AFFINE APPROXIMATION OF ISOPARAMETRIC ELEMENTS

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Abstract—The use of rectangular isoparametric elements in finite element analysis of second-order boundary-value problems requires evaluating integrals of rational polynomial functions. Gaussian quadrature formulas are currently the most popular method of obtaining approximations to the exact integrals. A new method is described in which the isoparametric finite element function spaces are approximated. The resulting integrals can be evaluated exactly, avoiding the computational expense of the Gaussian quadrature schemes, particularly the 27 point formula used in three-dimensional elements.

INTRODUCTION

Rectangular isoparametric finite elements were discovered as early as 1959 by Taij[1], and were introduced by Ergatoudis et al.[2] in 1968. Linear, elliptic, second-order boundary-value problems defined on irregular domains in two and three dimensions can be solved in principle using these elements. However, practical applications of the method to problems with complex geometries must face the issue of computational cost. For three-dimensional problems in particular, the cost of generating the finite element mesh and computing stiffness and loads may be a deciding factor in the utility of the method as an engineering tool. Of special importance is the scheme used to numerically integrate element stiffness matrices, since such calculations can dominate the time required for pre-processing calculations.

In this paper, alternatives to the standard rectangular isoparametric maps are presented which greatly reduce the effort required to integrate stiffness matrix and load vector components. Indeed, in certain cases, the mapping strategies proposed here may result in computation times which are an order of magnitude smaller than that required to employ full Gaussian quadrature. It suffices to describe the methods in the context of a model second-order problem in two dimensions using the four-node quadrilateral element.

MODEL PROBLEM

Let \( \Omega \) denote a smooth bounded domain in \( \mathbb{R}^2 \) with boundary \( \partial \Omega \), and consider the model Poisson problem:

\[ \text{find } u \in H^2(\Omega) \cap H_0^1(\Omega) \text{ such that } -\Delta u = f \text{ in } \Omega \]

where

- \( f \) is given data in \( H^m(\Omega) \)
- \( H^m(\Omega) \) is the space of functions which have square integrable \( m \)th order derivatives on \( \Omega \)
- \( H_0^1(\Omega) \) is the space of functions contained in \( H^1(\Omega) \) which are zero on \( \partial \Omega \)

The variational formulation[3] of this problem is:

\[ \text{find } u \in H_0^1(\Omega) \text{ such that } \int_{\Omega} (\nabla u \cdot \nabla v - f v) \, d\Omega = 0 \text{ for every } v \in H_0^1(\Omega). \]
Galerkin's method of approximation replaces the space \( H^1(\Omega) \) with a finite dimensional subspace \( \tilde{H}(\Omega) \) spanned by \( N \) linearly independent functions \( \{\phi_i\} \):

\[
\text{find } \tilde{u} \in \tilde{H}(\Omega) \text{ such that } \iint_{\Omega} (\nabla \tilde{u} \cdot \nabla \tilde{v} - f \tilde{v}) \, d\Omega = 0 \text{ for every } \tilde{v} \in \tilde{H}(\Omega). \tag{3}
\]

By construction, \( \tilde{u} \) can be expressed in the form

\[
\tilde{u} = \sum \alpha_i \phi_i, \quad \alpha_i \in \mathbb{R}. \quad i = 1, 2, \ldots, N. \tag{4}
\]

Satisfaction of (3) requires that for each \( i \),

\[
\sum_j \left( \iint_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \, d\Omega \right) \alpha_j = \iint_{\Omega} f \phi_i \, d\Omega. \tag{5}
\]

Relations (5) form a system of \( N \) linear equations in the unknown constants \( \{\alpha_i\} \). In matrix form,

\[
K \alpha = F \tag{6}
\]

where the individual terms in the stiffness matrix \( K \) are

\[
k_{mn} = \iint_{\Omega} \nabla \phi_m \cdot \nabla \phi_n \, d\Omega, \quad 1 \leq m, n \leq N. \tag{7}
\]

The finite element method of solving (1) consists of partitioning the domain \( \Omega \) into a mesh of finite elements \( \{\Omega_j\} \) and applying Galerkin's method. The set \( \{\phi_i\} \) is constructed by piecing together element basis functions.

