R_1(\sigma, \tilde{v}_1)$ and $R'_1(\sigma, \tilde{w}_1)$ belong to K_g. Suppose, also, that

$$R_1(\sigma,\, \tilde{v}_1) = g(R_{11}(\tilde{v}_{11}),\ldots,R_{1k}(\tilde{v}_{1k}))\,,$$

$$R'_1(\sigma, \tilde{w}_1) = g(R'_{11}(\tilde{w}_{11}),\ldots,R'_{1l}(\tilde{w}_{1l}))\,.$$

Then

$$R_1(x, \tilde{v}_1) = \varphi(x^\sigma, g(R_{11}(\tilde{v}_{11}),\ldots,R_{1k}(\tilde{v}_{1k})))\,,$$

$$R'_1(x, \tilde{w}_1) = \varphi(x^\sigma, g(R'_{11}(\tilde{w}_{11}),\ldots,R'_{1l}(\tilde{w}_{1l})))\,.$$

The first term of the both φ-representations is x^σ, since the variable x has the same ramming value in R_1 and R'_1. Clearly, by replacing $R_1(\sigma, \tilde{v}_1)$ and $R'_1(\sigma, \tilde{w}_1)$ by their g-representations in

$${}^\sigma f_1^x = g(R_1(\sigma, \tilde{v}_1), R_2(\tilde{v}_2),\ldots,R_p(\tilde{v}_p))\,,$$

$${}^\sigma f_2^x = g(R'_1(\sigma, \tilde{w}_1), R'_2(\tilde{w}_2),\ldots,R'_q(\tilde{w}_q))$$

respectively, we obtain g-representations of ${}^\sigma f_1^x$ and ${}^\sigma f_2^x$. Since

$${}^\sigma f_1^x = g(R_{11}(\tilde{v}_{11}),\ldots,R_{1k}(\tilde{v}_{1k}), R_2(\tilde{v}_2),\ldots,R_p(\tilde{v}_p))\,,$$

$${}^\sigma f_2^x = g(R'_{11}(\tilde{w}_{11}),\ldots,R'_{1l}(\tilde{w}_{1l}), R'_2(\tilde{w}_2)\ldots,R'_q(\tilde{w}_q))\,,$$

then due to the uniqueness of g-representation we have the three possibilities:

- $R_i(\tilde{v}_i) = R'_j(\tilde{w}_j)$ for some $i \in \{2,\dots,p\}$ and $j \in \{2,\dots,q\}$,
- $R_{1s}(\tilde{v}_{1s}) = R'_i(\tilde{w}_i)$ and $R_j(\tilde{v}_j) = R'_{1t}(\tilde{w}_{1t})$ for some $i \in \{2,\dots,q\}$, $s \in \{1,\dots,k\}$ and $j \in \{2,\dots,p\}, t \in \{1,\dots,l\}$, or
- $R_{1i}(\tilde{v}_{1i}) = R'_{1j}(\tilde{w}_{1j})$ for some $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,l\}$.

In the first case we act in the same way as in Case 1.3.2.1 and in the second case as in Case 1.3.2.2. In the third case it is necessary to replace $\tilde{v}_{1i}$ in f_1 and f_2 by such constants $\tilde{\sigma}$ that $R_{1i}(\tilde{\sigma}) = R'_{1j}(\tilde{\sigma}) = c_\varphi$. It is comparatively easy to check that in each of the three cases we obtain a B-pair with a number of variables less than (f_1, f_2). The verification is left to the reader.

Case 2: Let $f_1 \in K_g$ and $f_2 \in K_\varphi$, and let

$$\begin{aligned} f_1(\tilde{x}) &= g(R_1(x,\tilde{v}_1), R_2(\tilde{v}_2), \dots, R_p(\tilde{v}_p)), \\ f_2(\tilde{x}) &= \varphi(R'_1(x,\tilde{w}_1), R'_2(\tilde{w}_2)\dots, R'_q(\tilde{w}_q)), \end{aligned} \tag{33}$$

where $|\{\tilde{x}\}| \geqslant 4$ and either of $\{\tilde{v}_1\}$ and $\{\tilde{w}_1\}$ may be empty.

