Variational Inequalities for Generalized Quasi-Monotone Maps

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(Received February 2002; accepted January 2004)

Abstract—in this paper, we introduce a generalized quasi-monotone map and consider existence of solutions to generalized variational inequality problems for generalized quasi-monotone maps. Our result generalizes some theorems in [1]. © 2004 Elsevier Ltd. All rights reserved.

Keywords—M-η-quasimonotone, η-quasimonotone, M-η-monotone, η-monotone, Variational inequality problem, KKM-Fan theorem, Inner points.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real Banach space, K a nonempty convex subset of X, T : K → X* a map from K to the topological dual X* of X, and η : K × K → X a map.

Since a line segment seg [x, y] := {αx + (1 − α)y : α ∈ [0, 1]} for x, y ∈ K is an extension of a point y, the following generalized Stampacchia-type variational inequality problem (GSVIP) is considerable:

(GSVIP) find y ∈ K such that for each x (≠ y) ∈ K, there exists a u ∈ seg (x, y] such that

⟨T(u), η(x, u)⟩ ≥ 0.

In particular, if u = y for all x ∈ K, then we obtain the following variational inequality problem (SVIP):

(SVIP) find y ∈ K such that

⟨T(y), η(x, y)⟩ ≥ 0, for all x ∈ K.
Quasimonotonicity, which is known as a generalized monotonicity concept, has been studied in the context of variational inequality problems, complementarity problems, and other applications [1-12]. The concept of quasimonotonicity in \( \mathbb{R}^n \) introduced in [8] is weaker than pseudomonotonicity in the sense of Karamardian [13]. In case of gradient maps, quasimonotonicity corresponds to quasiconvexity of the underlying map, just as pseudomonotonicity corresponds to pseudoconvexity [8]. Since quasi-convex maps form a much broader class than pseudoconvex maps, the class of quasi-monotone maps is significantly larger than the class of pseudomonotone maps.

In [1], Hadjisavvas and Schaible considered the existence results of solutions to the following classic Stampacchia-type variational inequality problems (VIP):

\[
\text{(VIP)} \quad \text{find } y \in K \text{ such that } \quad (T(y), x - y) \geq 0, \quad \text{for all } x \in K,
\]

for quasi-monotone operators in Banach spaces. In the proof, they used some arguments significantly different from those in the pseudomonotone case, because the Minty-type lemma [14] does not hold for quasi-monotone maps. Furthermore, they considerably weakened the subset \( K \) by using the concept of inner points of \( K \) in some results of their paper. Recently, Luc [11] showed a sufficient condition under which quasimonotonicity coincides with pseudomonotonicity, and considered the existence of solutions to (VIP) for quasi-monotone maps and for densely pseudomonotone maps in Hausdorff topological vector spaces. He generalized Cottle and Yao’s result [15] and strengthened Hadjisavvas and Schaible’s result [1] using the concepts of positive points and segment-density.

In this paper, we introduce an \( M, \eta \)-quasi-monotone map, and then consider the existence of solutions to (GSVIP) for \( M, \eta \)-quasi-monotone hemicontinuous maps, and (SVIP) for \( \eta \)-quasi-monotone hemicontinuous maps in Banach spaces. Our result generalizes some theorems in [1].

Now we introduce some kinds of generalized quasi-monotone maps which generalize the classic quasi-monotone map.

**Definition 1.1.** Let \( K \) be a nonempty convex subset of a real Banach space \( X \), and let \( \eta : K \times K \rightarrow X \) and \( T : K \rightarrow X^* \) be maps.