**ISOPARAMETRIC ELEMENT BASIS FUNCTIONS**

The fundamental component of any finite element is the set of element basis functions \( \{\psi_i\} \) defined separately over each finite element domain \( \Omega_j \). In the case of isoparametric elements, the basis functions are defined via three well known concepts, namely the master element domain \( \Omega_m \), the master element basis functions \( \{\hat{\psi}_i\} \), and the isoparametric map \( F_i \). For the four-node isoparametric quadrilateral element, these are defined as follows:

\[
\Omega = \{\xi; \xi = (\xi, \eta) \in [-1,1]^2\} \tag{8}
\]

\[
\hat{\psi}_1 = \hat{\psi}_1(\xi, \eta) = (1/4)(1 - \xi)(1 - \eta) \\
\hat{\psi}_2 = \hat{\psi}_2(\xi, \eta) = (1/4)(1 + \xi)(1 - \eta) \\
\hat{\psi}_3 = \hat{\psi}_3(\xi, \eta) = (1/4)(1 + \xi)(1 + \eta) \\
\hat{\psi}_4 = \hat{\psi}_4(\xi, \eta) = (1/4)(1 - \xi)(1 + \eta) \tag{9}
\]

\[
F_i = \left\{F_i; \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ F_i \xi = x = (x, y) \in \mathbb{R}^2; \right. \tag{10}
\]

\[
x = \sum x_i \psi_i(\xi, \eta), \quad y = \sum y_i \psi_i(\xi, \eta), \quad i = 1, 2, 3, 4; \quad a_i = (x_i, y_i) = \text{coordinates of the } i\text{'th node of } \Omega_i. \left. \right\}
\]

The element basis functions are defined as follows:

\[
\psi_i = \hat{\psi}_i \cdot F_i^{-1} \quad i = 1, 2, 3, 4. \tag{11}
\]
Affine approximation of isoparametric elements

$F_i$ is often referred to as the bilinear map, and is invertible for convex $\Omega_i$.

Several properties of this method of construction are important to this study.

1. The global basis functions $\{\phi_i\}$ formed by piecing together the element basis functions are continuous. Galerkins method applied to second-order problems involves products of derivatives of the $\{\phi_i\}$. Thus continuity is desired, since products of Dirac-delta type distributions are not defined mathematically[5].

2. First degree polynomials are interpolated exactly[2], i.e. if $p(x, y)$ is any first degree polynomial, then by setting $\alpha_i = p(x, y_i)$ in (4) the following holds:

$$\bar{u}(x, y) = p(x, y).$$

In the context of the linear elasticity problem, this property is referred to as the ability to represent states of constant strain, and is important in considerations of convergence.

3. Compact support of the global basis functions allows for efficient calculation of (7) on an element-by-element basis since the restriction of an individual $\phi_i$ to a particular element domain is either an element basis function or the zero function.

**FORM OF THE INTEGRAND**

Difficult in inverting the isoparametric map makes integration over the master element domain more convenient than over the physical element domain. Thus a typical nonzero contribution to (7) is

$$k_{mn}^{\text{int}} = \int_{\Omega_i} \nabla \phi_m \cdot \nabla \phi_n \, d\Omega, \quad 1 \leq m, n \leq N$$

$$= \int_{\Omega_i} \nabla \psi_i \cdot \nabla \psi_j \, d\Omega, \quad 1 \leq i, j \leq 4$$

$$= \int_{-1}^{1} \int_{-1}^{1} J_i (\psi_i \xi, + \psi_i \eta, + \psi_i \eta, + \psi_i \eta)$$

$$+ (\psi_i \xi, + \psi_i \eta, + \psi_i \eta, + \psi_i \eta) d\xi d\eta$$

$$= \int_{-1}^{1} \int_{-1}^{1} (1/J_i) (\psi_i \psi_j (y_i^2 + x_i^2) + \psi_i \psi_j (y_i^2 + x_i^2))$$

$$- (\psi_i \psi_j + \psi_i \psi_j) (y_i y_j + x_i x_j)) d\xi d\eta, \quad (13)$$

where $J_i$ is the Jacobian determinant of the isoparametric map:

$$J_i = x_i y_j - x_j y_i. \quad (14)$$

For the four-node isoparametric quadrilateral element, $J_i$ can be expressed as[7]:

$$J_i = 1/4(A + B\xi + C\eta). \quad (15)$$

Rearranging (13) and substituting (15) results in:

$$k_{mn}^{\text{int}} = \int_{-1}^{1} \int_{-1}^{1} \frac{d_i + d_j \xi + d_{ij} \eta + d_{ij} \xi^2 + d_{ij} \xi \eta + d_{ij} \eta^2}{(1/4)(1 + B\xi + C\eta)} \, d\xi d\eta, \quad (16)$$

where the constants $\{d_i\}$ are combinations of the coordinates of the nodes.