First, note that since (x,σ) forms a special pair into (f_1, f_2), ${}^\sigma f_1^x = {}^\sigma f_2^x$. On the other hand, one can easily show that if at least one of the two conditions:

$$R_1(x,\tilde{v}_1) = x^{\sigma_1} \quad \text{and} \quad p = 2, \quad \text{or} \quad R'_1(x,\tilde{w}_1) = x^{\sigma_2} \quad \text{and} \quad q = 2$$

is not satisfied, then ${}^\sigma f_1^x \in K_g$ and ${}^\sigma f_2^x \in K_\varphi$ and so, by virtue of Lemma 4.3, ${}^\sigma f_1^x \neq {}^\sigma f_2^x$. Thus, it suffices to consider only the two possibilities:

$$R_1(x,\tilde{v}_1) = x^{\sigma_1} \quad \text{and} \quad p = 2, \quad \text{or} \quad R'_1(x,\tilde{w}_1) = x^{\sigma_2} \quad \text{and} \quad q = 2\,.$$

As a matter of fact, they are dual and so it suffices consider only one of them, say the first.

Case 2.1: Let $R_1(x,\tilde{v}_1) = x^{\sigma_1}$ and $p = 2$. In this case we can clearly rewrite (33) in the form:

$$\begin{aligned} f_1(\tilde{x}) &= g(x^\sigma, R_2(\tilde{v}_2)), \\ f_2(\tilde{x}) &= \varphi(R'_1(x,\tilde{w}_1), R'_2(\tilde{w}_2)\dots, R'_q(\tilde{w}_q)). \end{aligned} \tag{34}$$

But, as noted above, $\{\tilde{w}_1\}$ may be empty, so here, in its turn, there are also the two possibilities: either

$$\{\tilde{w}_1\} = \emptyset \quad \text{or} \quad \{\tilde{w}_1\} \neq \emptyset.$$

If $\{\tilde{w}_1\} = \emptyset$, then $R'_1(x,\tilde{w}_1) = x^{\bar{\sigma}}$ because the variable x has the same ramming value in f_1 and f_2. If $\{\tilde{w}_1\} \neq \emptyset$ then $R'_1(x,\tilde{w}_1) \in K_g$. Thus these two possibilities imply the two possibilities:

$$R'_1(x,\tilde{w}_1) = x^{\bar{\sigma}} \quad \text{or} \quad R'_1(x,\tilde{w}_1) \in K_g\,.$$

Let us consider separately each of them.

Case 2.1.1: Let $\{\tilde{w}_1\} = \emptyset$ then $R'_1(x, \tilde{w}_1) = x^{\bar{\sigma}}$. Then, clearly, (34) can be rewritten in the form:

$$f_1(\tilde{x}) = g(x^{\sigma}, r_2(\tilde{v}_2)),$$
$$f_2(\tilde{x}) = \varphi(x^{\bar{\sigma}}, R'_2(\tilde{w}_2)\ldots, R'_q(\tilde{w}_q)) \tag{35}$$

Since (x, σ) is a special pair into (f_1, f_2), ${}^{\sigma}f_1^x = {}^{\sigma}f_2^x$. Taking this equality into account, from (35) we see that

$$R_2(\tilde{v}_2) = \varphi(R'_2(\tilde{w}_2)\ldots, R'_q(\tilde{w}_q)).$$

It follows from $|\{\tilde{x}\}| \geqslant 4$ that

$$|\{\tilde{v}_2\}| = |\{\tilde{w}_2\}|\mathring{\cup} \cdots \mathring{\cup}|\{\tilde{w}_q\}| \geqslant 3.$$

Therefore, there are constants $\tilde{\beta}$ such that, by replacing some variables of $\tilde{w}_2, \ldots, \tilde{w}_q$ in R_2 by $\tilde{\beta}$, we obtain x_i^{τ}. Then, by replacing the same variables by $\tilde{\beta}$ in f_1 and f_2, we obtain the functions:

$$f_1^3 = g(x^{\sigma}, x_i^{\tau}) \quad \text{and} f_2^3 = \varphi(x^{\sigma}, x_i^{\tau})$$

which, as it is easy to check, form a B-pair with a number of variables less than (f_1, f_2).