(i) \( T \) is \( \eta \)-monotone on \( K \) if for every pair of distinct points \( x, y \in K \), we have

\[
\langle T(x) - T(y), \eta(x, y) \rangle \geq 0.
\]

(ii) \( T \) is \( M, \eta \)-monotone on \( K \) if for every pair of distinct points \( x, y \in K \), for \( \varepsilon \in [0, 1] \), there exists a \( u \in \text{seg} [\varepsilon x + (1 - \varepsilon) y, y] \) such that

\[
\langle T(x) - T(u), \eta(x, u) \rangle \geq 0.
\]

(iii) \( T \) is \( \eta \)-quasi-monotone on \( K \) if for every pair of distinct points \( x, y \in K \), we have

\[
\langle T(y), \eta(x, y) \rangle > 0 \iff \langle T(x), \eta(x, y) \rangle \geq 0.
\]

(iv) \( T \) is \( M, \eta \)-quasi-monotone on \( K \) if for every pair of distinct points \( x, y \in K \), for \( \varepsilon \in [0, 1] \), there exists a \( u \in \text{seg} [\varepsilon x + (1 - \varepsilon) y, y] \) such that

\[
\langle T(u), \eta(x, u) \rangle > 0 \iff \langle T(x), \eta(x, u) \rangle \geq 0.
\]
An $M$-$\eta$-quasi-monotone (respectively, $M$-$\eta$-monotone) map is called $M$-quasi-monotone (respectively, $M$-monotone) if $\eta(x, y) = x - y$, for all $x, y \in K$.

**Remark 1.1.** An $M$-quasi-monotone (respectively, $M$-monotone) map also can be regarded as an extension of a classic quasi-monotone (respectively, monotone) map.

Indeed, examples of an $M$-$\eta$-quasi-monotone, but not an $\eta$-quasi-monotone map can be found, even in two dimensions.

**Example 1.1.** Let $X = \mathbb{R}^2$ and $K = [0, \infty) \times \mathbb{R} \subset X$. Let $\eta : K \times K \to X$ be a map defined by

$$\eta((x, x'), (y, y')) = \max\{0, x^2 - y^2\},$$

for $(x, x'), (y, y') \in K$. Define $T : K \to X^*$ by

$$T((x, x')) = \begin{cases} (-(x + 1), 0), & \text{if } x \in [0, \infty) \cap \mathbb{Q}, \\ (x, 0), & \text{otherwise}, \end{cases}$$

then $T$ is $M$-$\eta$-quasi-monotone, but not $\eta$-quasi-monotone.

**Proof.** Let $(x, x')$, $(y, y')$ be any distinct elements of $K$.

If $x$ is an irrational number, then

$$\langle T((x, x')), \eta((x, x'), (y, y')) \rangle = x \cdot \max\{0, x^2 - y^2\} \geq 0.$$

If $x$ is a rational number, for $\varepsilon \in [0, 1]$ choose a rational number $u$ such that $(u, u')$ is a point on the seg $[\varepsilon(x, x') + (1 - \varepsilon)(y, y'), (y, y')]$ for $u' \in \mathbb{R}$, then

$$\langle T((u, u')), \eta((x, x'), (u, u')) \rangle \leq 0.$$

Hence, $T$ is $M$-$\eta$-quasi-monotone.

On the other hand, if $x$ is a positive rational number and $y$ is a positive irrational number less than $x$, then

$$\langle T((y, y')), \eta((x, x'), (y, y')) \rangle > 0,$$

but

$$\langle T((x, x')), \eta((x, x'), (y, y')) \rangle < 0.$$

Therefore, $T$ is not $\eta$-quasi-monotone.

**Example 1.2.** Define $\eta : K \times K \to X$ by $\eta((x, x'), (y, y')) = (x - y, x' - y')$ for $(x, x'), (y, y') \in K$ with the other same conditions in Example 1.1. We can easily show that $T$ is also $M$-$\eta$-quasi-monotone but not $\eta$-quasi-monotone.

**Definition 1.2.** A map $T : K \to X^*$ is said to be hemicontinuous if $T$ is continuous from the line segment of $K$ to the weak* topology of $X^*$.

### 2. VARIATIONAL INEQUALITY PROBLEMS FOR $M$-$\eta$-QUASI-MONOTONE MAPS

In this section, we consider the concept of inner points of $K$ and show the existence of solutions to (GSVIP) for an $M$-$\eta$-quasi-monotone map on the set having inner points.