In the special case of a parallelogram shaped physical element, $F_i$ is affine, $B$ and $C$ are zero, and $k_{mn}^{\text{int}}$ is the integral of a polynomial in $(\xi, \eta)$. In the general case of a trapezium, $F_i$
is nonlinear, $B$ and $C$ are nonzero, and $k_{\alpha\beta}$ is the integral of a rational polynomial function in $(\xi, \eta)$.

Okabe[7] has obtained a closed form solution to (16). Considerations of computational cost and numerical stability cause approximation methods, in particular Gaussian quadrature, to be the most popular methods of evaluating (16).

**AFFINE APPROXIMATION**

Consider quadrilateral and brick elements configured such that they can be expressed as the image of a master element under the action of bilinear and trilinear mappings, respectively. It is to these elements that affine approximation is to be applied. The method consists of partitioning the master element domain into simplicial subdomains (triangles in 2-D and tetrahedra in 3-D). The physical element is then considered to be the image of the master element under the action of a composite map, defined separately over each subdomain. The basis functions in the physical element become the image of the master basis functions under the action of the composite map.

In this paper, the properties of linear (affine) mappings defined over each subdomain are examined, and the composite map is termed multi-linear. The multi-linear maps and the Galerkin function spaces they generate are considered approximations of the isoparametric map and its Galerkin function space. Piecewise linearity of the multi-linear map is seen to allow for exact integration over each simplicial subdomain, as the Jacobian determinant of a linear mapping is constant.

**QUADRILATERALS: FIRST APPROXIMATION**

As previously defined, let $\Omega$ and $\Omega_i$ denote the respective master and physical quadrilateral element domains, $\{\hat{a}_i\}$ and $\{a_i\}$ denote the corners (nodes) of $\Omega$ and $\Omega_i$, $\{\hat{\psi}_i\}$ denote the master element basis functions, $F_i$ denote the bilinear map, and $J$ denote the Jacobian determinant of $F_i$. Consider the partition of $\Omega$ depicted in Fig. 1(a). Over each triangle a linear mapping is constructed with the property that the image of the vertices $\{\hat{a}_i\}$ coincides with $\{a_i\}$, i.e. nodes are mapped into nodes. This composite map is termed double-linear and denoted by $F_D$. $J_D$ denotes the (discontinuous) Jacobian determinant of $F_D$. The two triangular master element domains are denoted $\Omega_{D1}$, $\Omega_{D2}$, and their images under $F_D$ denoted $\Omega_{D1}$, $\Omega_{D2}$. The element basis functions are defined as:

$$\psi_i = \hat{\psi}_i \cdot F_D^{-1} \quad i = 1, 2, 3, 4. \quad (17)$$

The partition depicted in Fig. 1(b) yields a different mapping, thus a different Galerkin function space. In practice, however, this partition can be achieved by simply re-ordering the nodes in the physical element and constructing $F_D$.

Several properties of this construction follow:

1. In the special case of parallelogram-shaped elements, $F_D = F_i$.
2. For $\xi \in \Omega_{D1}$, $J_D = 1/2$ (area of $\Omega_{D1}$) = $J_i$ evaluated at $\hat{a}_i$[7].
3. The element basis functions are continuous on $\Omega_{D1} \cup \Omega_{D2}$, where $\Omega_{D1}$ is the closure of $\Omega_{D1}$. Partial (of the element basis functions) with respect to the physical coordinate variables are discontinuous across $\Omega_{D1} \cap \Omega_{D2}$ (except when $\Omega_i$ is a parallelogram).
4. First degree polynomials are not interpolated exactly (except when $\Omega_i$ is a parallelogram).
5. $F_i$ is linear along element boundaries; thus the image of the boundary of $\hat{\Omega}$ is the same under the action of $F_i$ and $F_D$. This condition guarantees continuity of the global basis functions for meshes containing isoparametric and/or affine approximation elements.

