Exceptional families of elements for set-valued mappings: An application to nonlinear complementarity problems

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Abstract

In this work, based upon the topological degree for set-valued mappings, instead of the Leray–Schauder type alternative and the technique of continuous selection, the notion of an exceptional family of elements for set-valued mappings in Isac’s sense is applied to nonlinear complementarity problems (for short, $S$-CP($F$, $K$, $H$)) and an existence result for solutions for $S$-CP($F$, $K$, $H$) is obtained.

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1. Introduction

It is well known that complementarity problems represent a wide class of mathematical models related to optimization, game theory, economics, engineering, mechanics, stochastic optimal control, etc. (see, for example, [2, 4,6,15]).

Because of the many important applications of complementarity problems, so far many authors have studied the explicit and implicit complementarity problems (see [10–12,16] and the references therein). Also, it is very important to study the feasibility of nonlinear complementarity problems. Recently, a variety of concepts of exceptional families of elements for continuous functions were introduced, and some feasibility and existence theorems for solutions for nonlinear complementarity problems defined by single-valued mappings were proved by many authors (see, for example, [5–7,13,19,20]). However, there are few papers dedicated to notions of exceptional families of elements for set-valued mappings and the study of the solutions of complementarity problems defined by set-valued mappings by using such notions (see [6,8,10]).

Inspired and motivated by the above research work, in this work we apply the notion of an exceptional family of elements for set-valued mappings in Isac’s sense [6] to nonlinear complementarity problems (for short,
S-CP\((F, K, H)\) and obtain an existence result for solutions for S-CP\((F, K, H)\) based upon the topological degree for set-valued mappings (see [1,3,14]), instead of the Leray–Schauder type alternative and the technique of continuous selection.

2. Preliminaries

Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space, where \(\langle \cdot, \cdot \rangle\) is the inner product defined on \(H\). We say that \(K \subset H\) is a closed convex cone if and only if \(K\) is a closed subset of \(H\) and the following properties are satisfied:

\[
(A_1) \ K + K \subseteq K; \quad (A_2) \lambda K \subseteq K, \quad \forall \lambda \in R_+.
\]

Whenever a closed convex cone \(K \subset H\) is defined, we have an ordering on \(H\) defined by \(x \leq y\) if and only if \(y - x \in K\). By definition the dual of \(K\) is \(K^* = \{y \in H : \langle x, y \rangle \geq 0\ \text{for all} \ x \in K\}\). Note that \(K^*\) is also a closed convex cone of \(H\). We say that an ordered Hilbert space \((H, \langle \cdot, \cdot \rangle, K)\) is a vector lattice if and only if, for every pair \((x, y)\) of elements of \(H\), the supremum \(x \vee y\) and the infimum \(x \wedge y\) exist in \(H\). If \((H, \langle \cdot, \cdot \rangle, K)\) is a vector lattice we define for every \(x \in H\), \(x^+ = x \vee 0\), \(x^- = (-x) \vee 0\), and \(|x| = x^+ - x^-\). Other properties of \(x^+, x^-\) and \(|x|\) are presented and proved in [17].

We say that an ordered Hilbert space \((H, \langle \cdot, \cdot \rangle, K)\) is a Hilbert lattice if and only if

\[
(B_1) \text{ \(H\) is a vector lattice;}
(B_2) ||x|| = ||x|| \text{ for every } x \in H;
(B_3) 0 \leq x \leq y \text{ implies that } ||x|| \leq ||y|| \text{ for every } x, y \in H.
\]

Let \(D \subset H\) be a closed convex set. We denote the projection onto \(D\) by \(P_D\), that is, for every \(x \in H\), \(P_D(x)\) is the unique element satisfying

\[
\|x - P_D(x)\| = \min_{y \in D} \|x - y\|.
\]