**Definition 2.1.** Let $K$ be a nonempty subset of a Banach space $X$ and $\eta : K \times K \to X$ a map. A point $x_0 \in K$ is an inner point of $K$ if and only if, for any $\xi \in X^* \setminus \{0\}$ and $y \in K$, the following implication is valid:

$$\text{if } \langle \xi, \eta(x, y) \rangle \leq \langle \xi, \eta(x_0, y) \rangle, \quad \text{for all } x \in K,$$

$$\text{then } \langle \xi, \eta(x, y) \rangle = \langle \xi, \eta(x_0, y) \rangle, \quad \text{for all } x \in K.$$

The set of all inner points of $K$ is denoted by $\text{inn } K$.

If $\eta(x, y) = x - y$, for all $x, y \in K$, the concept of inner points is the same as one introduced by Hadjisievaas and Schaebe [1]. Every relative algebraic interior point [1] of $K$ is an inner point of $K$ if $\eta$ satisfies the following equality: $\eta(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 \eta(x_1, y) + \alpha_2 \eta(x_2, y)$. Propositions 2.2 and 2.4 with the KKM-Fan theorem are crucial tools in showing the existence of solutions to (GSVIP).
**Proposition 2.2.** Let $K$ be a convex subset of a Banach space $X$ and $\eta : K \times K \to X$ a map satisfying the following condition:

$$
\eta \left( \sum_{i=1}^{n} \alpha_i x_i, y \right) = \sum_{i=1}^{n} \alpha_i \eta(x_i, y), \quad \text{for } x_i, y \in K,
$$

\[ (*) \]

$$
\alpha_i \geq 0 \quad (i = 1, 2, \ldots, n), \quad \sum_{i=1}^{n} \alpha_i = 1.
$$

If $x_0 \in K$ and $z \in \text{inn} K$, then $\text{seg} \ (x_0, z) \subset \text{inn} K$.

**Proof.** Let $y$ be a point of $K$ and suppose that, for $\alpha \in (0, 1)$, there is some $\xi \in X^* \setminus \{0\}$ such that

$$
\alpha_i \geq 0 \quad (i = 1, 2, \ldots, n), \quad \sum_{i=1}^{n} \alpha_i = 1,
$$

$$
\langle \xi, \eta(x, y) \rangle \leq \langle \xi, \eta(\alpha x_0 + (1 - \alpha)z, y) \rangle, \quad \text{for all } x \in K.
$$

Then,

$$
\langle \xi, \eta(x, y) \rangle \leq \alpha \langle \xi, \eta(x_0, y) \rangle + (1 - \alpha)\langle \xi, \eta(z, y) \rangle, \quad \text{for all } x \in K. \quad (2.1)
$$

Substituting $z$ for $x$ in (2.1), we have

$$
\langle \xi, \eta(z, y) \rangle \leq \langle \xi, \eta(x_0, y) \rangle.
$$

Also, setting $x = x_0$ in (2.1), we get

$$
\langle \xi, \eta(x_0, y) \rangle \leq \langle \xi, \eta(z, y) \rangle.
$$

Since $\langle \xi, \eta(x_0, y) \rangle = \langle \xi, \eta(z, y) \rangle$, from (2.1) we have $\langle \xi, \eta(x_0, y) \rangle \leq \langle \xi, \eta(z, y) \rangle$, for all $x \in K$. By the assumption that $z$ belongs to $\text{inn} K$,

$$
\langle \xi, \eta(x, y) \rangle = \langle \xi, \eta(z, y) \rangle, \quad \text{for all } x \in K.
$$

Hence,

$$
\langle \xi, \eta(x, y) \rangle = \langle \xi, \alpha \eta(z, y) + (1 - \alpha)\eta(z, y) \rangle = \langle \xi, \alpha \eta(x_0, y) + (1 - \alpha)\eta(z, y) \rangle
$$

$$
= \langle \xi, \eta(\alpha x_0 + (1 - \alpha)z, y) \rangle, \quad \text{for all } x \in K.
$$

Thus, $\text{seg} \ (x_0, z) \subset \text{inn} K$. Since $z \in \text{inn} K$, $\text{seg} \ (x_0, z) \subset \text{inn} K$.