Use of the basis functions formed in this manner to obtain approximate solutions to boundary-value problems leads to integrals of the form (13). By construction, the integral over $\hat{\Omega}$ is replaced by the sum of integrals over $\hat{\Omega}_{D1}$ and $\hat{\Omega}_{D2}$ which possess polynomial integrands.
due to the piecewise constant nature of \( J_0 \). This represents a computational speed advantage as compared to a \( 2 \times 2 \) Gaussian quadrature scheme.

**QUADRILATERALS: SECOND APPROXIMATION**

The nonlinear nature of the bilinear map was noted previously and can be demonstrated by drawing the image of the diagonals of \( \Omega \). Thus the image of the diagonals of \( \Omega \) under affine approximation mappings may indicate, in a qualitative sense, the accuracy of the approximation. With this idea in mind, a logical strategy for increasing accuracy is to improve the resolution of the curvature of the diagonals by partitioning \( \Omega \) as shown in Fig. 1(c). The result is a composite map defined separately over four subdomains. This map will be termed quadruple-linear and denoted by \( F_0 \). \( J_0 \) denotes the piecewise constant Jacobian determinant of \( F_0 \). Two aspects of \( F_0 \) are apparent:

1. \( F_0 \) appears to be a better approximation of \( F \) than \( F_0 \), as freedom exists in choosing where to map \((\xi, \eta) = (0, 0) = 0\). Letting \( F_0 = F_0 \), it can be shown that for \( \xi \in \hat{\Omega}_0 \) (see Fig. 1(c)), \( J_0 = 1/2(J_0 \text{ at } \hat{a}_1 \text{ plus } J_0 \text{ at } \hat{a}_2) = J_0 \text{ at } (\xi, \eta) = (0, -1) \).
2. The computational speed advantage of affine approximation is weakened, as integrals over four domains must now be evaluated.

**BRICKS: FIRST APPROXIMATION**

Let \( \Omega \) and \( \Omega_r \) denote the respective master and physical brick element domains, \( \{\hat{a}_i\} \) and \( \{a_i\} \) denote the coordinates of the corners (nodes) of \( \hat{\Omega} \) and \( \Omega_r \), \( \{\hat{w}_i\} \) denote the brick master element basis functions[8], \( G_i \) denote the trilinear isoparametric mapping, and \( J_i \) denote the Jacobian determinant of \( G_i \):

\[
\Omega = \{ \xi = (\xi, \eta, \zeta) \in [-1, 1]^3 \} \\
G_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3, G_i\xi = x = (x, y, z); \quad x = \sum x_i \hat{\psi}_i, \\
y = \sum y_i \hat{\psi}_i, \quad i = 1, 2, \ldots, 8; \\
z = \sum z_i \hat{\psi}_i, \quad a_i = (x_i, y_i, z_i) \}
\]

An eight-node brick element can be subdivided into a minimum of five tetrahedra. The piecewise linear composite map is constructed separately over each tetrahedron by requiring that the image of the vertices \( \{\hat{a}_i\} \) coincides with \( \{G_i \hat{a}_i\} \), i.e. nodes are mapped into nodes.

As with quadrilaterals, exactly two such partitions exist (Fig. 2(a), 2(b)). The two partitions yield different mappings and different Galerkin function spaces. The map generated by the partition of Fig. 2(a) is termed quintuple-linear and denoted \( G_0 \). \( J_0 \) denotes the Jacobian determinant of \( G_0 \). Several properties of this construction follow:

1. In the special case of parallelepipeds, that is bricks whose faces are plane parallelograms, \( G_0 = G_i \).
2. Let \( \hat{\Omega}_0 \) denote the tetrahedron formed by nodes \( \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4 \), and let \( \Omega_0 \) denote the image of \( \hat{\Omega}_0 \) under the action of \( G_0 \). Then for \( \xi \in \hat{\Omega}_0, J_0 = 3/4(\text{area of } \Omega_0) = J_i \text{ evaluated at } \hat{a}_i \).
3. Continuity of the element basis functions is maintained throughout the interior of \( \Omega_r \). Partial derivatives (of the element basis functions) with respect to physical coordinate variables are discontinuous on the interior of \( \Omega_r \), (except when \( \Omega_r \) is a parallelepiped).
4. First degree polynomials are not reproduced exactly (except when \( \Omega_r \) is a parallelepiped).