As is well known, if \(K \subset H\) is a closed convex cone, then the projection operator \(P_K\) is characterized by the following result: for every \(x \in H\), \(P_K(x)\) is the (unique) element in \(K\) satisfying

\[
(i) \ \langle P_K(x) - x, y \rangle \geq 0 \quad \text{for all } y \in K;
(ii) \ \langle P_K(x) - x, P_K(x) \rangle = 0.
\]

The operator \(P_K\) has several special properties and the following notion was defined and studied in [9]. We say that \(K\) is an isotone projection cone if and only if, for every \(x, y \in H\), \(x \leq y\) implies that \(P_K(x) \leq P_K(y)\). As is well known [9], if \((H, \langle \cdot, \cdot \rangle, K)\) is a Hilbert lattice, then \(K\) is an isotone projection cone and moreover, \(P_K(x) = x^+\) for every \(x \in H\).

Without other specifications, we will denote by \(B(H)\) the family of all nonempty closed bounded subsets of \(H\), and \(CB(H)\) a subspace of \(B(H)\) defined as follows:

\[
CB(H) = \{A \in B(H) : A \text{ is convex}\}.
\]

Let \(B_r = \{x \in H : ||x|| < r\}\) and \(S_r = \partial B_r = \{x \in H : ||x|| = r\}\). Moreover, we shall use the following notation. A set-valued mapping \(W : B_r \to B(H)\) is upper semicontinuous (u.s.c.) at \(x_0 \in B_r\) if, for every open set \(V\) containing \(W(x_0)\), there exists an open set \(U\) containing \(x_0\) such that \(W(U) \subseteq V\). \(W\) is lower semicontinuous (l.s.c.) at \(x_0 \in B_r\) if, for every open set \(V\) intersecting \(W(x_0)\), there exists an open set \(U\) containing \(x_0\) such that \(W(x) \cap V \neq \emptyset\) for every \(x \in U\). \(W\) is u.s.c (l.s.c.) on \(B_r\) if it is u.s.c. (l.s.c.) at every point of \(B_r\). If \(W\) is a single-valued mapping, both the definitions above provide the ordinary definition of continuity. Furthermore, \(W\) is continuous on \(B_r\) if it is both u.s.c and l.s.c. on \(B_r\). Let

\[
\tilde{J}_C(B_r, H) = \{F : B_r \to B(H) : F \text{ is continuous, compact, } F(x) \in \tilde{C}(H) \text{ and } x \notin F(x) \text{ for every } x \in S_r\}.
\]

As in [3] by \(\tilde{J}_{CV}(B_r, H)\) we shall define the set of all associated vector fields, i.e., a set-valued mapping \(T : B_r \to B(H)\) belongs to \(\tilde{J}_{CV}(B_r, H)\) if and only if there is \(F \in \tilde{J}_C(B_r, H)\) such that \(T(x) = x - F(x)\). Noting that for \(T \in \tilde{J}_{CV}(B_r, H)\), we have \(\theta \notin T(x)\) for all \(x \in S_r\). Two compact vector fields \(T, G \in \tilde{J}_{CV}(B_r, H)\) are homotopic [3]
Lemma 3.1. Given a set-valued mapping $F : H \rightarrow 2^H$ a set-valued mapping with nonempty values. In this section, we consider the following nonlinear complementarity problem for set-valued mapping:

$$S-\text{CP}(F, K, H) : \begin{cases}
\text{find } x_0 \in K \text{ and } v_0 \in F(x_0) \text{ such that } \\
v_0 \in K^* \text{ and } \langle x_0, v_0 \rangle = 0.
\end{cases}$$

We say that $\{x_0, v_0\}$ is a solution of $S-\text{CP}(F, K, H)$.

A set-valued mapping $F : H \rightarrow 2^H$ has a fixed point in $H$ if there exists an $x \in H$ such that $x \in F(x)$. The following conclusions are useful in the proof of main result of this work, which generalizes the corresponding result of [7] from single-valued mappings to set-valued mappings.