**Proposition 2.3.** Let $K$ be a convex subset of a Banach space $X$. Let $T : K \to X^*$ be an $M$-$\eta$-quasi-monotone map and $\eta : K \times K \to X$ a map such that $x \mapsto \eta(\cdot, x)$ is weakly continuous. If $T$ is hemicontinuous, then it is $\eta$-quasi-monotone.

**Proof.** Let $x$ and $y$ be distinct points of $K$ such that $\langle T(y), \eta(x, y) \rangle > 0$. Suppose that $\langle T(x), \eta(x, y) \rangle < 0$. For each $n \in \mathbb{N}$, since $T$ is $M$-$\eta$-quasi-monotone, there exists a point $u_n \in \text{seg} \ [(1/n)x + (1 - 1/n)y, y]$ such that

$$
\langle T(x), \eta(x, u_n) \rangle < 0 \text{ implies } \langle T(u_n), \eta(x, u_n) \rangle \leq 0.
$$

Since $u_n \to y$ along a line segment $\text{seg} \ [x, y]$ as $n \to \infty$, from the weak continuity of $x \mapsto \eta(\cdot, x)$, we have $\eta(x, u_n) \to \eta(x, y)$ weakly. Thus, $\langle T(x), \eta(x, u_n) \rangle \to \langle T(x), \eta(x, y) \rangle$ as $n \to \infty$, so there exists $n_0 \in \mathbb{N}$ such that

$$
\langle T(x), \eta(x, u_n) \rangle < 0, \quad \text{for } n \geq n_0.
$$

Therefore, $\langle T(u_n), \eta(x, u_n) \rangle \leq 0$ for $n \geq n_0$, since $T$ is hemicontinuous and $u_n \to y$ on $\text{seg} \ [x, y]$, $T(u_n) \to T(y)$ weakly*. Thus, $\langle T(y), \eta(x, u_n) \rangle \leq 0$ for $n \geq n_0$ and thus, $\langle T(y), \eta(x, y) \rangle \leq 0$, which is a contradiction.

Luc [11, Proposition 3.5] showed that if $T$ is hemicontinuous on a convex and closed set $K$, then quasimonotonicity is equivalent to dense quasimonotonicity. Hence, from Proposition 2.3, we know that if $T$ is hemicontinuous on a convex set $K$, then $M$-quasimonotonicity is equivalent to dense quasimonotonicity.
PROPOSITION 2.4. Let $K$ be a convex subset of a Banach space $X$ and $y \in K$. Let $T : K \to X^*$ be a hemicontinuous map and $\eta : K \times K \to X$ a map satisfying the condition $(\ast)$. 

(a) If $\langle T(x), \eta(x,y) \rangle \geq 0$, for all $x \in K$ and $\eta(y,y) = 0$, then $\langle T(y), \eta(x,y) \rangle \geq 0$, for all $x \in K$. 
(b) Let $T$ be $M, \eta$-quasi-monotone. If $x \mapsto \eta(\cdot, x)$ and $x \mapsto \eta(x, \cdot)$ are weakly continuous, then for every $x, y \in K$ with $\langle T(y), \eta(x,y) \rangle \geq 0$, one has either $\langle T(x), \eta(x,y) \rangle \geq 0$ or $\langle T(y), \eta(x,y) \rangle \leq 0$, for all $z \in K$.

PROOF. 