Continuity of the global basis functions is complicated by the fact that element boundaries in three dimensions consist of faces rather than the line segments of two-dimensional elements.
Fig. 1. (a, b) Double-linear mappings. (c) Quadruple-linear mapping.
Two connected and properly shaped brick elements intersect at most across one face, thus two situations will require further detail, namely connection of an isoparametric element to an affine approximation element, and the connection of an affine approximation element to another affine approximation element.

Along any edge, $G_i$ is linear. Thus the images of master element edges are the same under the action of $G_i$ and $G_j$. This condition guarantees continuity of global basis functions on the edges of the elements.

The image of any face of the master element under the action of $G_i$ is, in general, a hyperbolic paraboloid. The image of any face of the master element under the action of $G_j$ is a pair of triangles. Thus, in the case of an isoparametric element and an affine approximation element sharing a face, continuity of the global basis functions exists if and only if the face is a plane parallelogram.

Inter-element continuity between two affine approximation elements which share a common face is also conditional. Consider a face as the sum of two triangles. It can be seen that in general, two separate definitions of the face exist, depending on which two nodes are connected to form the two triangles. For faces which are plane parallelograms, continuity of the global basis functions is satisfied. For faces which are not plane parallelograms, continuity exists if and only if the two element partitions split the face across the same diagonal.

**BRICKS: SECOND APPROXIMATION**

Many possibilities exist for defining a second approximation. However, only one will be described as it possesses advantages over other forms.

Consider the partition of $\Omega$ into the six square-bottom pyramids depicted in Fig. 2(e). Three aspects of this partition are apparent:

1. Each face represents the bottom of a square-bottom pyramid. Splitting each pyramid results in a total partition of 12 tetrahedra. By construction, each face may be split independently of the remaining faces. This property is useful when constructing meshes of compatible elements.

2. Analogous to the quadruple-linear map for quadrilaterals, freedom exists in choosing where to map $\xi = (0, 0, 0)$.

3. The computational speed advantage of affine approximation over $3 \times 3 \times 3$ Gaussian quadrature is weakened.

**SMOOTHING**

As noted previously, the affine approximation basis functions for distorted elements possess discontinuous derivatives on the interior of the elements. In practice, the calculation of derivatives from an isoparametric element solution usually involves some form of extrapolation and averaging between elements. It is reasonable then to expect improvement of affine approximation element solutions when some form of averaging is performed on the interior of element domains.

The process of manipulating the solution on the interior of affine approximation elements will be termed smoothing. The form of smoothing to be examined consists of replacing the affine approximation basis functions with the isoparametric basis functions for all post-processing operations. An important advantage of this technique is that the smoothing process can be performed by simply applying existing isoparametric element post-processing software.

**THEORETICAL ELEMENT BEHAVIOR**

Mathematical analysis of the finite element method has produced definitions of element performance in terms of asymptotic behavior. Asymptotic convergence is generally accepted to be an important criteria in determining the value of any method of approximation. In the present section, the effects of affine approximation on the classical "$h$ convergence" of the finite element method is presented.
Fig. 2. (a, b) Quintuple-linear mappings, (c) brick element partition for second approximation.
Consider the model problem (2). Let \( \{ \Omega_i \} \) denote a given partition of \( \Omega \) into convex quadrilaterals, and let \( h \) denote the associated mesh parameter:

\[
h = \sup_{\Omega_i \in \{ \Omega_i \}} \text{diameter}(\Omega_i).
\]  

(20)

Let \( T^h \) denote the set of all possible meshes with mesh parameter \( h \), and let \( r^h \) denote an arbitrary member of \( T^h \).

By the Lax–Milgram Theorem\[4\], there exists a unique solution \( u \) which depends continuously on the data \( f \). Similarly, the discrete variational boundary-value problem (3) generated by the partition \( r^h \) possesses a unique solution \( u^h \). Let the following hold:

- \( u^h_i = \) the solution to (3) using isoparametric elements on \( r^h_i \).
- \( u^h_D = \) the solution to (3) using double-linear elements on \( r^h_D \).