Case 2.1.2: Now let $\{\tilde{w}_1\} \neq \emptyset$, then according to Lemma 4.4 one can assume that

$$R'_1(x, \tilde{w}_1) = g(R'_{11}(x, \tilde{w}_{11}), R'_{12}(\tilde{w}_{12}), \ldots, R'_{1k}(\tilde{w}_{1k})). \tag{36}$$

where $\{\tilde{w}_{11}\}$ may be empty.

Then, by replacing $R'_1(x, \tilde{w}_1)$ in (34) by (36), we have

$$f_2(\tilde{x}) = \varphi(g(R'_{11}(x, \tilde{w}_{11}), R'_{12}(\tilde{w}_{12}), \ldots, R'_{1k}(\tilde{w}_{1k}), \ldots, R'_2(\tilde{w}_2)\ldots, R'_q(\tilde{w}_q)). \tag{37}$$

Since ${}^{\sigma}f_1^x = {}^{\sigma}f_2^x$ again, using (37), we have

$$R_2(\tilde{v}_2) = \varphi(g(R'_{11}(\sigma, \tilde{w}_{11}), R'_{12}(\tilde{w}_{12}), \ldots, R'_{1k}(\tilde{w}_{1k}), \ldots, R'_2(\tilde{w}_2)\ldots, R'_q(\tilde{w}_q)). \tag{38}$$

By replacing $R_2(\tilde{v}_2)$ in the representation

$$f_1(\tilde{x}) = g(x^{\sigma}, R_2(\tilde{v}_2))$$

by (38), we have obtained

$$f_1(\tilde{x}) = g(x^{\sigma}, \varphi(g(R'_{11}(\sigma, \tilde{w}_{11}), R'_{12}(\tilde{w}_{12}), \ldots, R'_{1k}(\tilde{w}_{1k}), \ldots,$$
$$R'_2(\tilde{w}_2)\ldots, R'_q(\tilde{w}_q))). \tag{39}$$

Since $\{\tilde{w}_{11}\}$ may be empty, we have the two possibilities: either

$$\{\tilde{w}_{11}\} = \emptyset \quad \text{or} \quad \{\tilde{w}_{11}\} \neq \emptyset.$$

Now we will consider separately each of these two possibilities.

Case 2.1.2.1: Suppose that $\{\tilde{w}_{11}\} = \emptyset$. Since the function

$$g(R'_{12}(\tilde{w}_{12}),\dots,R'_{1k}(\tilde{w}_{1k}))$$

differs from any constant, there are constants $\tilde{\beta}_1$ such that, by replacing some variables of this function by $\tilde{\beta}_1$, we obtain $x_i^{\sigma_i}$. Similarly, there are constants $\tilde{\beta}_2$ such that, by replacing some variables of the function

$$\varphi(R'_2(\tilde{w}_2)\dots,R'_q(\tilde{w}_q))$$

by $\tilde{\beta}_2$, we obtain $x_j^{\sigma_j}$. Obviously, by replacing the same variales by $\tilde{\beta}_1$ and $\tilde{\beta}_2$ in f_1 and f_2, we obtain the functions

$$f_1^4 = g(x^{\sigma}, \varphi(x_i^{\sigma_i}, x_j^{\sigma_j}) \quad \text{and} \quad f_2^4 = \varphi(g(x^{\sigma}, x_i^{\sigma_i}), x_j^{\sigma_j})$$

forming a B-pair. Since $|\{\tilde{x}\}| \geqslant 4$, at least one of the sets

$$\{\tilde{w}_{12}\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{w}_{1k}\} \quad \text{and} \quad \{\tilde{w}_2\}\mathring{\cup}\cdots\mathring{\cup}\{\tilde{w}_q\}$$

contains no less that two variables, hence the B-pair (f_1^4, f_2^4) depends on a number of variables less than (f_1, f_2).

