Abstract

The paper is concerned with the existence of almost periodic mild solutions to evolution equations of the form
\[
\dot{u}(t) = Au(t) + f(t),
\]
where \(A\) generates a \(C\)-semigroup and \(f\) is almost periodic. Using the evolution \(C\)-semigroup associated with the equation under consideration, we obtain various sufficient conditions for the existence of almost periodic solutions to (1) which extend previous ones to a more general class of ill-posed equations involving \(C\)-semigroups.

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In this paper we are concerned with the existence of almost periodic solutions to equations of the form
\[
\frac{du}{dt} = Au + f(t),
\]
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where $A$ is a (unbounded) linear operator which generates a $C$-semigroup of linear operators on the Banach space $\mathbb{X}$ and $f$ is an almost periodic function in the sense of Bohr (for the definition and properties see \cite{1, 11, 16}).

The above-mentioned problem has been extensively studied by many authors in the case where $A$ generates a $C_0$-semigroups (see, e.g., \cite{2–6, 9, 13, 16, 22–27, 29, 34, 35} and our list of references). On the other hand, as shown in many works, the notion of $C$-semigroups introduced by Da Prato allows one to approach a larger class of evolution equations which are ill-posed (see, e.g., \cite{8, 14, 15, 30–33}). Recently, there has been an increasing interest in extending results on the asymptotic behavior of solutions in the case of $C_0$-semigroups to the ill-posedness case which involves $C$-semigroups (see \cite{9, 12, 36} and the references therein for more information).

In this paper we extend recent results on the existence of almost periodic solutions in \cite{6, 13, 23, 24, 34}, to the case where $A$ is assumed to generate only a $C$-semigroup of linear operators. To this end, we consider the evolution $C$-semigroup associated with the $C$-semigroup generated by the operator $A$ in Eq. (1) (see definition in Section 3). We refer the reader to \cite{4, 7, 10, 20, 21} and the references therein for more information on the history and further applications of evolution semigroups to the study of the asymptotic behavior of dynamical systems and differential equations such as exponential dichotomy and stability. Recently, evolution semigroups have been applied to study almost periodic solutions of evolution equations in \cite{23}. In this direction see also \cite{6, 22}, and especially the monograph \cite{13} in which a systematic presentation has been made. It turns out that under appropriate conditions this method works well for evolution equations associated with $C$-semigroups.

As results we obtain Theorems 2.7, 2.12 which yield Corollaries 2.8, 2.13, the main results of this paper.

1. Preliminaries

1.1. Notation

Throughout the paper, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{X}$ stand for the sets of real, complex numbers, and a complex Banach space, respectively; $L(\mathbb{X})$, $BUC(\mathbb{R}, \mathbb{X})$, $AP(\mathbb{X})$ denote the spaces of all linear bounded operators on $\mathbb{X}$, all $\mathbb{X}$-valued bounded uniformly continuous, and all almost periodic functions in Bohr’s sense (see \cite[p. 4]{16}) with sup-norm, respectively. For a linear operator $A$, we denote by $D(A)$, $\sigma(A)$ the domain and the spectrum of $A$. Let $(S(t))_{t \in \mathbb{R}}$ be the translation group on $BUC(\mathbb{R}, \mathbb{X})$ given by

$$(S(t)u)(s) := u(t + s), \quad \forall s, t \in \mathbb{R}, \forall u \in BUC(\mathbb{R}, \mathbb{X}),$$

whose infinitesimal generator is $D := d/dt$, defined on $D(D) := BUC^1(\mathbb{R}, \mathbb{X})$ which consists of all functions $f \in BUC(\mathbb{R}, \mathbb{X})$ such that the derivative $f'$ exists as an element of $BUC(\mathbb{R}, \mathbb{X})$.

1.2. Spectral theory of functions

In the present paper $sp(u)$ stands for the Beurling spectrum of a given bounded uniformly continuous function $u$, which is defined by
\[ \text{sp}(u) := \{ \xi \in \mathbb{R} : \forall \varepsilon > 0, \, \exists \varphi \in L^1(\mathbb{R}) : \text{supp} \, \tilde{\varphi} \subset (\xi - \varepsilon, \xi + \varepsilon), \, \varphi \ast u \neq 0 \}, \]

where

\[ \tilde{\varphi}(s) := \int_{-\infty}^{\infty} e^{-ist} f(t) dt; \quad \varphi \ast u(s) := \int_{-\infty}^{\infty} \varphi(s-t)u(t) dt. \]

The notion of Beurling spectrum of a function \( u \in BUC(\mathbb{R}, \mathbb{X}) \) coincides with the one of Carleman spectrum, which consists of all \( \xi \in \mathbb{R} \) such that the Carleman–Fourier transform of \( u \), defined by

\[ \hat{u}(\lambda) := \begin{cases} \int_{0}^{\infty} e^{-\lambda t} u(t) dt & (\text{Re} \lambda > 0), \\ -\int_{0}^{\infty} e^{\lambda t} u(-t) dt & (\text{Re} \lambda < 0), \end{cases} \]

has no holomorphic extension to any neighborhood of \( i\xi \) (see [27, Proposition 0.5, p. 22]).