With these definitions in place, the following sequences are constructed:

\[
\{ r_i \} = \{ r^h_i, r^h_{i-1}, \ldots \}, \quad h_1 > h_2 > h_3 > \ldots > 0
\]

(21)

\[
\{ u_i \} = \{ u^h_i, u^{h-1}_i, \ldots \}
\]

(22.a)

\[
\{ u_D \} = \{ u^h_D, u^{h-1}_D, \ldots \}
\]

(22.b)

The "h convergence" concept of the finite element method is based on the following\[3–6\]: If the elements don’t become excessively distorted, i.e., they shrink in size faster than they deviate from a square (see Ciarlet \[6 pg. 247\]), and the element basis functions interpolate exactly polynomials of degree \( k \), then there exists a constant \( C \) such that

\[
\| \lambda_i \|_{1, \Omega} \leq C h^k |u|_{2, \Omega} \quad \text{as } h \to 0
\]

(23)

where

\[
|\cdot|_{m, \Omega} = \text{square root of the integral over } \Omega \text{ of the sum of the squares of all } m^{\text{th}} \text{ order derivatives}
\]

\[
\| \cdot \|_{m, \Omega} = \sum |\cdot|_{i, \Omega} \quad i = 0, \ldots, m.
\]

Recall that for the isoparametric elements,

\[
\psi_i = \hat{\psi}_i \cdot F_i^{-1}
\]

while for affine approximation elements

\[
\psi_i = \hat{\psi}_i \cdot F_i^{-1} = \hat{\psi}_i \cdot F_D^{-1} \cdot F_i \cdot F_i^{-1} = \hat{\psi}_i^* \cdot F_i^{-1}
\]

where

\[
\hat{\psi}_i^* = \hat{\psi}_i \cdot F_D^{-1} \cdot F_i.
\]

(24)

Thus an affine approximation element can be treated as a parametric element with the master element basis functions defined by (24).

For the four-node quadrilateral isoparametric element, it can be shown that \( k = 1 \). Therefore, from (23)

\[
\| \lambda_i \|_{1, \Omega} \leq C h |u|_{2, \Omega} \quad \text{as } h \to 0.
\]

(25)

For affine approximation elements on meshes composed of parallelograms, \( F_i = F_n \) and (25)
holds for \(\{\lambda_0\}\). However, if the mesh contains elements which aren’t parallelograms, then \(k = 0\) and

\[
\|\lambda_0\|_{1,1} \leq C|u|_{2,0} \text{ as } h \rightarrow 0. \tag{26}
\]

Numerical experiments in the next section test the reliability of these results.

**EXPERIMENTAL ASYMPTOTIC BEHAVIOR**

Let \(\Omega\) consist of the unit square:

\[
\Omega = \{x \in ]0,1[^2\}.
\]

Problem (2) is defined on \(\Omega\) where \(f\) is chosen such that

\[
u = \sin (\pi x) \sin (\pi y).
\]

Consider a sequence of partitions of the form (21) as depicted in Fig. 3. Note that this construction allows for mesh refinement in which the elements shrink in size, while their geometric distortion remains constant. The variable \(\nu\) will be treated as a parameter which assumes the following values:

\[
\nu \in \{0, .1, .2, .3, .4, .5\}.
\]

For each fixed value of \(\nu\), the sequences (22) are constructed.

Fig. 3. Distortion preserving mesh refinement.
Figure 4 contains plots of the natural logarithm of the $H^1$ norm of the terms in $\{\lambda_i\}$ verses the natural logarithm of $(1/h)$, with a different curve for each of the six different values of $\nu$. Figure 5 depicts the same measurements made on solutions obtained from unsmoothed double-linear affine approximation elements while Fig. 6 depicts corresponding measurements made on smoothed quadruple-linear affine approximation element solutions. Each curve represents meshes containing from 4 to 900 elements. Several observations follow:

1. For $\nu = 0$, the plots for affine approximation and isoparametric elements are identical, and exhibit slopes of $-1$ as predicted by (25).
2. As the mesh is distorted, the solutions obtained by affine approximation do not appear to be converging to the exact solution as suggested by (26).
3. Solutions obtained using smoothed quadruple-linear elements are practically indistinguishable from the solutions obtained using isoparametric elements over the range of meshes tested. However, it seems likely that they too will not converge to the exact solution as the mesh is further refined.