Case 2.1.2.2: Now suppose that $\{\tilde{w}_{11}\} \neq \emptyset$. Since each of the functions $R'_{12},\dots,R'_{1k}$ is monotone (which follows from (39) if we recall that f_1 is monotone), and differs from any constant, there are constants $\tilde{\beta}_2,\dots,\tilde{\beta}_k$ such that

$$R'_{12}(\tilde{\beta}_2) = \cdots = R'_{1k}(\tilde{\beta}_k) = c_g\,.$$

By replacing $\tilde{w}_{12},\dots,\tilde{w}_{1k}$ by $\tilde{\beta}_2,\dots,\tilde{\beta}_k$ in f_1 and f_2, we obtain the functions

$$f_1^5 = g(x^{\sigma}, \varphi(R'_{11}(\sigma, \tilde{w}_{11}), R'_2(\tilde{w}_2),\dots,R'_q(\tilde{w}_q)))\,,$$

$$f_2^5 = \varphi(R'_{11}(x, \tilde{w}_{11}), R'_2(\tilde{w}_2),\dots,R'_q(\tilde{w}_q))\,,$$

which, as it is easy to check, form a B-pair. Since $k \geqslant 2$, the B-pair (f_1^5, f_2^5) depends on the number of variables less than (f_1, f_2). Because the above cases exhaust all possibilities which can occur, we have completely proved the lemma. □

Lemma 7.23. *If* $(f_1, f_2) \in P_2^n \times P_2^n$ *is a B-pair, then either it is one-type to one of the following pairs:*

$$(x_1\bar{x}_2, x_1 \vee x_2), \qquad (x_1 \vee x_2 \vee x_3, \bar{x}_1(x_2 \vee x_3)), \qquad (x_1 \vee x_2x_3, (x_1 \vee x_2)x_3))$$

or, by replacing the same variables in f_1 *and* f_2 *by the same constants of* $\{0, 1\}$, *one can obtain a pair which is one-type to one of the above pairs.*

Proof. In case $n \leqslant 3$, by virtue of Lemma 7.21, the lemma is true, otherwise Lemma 7.22 is repeatedly applied until n is smaller than 4. □

Note, henceforth we do not assume that in the representation of an s-function

$$f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x}),$$

the functions f_1 and f_2 must have different ramming values. However we have assumed that f_1 and f_2 are different read-once functions depending essentially on all their variables.

Lemma 7.24. *Let $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ be an s-function with f_1 and f_2 satisfying*

(i) $|\{\tilde{x}\}| \geqslant 2$,
(ii) *f_1 and f_2 depend essentially on all their variables,*
(iii) *f_1 and f_2 are different read-once functions,*
(iv) *there is a variable $x \in \{\tilde{x}\}$ and a constant $\sigma \in \{0, 1\}$ such that the functions ${}^{\sigma}f_1^x$ and ${}^{\sigma}f_2^x$ depend essentially on all their variables and are equal. Then (f_1, f_2) is one-type to one of the pairs*

$$(x_1\bar{x}_2, x_1 \vee x_2), \qquad (x_1 \vee x_2x_3, (x_1 \vee x_2)x_3).$$

Proof. Since (f_1, f_2) is clearly a B-pair, it follows directly from the preceding lemma that either (f_1, f_2) is one-type to one of the following pairs:

$$(x_1\bar{x}_2, x_1 \vee x_2), \qquad (x_1 \vee x_2 \vee x_3, \bar{x}_1(x_2 \vee x_3)), \qquad (x_1 \vee x_2x_3, (x_1 \vee x_2)x_3)$$