(a) Assume that $\langle T(x), \eta(x,y) \rangle \geq 0$, for all $x \in K$ and for each $n \in \mathbb{N}$, set $y_n = (1/n)x + (1 - 1/n)y$. Then,

$$\langle T(y_n), \eta(y_n,y) \rangle = \frac{1}{n} \langle T(y_n), \eta(x,y) \rangle + \left(1 - \frac{1}{n}\right) \langle T(y_n), \eta(y,y) \rangle = \frac{1}{n} \langle T(y_n), \eta(x,y) \rangle,$$

which implies $\langle T(y_n), \eta(x,y) \rangle \geq 0$. Since $T$ is hemicontinuous and $y_n \to y$ as $n \to \infty$ on $\text{seg} [x, y]$, $T(y_n) \to T(y)$ in the weak* topology, and thus, $\langle T(y), \eta(x,y) \rangle \geq 0$.

(b) Suppose that $\langle T(y), \eta(x_0,y) \rangle > 0$ for some $x_0 \in K$. Set $x_\alpha = \alpha x_0 + (1 - \alpha)x$ for some $\alpha \in (0, 1]$. Then, $\langle T(y), \eta(x_\alpha,y) \rangle > 0$ by the condition $(\ast)$. By Proposition 2.3, $T$ is $\eta$-quasi-monotone, so $\langle T(x_\alpha), \eta(x_\alpha,y) \rangle \geq 0$. Since $x_\alpha \to x$ on $\text{seg} [x, x_0]$ as $\alpha \to 0$, by the hemicontinuity of $T$, $T(x_\alpha)$ converges weakly* to $T(x)$. Thus, $\langle T(x_\alpha), \eta(x_\alpha,y) \rangle$ converges to $\langle T(x), \eta(x_\alpha,y) \rangle$ in $\mathbb{R}$. From the weak continuity of $x \mapsto \eta(x, \cdot)$, we have

$$\langle T(x), \eta(x_\alpha,y) \rangle \to \langle T(x), \eta(x,y) \rangle.$$ 

Therefore, $\langle T(x), \eta(x,y) \rangle \geq 0$, which implies that either $\langle T(x), \eta(x,y) \rangle \geq 0$ or $\langle T(y), \eta(z,y) \rangle \leq 0$, for all $z \in K$.

Let $K$ be a nonempty subset of a topological vector space $X$. Then, a map $F : K \to 2^X$ is said to be a Knaster-Kuratowski-Mazurkiewicz (in short, KKM) map if for each nonempty finite subset $N$ of $K$, co $N \subset F(N)$, where co $N$ is the convex hull of $N$.

The following KKM-Fan theorem in [16] is essential in our results.

THEOREM 2.5. (See [16].) Let $X$ be a topological vector space, $K$ an arbitrary nonempty subset of $X$, and $F : K \to 2^X$ a KKM map. If all the sets $F(x)$ are closed in $X$ and at least one is compact, then $\bigcap_{x \in K} F(x) \neq \emptyset$.

THEOREM 2.6. Let $X$ be a real Banach space and $K$ be a nonempty convex subset of $X$ with $\text{inn} K \neq \emptyset$. Let $T : K \to X^*$ be an $M, \eta$-quasi-monotone and hemicontinuous map, and $\eta : K \times K \to X$ a map satisfying the condition $(\ast)$ such that

(i) $x \mapsto \eta(x, \cdot)$ and $x \mapsto \eta(\cdot, x)$ are weakly continuous, and
(ii) $\eta(x,y) + \eta(y,x) = 0$ for $x, y \in K$.

Suppose that at least one of the following assumptions holds:

(a) $K$ is weakly compact, or
(b) $X$ is reflexive, $K$ is closed, and there exists $\rho > 0$ such that for each $x \in K$ with $\|x\| \geq \rho$, there exists $y \in K$ satisfying $\|y\| < \rho$ and $\langle T(x), \eta(x,y) \rangle \geq 0$.

Then, $(\text{GSVIP})$ is solvable.