As a second asymptotic behavior test, consider the same problem solved using the element subdivision strategy depicted in Fig. 7. The two curves in Fig. 8 represent the error using isoparametric and unsmoothed double-linear elements on meshes of 4 to 1024 elements. Figure 9 represents the same measurements made on isoparametric and smoothed double-linear element solutions.

The unsmoothed double-linear solutions appear to be converging at the same rate as the isoparametric solutions. The smoothed double-linear solutions are practically identical to the isoparametric solutions. Proof of convergence using double-linear elements with the element subdivision strategy of Fig. 7 has not been achieved, although a deeper study of the problem is contained in [9].

**TIMING TESTS**

Reducing the computational expense associated with element stiffness matrix formulation is the primary advantage of affine approximation. Qualitatively, calculation of the element stiffness matrix for isoparametric elements can be considered as forming the sum of $L$ separate element stiffness matrices, where $L$ is the number of Gauss points. Calculation of each separate stiffness matrix requires evaluation of transformation metrics from the isoparametric map.

For affine approximation elements, calculation of the total element stiffness matrix can be considered as forming the sum of $M$ separate stiffness matrices, where $M$ is the number of master element subdomains. Each individual stiffness matrix requires the calculation of metric terms from the multi-linear map.

Calculation of metric terms from a linear map is less expensive than calculation of metric terms from an isoparametric map. Thus a reduction in total computational expense should be achieved using affine approximation provided

$$M \leq L.$$  

Table 1 lists the relative speed increases obtained for the linear elasticity problem using affine approximation elements. The speed increases are somewhat conservative. For example, when $k_f$ is obtained using the standard $RDB$ triple matrix product method[8] on element $E3$, then $k_f/k_{A3} = 8.7$ due to the large number of zeros in the $B$ matrix.

**CONCLUDING REMARKS**

The performance of affine approximation elements in the context of practical situations is of prime interest. The reduction in computational expense can be predicted a priori, however the accuracy of the approximate solution for practical computations using practical meshes is much more difficult to quantify. One generally accepted property of isoparametric elements is that accuracy usually decreases as the elements deviate from simple squares and cubes. For the
Fig. 4. Isoparametric asymptotic behavior using distortion-preserving mesh refinement.

Fig. 5. Unsmoothed double-linear asymptotic behavior using distortion-preserving mesh refinement.
Fig. 6. Smoothed quadrilateral asymptotic behavior using distortion-preserving mesh refinement.

Fig. 7. Mesh refinement by element subdivision.
Fig. 8. Isoparametric and unsmeared double-linear asymptotic behavior using element subdivision mesh refinement.

Fig. 9. Isoparametric and smoothed double-linear asymptotic behavior using element subdivision mesh refinement.
Affine approximation of isoparametric elements

Table 1: Speed increases using affine approximation on the linear elasticity problem. $m_i$ = the computer time required to calculate metrics from the isoparametric map, $m_i$ = the computer time required to calculate metrics from a linear map, $k_i$ = the computer time required to formulate the isoparametric element stiffness matrix, $k_{i1}$ = the computer time required to formulate the affine approximation element stiffness matrix using first order approximation, $k_{i2}$ = the computer time required to formulate the affine approximation element stiffness matrix using second order approximation. $E_i$ = the 8 node two-dimensional element with 4 Gauss points, $E_2$ = the 9 node two-dimensional element with 9 Gauss points, $E_3$ = the 20 node three-dimensional element with 27 Gauss points.

<table>
<thead>
<tr>
<th>E_i</th>
<th>E_2</th>
<th>E_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_i/m_i$</td>
<td>4.40</td>
<td>4.80</td>
</tr>
<tr>
<td>$k_i/k_{i1}$</td>
<td>7.15</td>
<td>4.70</td>
</tr>
<tr>
<td>$k_i/k_{i2}$</td>
<td>1.13</td>
<td>2.46</td>
</tr>
</tbody>
</table>

linear elasticity problem in particular, distorted elements are often observed to be "stiffer" than undistorted elements.

Since affine approximation elements are identical to isoparametric elements whenever the element domains are parallelograms/parallelepipeds, it is likely that they too will be adversely affected by element distortion. The extent to which they may be distorted and still produce satisfactory results will almost certainly rely on experience and engineering judgement.

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REFERENCES