Lemma 3.1 ([6]). Given a set-valued mapping $F : H \rightarrow 2^H$ with nonempty values, then $S-\text{CP}(F, K, H)$ has a solution if and only if the set-valued mapping

$$\Phi(x) = P_K(x) - F(P_K(x))$$

for all $x \in H$,

has a fixed point in $H$. If $x_0$ is a fixed point of $\Phi$, then $(x_0^+, x_0^-)$ is a solution of $S-\text{CP}(F, K, H)$, where $x_0^+ = P_K(x_0)$ and $x_0^- = x_0^+ - x_0 \in F(x_0^+)$.

From Lemma 3.1, it is easy to prove that the following conclusion holds.

Lemma 3.2. Given a set-valued mapping $F : H \rightarrow 2^H$, $S-\text{CP}(F, K, H)$ has a solution if and only if the nonlinear equation

$$\theta \in x - P_K(x - F(x))$$

is solvable in $K$, where $\theta$ denotes the zero point of $H$.

In [6], Isac introduced the concept of an exceptional family of elements for set-valued $F$ with respect to the convex cone $K \subset H$.

Definition 3.1 ([6]). We say that a family of elements $\{x^r\}_{r>0} \subset K$ is an exceptional family of elements for $F : H \rightarrow 2^H$, with respect to the convex cone $K \subset H$, if and only if for every real number $r > 0$ there exists a real number $\mu_r > 0$ and $v^r \in F(x^r)$ such that the vector $u_r := v^r + \mu_r x^r$ satisfies the following conditions:

$$(C_1) \ u_r \in K^*; \quad (C_2) \ \langle u_r, x^r \rangle = 0; \quad (C_3) \ \|x^r\| \rightarrow \infty \text{ as } r \rightarrow \infty.$$
The concept of a regular exceptional family of elements for a single-valued mapping \( f \) was independently discovered by Smith \cite{18} under the name exceptional sequence and a few years ago by G. Isac under the name of opposite radial sequence, in an unpublished note. Remark that the notion of an exceptional sequence introduced by Smith cannot be defined with respect to an arbitrary convex cone, and cannot be related to topological degree.

In \cite{6,10}, the authors studied \( S\text{-CP}(F, K, H) \) by using the Leray–Schauder type alternative and the technique of continuous selection. Now, we consider \( S\text{-CP}(F, K, H) \) through the topological degree for set-valued mappings.

**Theorem 3.1.** For any set-valued mapping \( F : H \rightarrow 2^H \) such that \( F(x) = x - G(x) \), where \( G : H \rightarrow B(H) \) is continuous compact with nonempty closed convex values, there exists either a solution for \( S\text{-CP}(F, K, H) \) or a regular exceptional family of elements for \( F \) with respect to \( K \).

**Proof.** Using Lemma 3.2 we know that the solvability of \( S\text{-CP}(F, K, H) \) is equivalent to the equation

\[
\theta \in x - P_K(x - F(x))
\]

being solvable in \( K \). Let \( T(x) = x - P_K(x - F(x)) \) and

\[
h(x, t) = tx + (1 - t)T(x), \quad t \in [0, 1]. \tag{3.1}
\]

From the definition of \( T \) we have

\[
h(x, t) = tx + (1 - t)(x - P_K(x - F(x))) \\
= x - (1 - t)P_K(x - F(x)) \\
= x - H(x, t), \tag{3.2}
\]

where \( H(x, t) = (1 - t)P_K(x - F(x)) = (1 - t)P_K(G(x)), h(x, 0) = T(x) \) and \( h(x, 1) = x \). It is easy to see that \( H : B_r \times [0, 1] \rightarrow B(H) \) is continuous compact with nonempty closed convex values and \( h \in \tilde{J}_V C(B_r \times [0, 1], H) \).