PROOF. Let $F : K \to 2^K$ be a map defined by, for $x \in K$, $F(x) = \{y \in K : \text{ for } \varepsilon \in [0, 1]$, there exists $u \in \varepsilon x + (1-\varepsilon) y$, such that $\langle T(u), \eta(x,u) \rangle \geq 0 \}$. Then, for $x \in K$, $x \in F(x)$, hence, $F(x)$ is nonempty, and $F$ is a KKM map. In fact, suppose that $F$ is not a KKM map. Then, there exist $x_1, x_2, \ldots, x_n \in K$ such that $\co \{x_1, x_2, \ldots, x_n \} \notin \bigcup_{i=1}^n F(x_i)$. Put $w := \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n F(x_i)$ with $\sum_{i=1}^n \alpha_i = 1$ for $\alpha_i \geq 0$ $(i = 1, 2, \ldots, n)$. Since $w \notin F(x_i)$ for $i = 1, 2, \ldots, n$,
there exists an \( \varepsilon_i > 0 \) such that for all \( u_i \in \text{seg} \left[ \varepsilon_i x_i + (1 - \varepsilon_i)w, w \right] \subset \text{seg} \left[ \varepsilon x_i + (1 - \varepsilon)w, w \right] \), where \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \} \),

\[ \langle T(u_i), \eta(x_i, u_i) \rangle < 0. \]

In particular, \( \langle T(w), \eta(x_i, w) \rangle < 0 \) for each \( i = 1, 2, \ldots, n \). Since \( \sum_{i=1}^{n} \alpha_i = 1 \) for \( \alpha_i \geq 0 \) \( (i = 1, 2, \ldots, n) \), we have

\[ \sum_{i=1}^{n} \alpha_i \langle T(w), \eta(x_i, w) \rangle < 0. \]

On the other hand,

\[ \sum_{i=1}^{n} \alpha_i \langle T(w), \eta(x_i, w) \rangle = \left\langle T(w), \eta \left( \sum_{i=1}^{n} \alpha_i x_i, w \right) \right\rangle = \left\langle T(w), \eta(w, w) \right\rangle = 0, \]

which shows a contradiction. Thus, \( F \) is a KKM map. Define a map \( \bar{F} : K \to 2^K \) by \( \bar{F}(x) = \overline{F(x)} \) for each \( x \in K \), where \( \overline{F(x)} \) is the closure of \( F(x) \) with respect to the weak topology of \( X \). Since \( F(x) \subset \overline{F(x)} \) for \( x \in K \), \( \bar{F} \) is also a KKM map.

(a) We assume that \( K \) is weakly compact. Since each \( \overline{F(x)} \) is a weakly closed subset of a weakly compact set \( K \), it is weakly compact. By Theorem 2.5, \( \bigcap_{x \in K} \overline{F(x)} \) is nonempty.

Moreover, \( \bigcap_{x \in K} F(x) \) is nonempty. In fact, let \( \bar{x} \in \bigcap_{x \in K} \overline{F(x)} \), \( \bar{x} \in \text{inn} K \) and \( \bar{x} \) be an arbitrary point of \( K \). For each \( n \in \mathbb{N} \), set \( x_n = (1/n)\bar{x} + (1 - 1/n)\bar{x} \), then \( x_n \in \text{inn} K \) by Proposition 2.2. For each fixed \( n \in \mathbb{N} \), since \( \bar{x} \in \overline{F(x_n)} \), there exists a net \( \{x^n_\alpha\}_{\alpha \in I_n} \) in \( F(x_n) \) such that \( x^n_\alpha \to \bar{x} \) weakly in \( K \). The fact that \( x^n_\alpha \) belongs to \( F(x_n) \) for each \( \alpha \in I_n \) guarantees the existence of \( u^n_{\alpha,m} \) in \( \text{seg} \left[ (1/m)x_n + (1 - 1/m)x_n, x_n \right] \), \( m \in \mathbb{N} \), satisfying the following inequality:

\[ \langle T(u^n_{\alpha,m}), \eta(x_n, u^n_{\alpha,m}) \rangle \geq 0. \quad (2.2) \]

By Proposition 2.4(b), either \( \langle T(x_n), \eta(x_n, u^n_{\alpha,m}) \rangle \geq 0 \) or \( \langle T(u^n_{\alpha,m}), \eta(x, u^n_{\alpha,m}) \rangle \leq 0 \), for all \( x \in K \).