Below we list some properties of the spectra of functions which we will need in the sequel.

**Proposition 1.1.** Let \( u, u_n, v \in BUC(\mathbb{R}, \mathbb{X}) \) such that \( \lim_{n \to \infty} \| u_n - u \| = 0 \) and \( \psi \in S \) (the space of all continuous functions that decay faster than any polynomials at infinity). Then

(i) \( \text{sp}(u) \) is closed;
(ii) \( \text{sp}(u + v) \subset \text{sp}(u) \cup \text{sp}(v) \);
(iii) \( \text{sp}(\psi \ast u) \subset \text{sp}(u) \cap \text{supp} \, \tilde{\psi} \);
(iv) \( \text{sp}(u - \psi \ast u) \subset \text{sp}(u) \cup \text{supp}(1 - \tilde{\psi}) \);
(v) if \( \tilde{\psi} \equiv 1 \) on a neighborhood of \( \text{sp}(u) \), then \( \psi \ast u = u \);
(vi) if \( \text{sp}(u) \cap \text{supp} \, \tilde{\psi} = \emptyset \), then \( \psi \ast u = 0 \);
(vii) if \( \text{sp}(u_n) \subset \Lambda, \, \forall n \), then \( \text{sp}(u) \subset \Lambda \);
(viii) if \( \text{sp}(u) \) is countable and \( \mathbb{X} \) does not contain any subspace which is isomorphic to the space of numerical sequences \( c_0 \), then \( u \) is almost periodic;
(ix) if \( u \) is uniformly continuous and \( \text{sp}(u) \) is discrete, then \( u \) is almost periodic.

**Proof.** We refer the reader to [27, Propositions 0.4 and 0.6, Theorem 0.8, pp. 20–25] and [16, Chapter 6] for the proofs of (i)–(viii) and [3] for (ix). \( \square \)

**Remark 1.2.** The condition \( \mathbb{X} \nsubseteq c_0 \) in Proposition 1.1(viii) can be replaced by one of the following conditions (see, e.g., [16, p. 92], [2, Section 3], and [29]):

(a) \( u(\mathbb{R}) \) is relatively weakly compact in \( \mathbb{X} \), or
(b) \( u \) is totally ergodic, i.e., the limit

\[ M_\eta u = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} e^{i\eta s} S(s)u(s) ds \]

exists in \( BUC(\mathbb{R}, \mathbb{X}) \) for all \( \eta \in \mathbb{R} \).
The Beurling spectrum of a function $u$ in $BUC(\mathbb{R}, \mathcal{X})$ coincides with its Arveson spectrum, defined by (see [2, Section 2])

$$i \text{ sp}(u) = \sigma(\mathcal{D}_u),$$

where $\mathcal{D}_u$ is the infinitesimal generator of the restriction of the translation group $(S(t)|_{\mathcal{M}_u})_{t \in \mathbb{R}}$ in $BUC(\mathbb{R}, \mathcal{X})$ to the closed subspace $\mathcal{M}_u := \text{span}\{S(\tau)u, \tau \in \mathbb{R}\}$. Set $\Lambda(\mathcal{X}) := \{u \in BUC(\mathbb{R}, \mathcal{X}) : \text{ sp}(u) \subset \Lambda\}$, where $\Lambda$ is a closed subset of the real line. By Proposition 1.1(vii), $\Lambda(\mathcal{X})$ is a closed subspace of $BUC(\mathbb{R}, \mathcal{X})$. In particular, we have (see, e.g., [22]) the following lemma.

**Lemma 1.3.** Let $\Lambda$ be a closed subset of the real line. Then

$$\sigma(\mathcal{D}_{\Lambda(\mathcal{X})}) = i \Lambda.$$  (2)

1.3. $C$-semigroups: definition and basic properties

**Definition 1.4.** Let $\mathcal{X}$ be a Banach space and let $C$ be an injective operator in $L(\mathcal{X})$. A family $\{S(t); t \geq 0\}$ in $L(\mathcal{X})$ is called an exponentially bounded $C$-semigroup if the following conditions are satisfied:

(i) $S(0) = C$,
(ii) $S(t + s)C = S(t)S(s)$ for $t, s \geq 0$,
(iii) $S(\cdot)x : [0, \infty) \to \mathcal{X}$ is continuous for any $x \in \mathcal{X}$,
(iv) there are $M \geq 0$ and $a \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{at}$ for $t \geq 0$.