By using the topological degree for set-valued mappings (denoted by \( \text{Deg}(\cdot) \)) and applying Lemma 2.1, we have the following two cases:

1. There exists an \( r > 0 \) such that

\[
\theta \notin h(x, t), \quad \text{i.e., } x \notin H(x, t), \quad \forall x \in S_r, \quad \forall t \in [0, 1].
\]

Thus, \( T, I \in \tilde{J}_V C(B_r, H) \) are homotopic. Then Lemma 2.1(2) and (3) imply that

\[
\text{Deg}(T) = \text{Deg}(I).
\]

It is well known that \( \text{Deg}(I) = \text{deg}(I) = 1 \) and so \( \text{Deg}(T) = 1 \). Again by Lemma 2.1(1), this means that the ball \( \tilde{B}_r \) contains at least one solution to the equation \( \theta \in T(x) \). Therefore \( S\text{-CP}(F, K, H) \) has a solution.

2. For each \( r > 0 \) there exist a point \( x^r \in S_r \) and a scalar \( t_r \in [0, 1] \) such that

\[
\theta \in h(x^r, t_r), \quad \text{i.e., } x^r \in H(x^r, t_r). \tag{3.3}
\]

We now claim that \( t_r \neq 0 \). Indeed, if \( t_r = 0 \), from (3.1) we have that \( \theta \in T(x^r) \) and hence the problem \( S\text{-CP}(F, K, H) \) has a solution. We also remark that \( t_r 
eq 1 \). In fact, if \( t_r = 1 \), using (3.3) and again (3.1), we deduce that \( \theta = x^r \), which is impossible since \( x^r \in S_r \). Hence, we can say that either \( S\text{-CP}(F, K, H) \) has a solution or for any \( r > 0 \) there exist \( x^r \in S_r \) and \( t_r \in (0, 1) \) such that \( \theta \in h(x^r, t_r) \). From (3.2) we have

\[
\theta \in x^r - (1 - t_r)P_K(x^r - F(x^r))
\]

or

\[
\frac{1}{1 - t_r}x^r \in P_K(x^r - F(x^r)).
\]

Since \( K \) is a cone, we know that \( x^r \in K \). Applying the properties (i) and (ii) of operator \( P_K \) we deduce that there exists \( v_r \in F(x^r) \) such that

\[
\left\langle \frac{1}{1 - t_r}x^r - (x^r - v^r), y \right\rangle 
\geq 0 \quad \text{for all } y \in K, \tag{3.4}
\]
and
\[
\left\langle \frac{1}{1-t_r}x^r - (x^r - v^r), \frac{1}{1-t_r}x^r \right\rangle = 0. \tag{3.5}
\]
Letting \(\mu_r = \frac{t_r}{1-t_r}\) in (3.4) and (3.5), we have
\[
\langle \mu_r x^r + v^r, y \rangle \geq 0 \quad \text{for all } y \in K, \tag{3.6}
\]
and
\[
\langle \mu_r x^r + v^r, x^r \rangle = 0. \tag{3.7}
\]
Considering (3.6) and (3.7) and the fact that for any \(r > 0, x^r \in K\) and \(\|x^r\| = r\), we know that \(\{x^r\}_{r>0}\) is a regular exceptional family of elements for \(F\) with respect to \(K\). This completes the proof. \(\Box\)

**Remark 3.2.** As an application of exceptional family of elements for set-valued mappings in Isac's sense, in **Theorem 3.1**, we establish the alternative theorem of the existence of a solution for \(S\text{-CP}(F, K, H)\) based on the topological degree for set-valued mappings, instead of the Leray–Schauder type alternative and the technique of continuous selection. As a consequence, the method used in **Theorem 3.1** is quite different from that in [6,10].

From **Theorem 3.1**, we have the following result.

**Theorem 3.2.** For any set-valued mapping \(F : H \rightarrow 2^H\) with \(F(x) = x - G(x)\), where \(G : H \rightarrow B(H)\) is continuous compact with nonempty closed convex values, if there does not exist a regular exceptional family of elements for \(F\) with respect to \(K\), then \(S\text{-CP}(F, K, H)\) is solvable.

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**References**