If \( \langle T(u^n_{\alpha,m}), \eta(x, u^n_{\alpha,m}) \rangle \leq 0 \), for all \( x \in K \), then by (2.2) we have

\[ \langle T(u^n_{\alpha,m}), \eta(x_n, u^n_{\alpha,m}) \rangle = 0. \]

So

\[ \langle T(u^n_{\alpha,m}), \eta(x, u^n_{\alpha,m}) \rangle \leq \langle T(u^n_{\alpha,m}), \eta(x_n, u^n_{\alpha,m}) \rangle, \]

for all \( x \in K \). Since \( x_n \in \text{inn} K \), \( \langle T(u^n_{\alpha,m}), \eta(x_n, u^n_{\alpha,m}) \rangle = 0 \), for all \( x \in K \). Therefore, \( u^n_{\alpha,m} \in \bigcap_{x \in K} F(x) \), so \( u^n_{\alpha,m} \) is a solution of (GSVIP).

Let us consider the other case, that is, assume that for each \( \alpha \in I_n \),

\[ \langle T(x_n), \eta(x_n, u^n_{\alpha,m}) \rangle \geq 0. \]

By the hemicontinuity of \( T \) and the weak continuity of \( x \mapsto \eta(x, \cdot) \), we have

\[ \langle T(x), \eta(x, u^n_{\alpha,m}) \rangle \geq 0, \quad \text{for all } \alpha \in I_n. \]

Since \( \{u^n_{\alpha,m}\}_{m \in \mathbb{N}} \) converges to \( x^n_\alpha \) as \( m \to \infty \) and \( \{x^n_\alpha\}_{\alpha \in I_n} \) converges to \( \bar{x} \), by the weak continuity of \( x \mapsto \eta(\cdot, x) \), we have

\[ \langle T(x), \eta(x, \bar{x}) \rangle \geq 0. \]

By Proposition 2.4(a), we obtain

\[ \langle T(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0. \]

Since \( x \) is arbitrary, \( \bar{x} \in \bigcap_{x \in K} F(x) \), which says that \( \bar{x} \) is a solution of (GSVIP). Consequently, (GSVIP) is solvable.
Suppose that Assumption (b) holds. And let $K_1 = \{x \in K : \|x\| \leq \rho\}$. First, we show that $\text{inn} \, K_1 \neq \emptyset$. By the assumption, $K_1$ is nonempty and so we can find $x_0 \in K$ with $\|x_0\| < \rho$.

Let $z \in \text{inn} \, K$ and choose $t > 0$ sufficiently small so that $z_0 = tz + (1 - t)x_0 \in K_1$, then by Proposition 2.2, $z_0 \in \text{inn} \, K$. Now we show that $z_0 \in \text{inn} \, K_1$. Suppose that there is some $\xi \in X^* \setminus \{0\}$ and let $y \in K_1$ such that

$$\langle \xi, \eta(x, y) \rangle \leq \langle \xi, \eta(z_0, y) \rangle,$$

for all $x \in K_1$.

For each $x \in K$, we can find $\alpha \in (0, 1]$ such that $\alpha x + (1 - \alpha)z_0 \in K_1$, so

$$\langle \xi, \eta(\alpha x + (1 - \alpha)z_0, y) \rangle \leq \langle \xi, \eta(z_0, y) \rangle.$$

By the condition ($\ast$), we deduce that $\langle \xi, \eta(x, y) \rangle \leq \langle \xi, \eta(z_0, y) \rangle$ for all $x \in K_1$. From the fact that $z_0 \in \text{inn} \, K$, we have

$$\langle \xi, \eta(x, y) \rangle = \langle \xi, \eta(z_0, y) \rangle,$$

for all $x \in K$, which implies that $z_0 \in \text{inn} \, K_1$. Since $K_1$ is a bounded closed and convex subset of a reflexive Banach space $X$, it is weakly compact [17]. From the fact that $\text{inn} \, K_1 \neq \emptyset$, by (a) there exists $\bar{x} \in K_1$ such that

$$\langle T(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0,$$

for all $x \in K_1$. (2.3)