or from (f_1, f_2) we can obtain another B-pair (g_1, g_2) with a number of variables less than (f_1, f_2) by replacing the same variables in f_1 and f_2 by the same constants of $\{0, 1\}$. But it will be shown now that the second statement must be rejected because it contradicts that f is an s-function. Indeed, this statement yields that the function

$$f'(y,\tilde{x}') = \bar{y}g_1(\tilde{x}') \vee yg_2(\tilde{x}')$$

is a proper subfunction of f. According to the definition of B-pair the functions g_1 and g_2 are, however, different and depend essentially on all their variables hence, by virtue of Lemma 5.2, y is a distinguished variable of f', and so f cannot be an s-function.

Let us also show that (f_1, f_2) cannot be one-type to

$$(x_1 \vee x_2 \vee x_3, \bar{x}_1(x_2 \vee x_3)).$$

Suppose to the contrary that, for example,

$$f(y,\tilde{x}) = \bar{y}(x_1 \vee x_2 \vee x_3) \vee y(\bar{x}_1(x_2 \vee x_3))$$

is an s-function. It is easy to check that y is a distinguished variable of this function. Hence f cannot be an s-function. Thus we have completely proved the lemma. □

Clearly, Lemma 7.24 completes Case 2 of Lemma 5.3.

Lemma 7.25. *Let $f(y,\tilde{x}) = \bar{y}f_1(\tilde{x}) \vee yf_2(\tilde{x})$ be an s-function. Then the pair (f_1, f_2) may be one-type only to one of the following pairs:*

$$(x_1 \vee x_2x_3, (x_1 \vee x_2)x_3)$$

$$(x_1x_2 \vee x_3x_4, (x_1 \vee x_3)(x_2 \vee x_4))$$

$$(x_1 \vee \cdots \vee x_n, x_1 \cdots x_n), \quad n \geqslant 2,$$

$$(x_1 \vee \cdots \vee x_n, x_1(\bar{x}_2 \vee \cdots \vee \bar{x}_n)), \quad n \geqslant 2,$$

$$(x_1 \vee x_2 \cdots x_n, x_1(x_2 \vee \cdots \vee x_n)), \quad n \geqslant 3,$$

$$(x_1 \vee \cdots \vee x_n, \bar{x}_1 \vee \cdots \vee \bar{x}_n), \quad n \geqslant 1,$$

$$(x_1 \vee \cdots \vee x_{n-2})x_{n-1} \vee x_n, (\bar{x}_1 \vee \cdots \vee \bar{x}_{n-2})x_n \vee x_{n-1}), \quad n \geqslant 4,$$

$$(x_1 \cdots x_{n-2}x_{n-1} \vee x_n, \bar{x}_1 \cdots \bar{x}_{n-2}x_n \vee x_{n-1}), \quad n \geqslant 3,$$

$$(x_1 \cdots x_{n-2}x_{n-1} \vee x_n, (x_1 \vee \cdots \vee x_{n-2})x_n \vee x_{n-1}), \quad n \geqslant 4,$$

$$(x_1 \cdots x_{n-2}x_{n-1} \vee x_n, (x_1 \vee \cdots \vee x_{n-2} \vee \bar{x}_n)\bar{x}_{n-1}), \quad n \geqslant 3,$$

$$((x_1 \vee \cdots \vee x_{n-3} \vee x_{n-1})x_{n-2} \vee x_n, (x_1 \cdots x_{n-3}x_n \vee x_{n-2})x_{n-1}), \quad n \geqslant 4.$$

Proof. In case $|\{\tilde{x}\}| < 2$ we can easily find the only appropriate pair $(x_1, \bar{x}_1)$. In case $|\{\tilde{x}\}| \geqslant 2$ it suffices to apply Lemmas 7.17 and 7.24. □

To complete the proof of the main theorem it is necessary to ascertain the following: (i) what pairs of the above list can in fact produce an s-function; (ii) what s-functions in them are not one-type functions. In other words what s-functions define in fact a class of distinct representatives of s-functions.

By a somewhat lengthy process, we can show the following;

(1) all pairs from the first to the seventh produce s-functions;

(2) the eighth pair produces an s-function only for $n = 3$;

(3) in no cases all pairs from the ninth to the eleventh produce s-functions. (We here mean a number in the above list.)

We will now prove only (2). The proof of the others cases is left as problems to the reader.

Consider the function:

$$g_1 = \bar{y}(x_1 \cdots x_{n-2}x_{n-1} \vee x_n) \vee y(\bar{x}_1 \cdots \bar{x}_{n-2}x_n \vee x_{n-1}), \quad n \geqslant 4.$$

Since $n \geqslant 4$,

$${}^0g_1^{x_1} = x_n(\bar{y} \vee \bar{x}_2 \cdots \bar{x}_{n-2}) \vee yx_{n-1}.$$

Obviously ${}^0g_1^{x_1}$ is not represented by a read-once formula over $\{\vee, \wedge, -\}$. Hence, by virtue of Lemma 2.3, there is a subfunction of ${}^0g_1^{x_1}$ which depends essentially on at least two variables and has a distinguished variable. But since any subfunction of ${}^0g_1^{x_1}$ is also a subfunction of g_1, then g_1 is not s-function.

Now, consider the function

$$g_2 = \bar{y}(x_1x_2 \vee x_3) \vee y(\bar{x}_1x_3 \vee x_2).$$

For example, by replacing x_1 by 0 in this function one can obtain the function

$$^0g_2^{x_1} = yx_2 \vee x_3,$$

which is, clearly, read-once. One can likewise verify that all subfunctions of g_2 are read-once. Therefore, g_2 is an s-function. This completes the proof of (2).

We also leave to the reader the verification that

(4) the third and the fifth pairs produce one-type functions for $n \geqslant 3$;

(5) the fourth and the seventh pairs produce one-type functions for $n \geqslant 4$, and the fourth and the eighth pairs for $n = 4$;

(6) the others of the above list produce s-functions which are not one-type.

As a result, we can present the following list as a class of the distinct representatives of s-functions:

$$\bar{y}(x_1 \vee x_2x_3) \vee y(x_1 \vee x_2)x_3,$$

$$\bar{y}(x_1x_2 \vee x_3x_4) \vee y(x_1 \vee x_3)(x_2 \vee x_4),$$

$$\bar{y}(x_1 \vee \cdots \vee x_n) \vee yx_1 \cdots x_n, \quad n \geqslant 2,$$

$$\bar{y}(x_1 \vee \cdots \vee x_n) \vee yx_1(\bar{x}_2 \vee \cdots \vee \bar{x}_n), \quad n \geqslant 2,$$

$$\bar{y}(x_1 \vee \cdots \vee x_n) \vee y(\bar{x}_1 \vee \cdots \vee \bar{x}_n), \quad n \geqslant 1.$$

We can easily verify that the list given in the theorem is merely a more convenient representation of the same functions.

This way we have completely proved the main theorem. □

8. Conclusion

As it is shown above, there is a convenient necessary condition for a basis to be premaximal.

- To prove that this condition is also sufficient is an open interesting problem. However, in our opinion, this problem is very difficult.
- Another open problem is to prove a statement like Pratt's one. Namely, we need to show that the equality $L_{B_0}(f) = O((L_B(f))^\alpha)$ with a small value of α for all $f \in P_2$ and all bases B of the type $B_0 \cup \{g\}$ where g is an s-function. This would be an additional argument for our assumption that the property *"to be a premaximal basis"* is really formalization of our intuitive concept of *"to differ very little from the de Morgan basis in the sense of computational possibilities"*.
- Finally, we would like to observe that a knowledge of peculiarities of premeximal bases really gives a possibility to obtain nonlinear lower bounds on the complexity of

Boolean functions in computing them by formulas. In particular, the author showed that full binary basis is premaximal in exactly the same way. So it is very important to go on in investigation of these peculiarities.

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