If $\|\bar{x}\| = \rho$, by the hypothesis, there exists $y \in K$ such that $\|y\| < \rho$ and $\langle T(\bar{x}), \eta(\bar{x}, y) \rangle \geq 0$. On the other hand, by Condition (ii), we have $\langle T(\bar{x}), \eta(\bar{x}, y) \rangle \leq 0$ from (2.3). Thus, $\langle T(\bar{x}), \eta(\bar{x}, y) \rangle = -\langle T(\bar{x}), \eta(\bar{x}, y) \rangle = 0$. For each $x \in K$, if we choose $t \in (0, 1]$ such that $tx + (1 - t)y \in K_1$, then $\langle T(\bar{x}), \eta(tx + (1 - t)y, \bar{x}) \rangle \geq 0$, which implies

$$\langle T(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0.$$

If $\|\bar{x}\| < \rho$, then for each $x \in K$, we can find $t \in (0, 1]$ such that

$$tx + (1 - t)x \in K_1.$$

Hence, $\langle T(\bar{x}), \eta(tx + (1 - t)x, \bar{x}) \rangle \geq 0$, which implies

$$\langle T(\bar{x}), \eta(x, \bar{x}) \rangle \geq 0.$$

Thus, $\bar{x}$ is a solution of (SVIP), hence, a solution of (GSVIP).

Actually, the solution set of (GSVIP) in Theorem 2.6 is also that of (SVIP).

In the following remark, it is shown that the solution set of (SVIP) is a proper subset of that of (GSVIP) for the defined map $T$, which is not hemicontinuous.

**Remark 2.1.** In Example 1.1, if we replace $K = [0, \infty) \times \mathbb{R}$ with $K = [0, \infty) \times \{0\}$, then the solution sets of (GSVIP) and (SVIP) are the domain $K$ itself and a set $\{(x, 0) : x$ is an irrational number in $[0, \infty)\}$, respectively.

In fact, if $y$ is a positive irrational number, then $(y, 0)$ is a common solution of (SVIP) and (GSVIP), since

$$\langle T((y, 0)), \eta((x, 0), (y, 0)) \rangle \geq 0,$$

for any point $(x, 0)$ of $K$.

Let $y$ be a nonnegative rational number. If $x$ is a number larger than $y$, we have

$$\langle T((y, 0)), \eta((x, 0), (y, 0)) \rangle < 0.$$

On the other hand, for any $(x, 0) \in K$ distinct from $(y, 0)$, we can choose an irrational number $u$ belonging to $\text{seg} \, (x, y)$ so that $\langle T((u, 0)), \eta((x, 0), (u, 0)) \rangle \geq 0$. Therefore, $(y, 0)$ is not a solution of (SVIP) but a solution of (GSVIP), where $y$ is a nonnegative rational number.

Since a closed convex bounded subset of a reflexive Banach space is weakly compact, we obtain the following theorems as corollaries.
THEOREM 2.7. Let $K$ be a nonempty closed convex bounded subset of a reflexive Banach space $X$ under the other same conditions except (a) and (b) in Theorem 2.6. We obtain the same result of Theorem 2.6.

Since a quasi-monotone map is $M$-quasi-monotone, we obtain the following result.

THEOREM 2.8. (See [1].) Let $K$ be a nonempty closed convex subset of a real reflexive Banach space $X$. Let $T : K \rightarrow X^*$ be a hemicontinuous quasi-monotone map. Suppose that $\text{inn} \; K$ is nonempty. Assume further that either $K$ is bounded or there exists $\rho > 0$ such that, for each $x \in K$ with $\|x\| \geq \rho$, there exists $y \in K$ satisfying

$$
\|y\| < \rho \quad \text{and} \quad \langle T(x), x - y \rangle \geq 0.
$$

Then, (VIP) is solvable.

REFERENCES