We define an operator $A$ as follows:

$$D(A) = \{x \in \mathcal{X} : \lim_{h \to 0^+}(S(h)x - Cx)/h \in R(C)\}$$

$$Ax = C^{-1} \lim_{h \to 0^+}(S(h)x - Cx)/h, \quad \forall x \in D(A).$$

This operator is called the generator of $(S(t))_{t \geq 0}$.

**Lemma 1.5.** Let $C$ be an injective linear operator in $L(\mathcal{X})$ and let $(S(t))_{t \geq 0}$ be a $C$-semigroup with generator $A$. Then the following assertions hold true:

(i) $S(t)S(s) = S(s)S(t)$ for $t, s \geq 0$;
(ii) if $x \in D(A)$, then $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$, and

$$\int_0^t S(\xi)Ax \, d\xi = S(t)x - Cx, \quad \forall t \geq 0;$$

(iii) $\int_0^t S(\xi)x \, d\xi \in D(A)$ and $A \int_0^t S(\xi)x \, d\xi = S(t)x - Cx$ for every $x \in \mathcal{X}$ and $t \geq 0$;
(iv) $A$ is closed and satisfies $C^{-1}AC = \Lambda$;
(v) $R(C) \subset D(A)$. 

In general, $A$ is not necessarily densely defined. For more information about $C$-semigroups we refer the reader to [8,14,17,18].

We will need the following lemma in the next section.

**Lemma 1.6.** Let $(T(t))_{t \geq 0}$ be a $C$-semigroup with generator $A$, and let the operator $T(1) - C$ be invertible. Then the generator $A$ is invertible as well.

**Proof.** By the strong continuity of $(T(t))_{t \geq 0}$, then a map $B: \mathbb{X} \ni x \mapsto \int_0^1 T(t) x \, d\xi$ is a bounded linear operator on $\mathbb{X}$. By (ii) and (iii) of Lemma 1.5 we have that

\[
(T(1) - C)x = \begin{cases} ABx, & \forall x \in \mathbb{X}, \\ BAx, & \forall x \in D(A). \end{cases}
\]

(3)

Since $T(1) - C$ is invertible, we have

\[(T(t) - C)^{-1} BA \subset AB(T(t) - C)^{-1} = I.\]

This means that $A$ is invertible and $A^{-1} = (T(t) - C)^{-1} B = B(T(t) - C)^{-1}$, completing the proof of the lemma. $\square$

2. Main results

2.1. Evolution $C$-semigroups

We consider the equation

\[
\frac{dx}{dt} = Ax + f(t), \quad x \in \mathbb{X}, \; t \in \mathbb{R},
\]

where $A$ generates a $C$-semigroup of bounded linear operators on $\mathbb{X}$. We always assume that $C$ is an injection if not stated otherwise. We first recall the following definitions.

**Definition 2.1.**

(i) An $\mathbb{X}$-valued function $u$ on $\mathbb{R}$ is said to be a *solution on $\mathbb{R}$* to Eq. (4) for given linear operator $A$ and $f \in BUC(\mathbb{R}, \mathbb{X})$ (or sometime, *classical solution*) if $u \in BUC^1(\mathbb{R}, \mathbb{X})$, $u(t) \in D(A)$, $\forall t$ and $u$ satisfies Eq. (4) for all $t \in \mathbb{R}$.

(ii) Let $A$ be the generator of a $C$-semigroup of linear operators $(T(t))_{t \geq 0}$. An $\mathbb{X}$-valued continuous function $u$ on $\mathbb{R}$ is said to be a *mild solution on $\mathbb{R}$* to Eq. (4) for a given $f \in BUC(\mathbb{R}, \mathbb{X})$ if $u$ satisfies

\[
Cu(t) = T(t - s)u(s) + \int_s^t T(t - r) f(r) \, dr, \quad \forall t \geq s.
\]

(5)

As shown later in Lemma 2.9, by the injectiveness of the operator $C$, every classical solution is a mild one. We introduce the following operator $L_M$. 

Definition 2.2. Let $\mathcal{M}$ be a closed subspace of $BUC(\mathbb{R}, X)$. We define the operator $L_M$ on $\mathcal{M}$ as follows: $u \in D(L_M)$ if and only if $u \in \mathcal{M}$ and there is $f \in \mathcal{M}$ such that

$$Cu(t) = T(t-s)u(s) + \int_s^t T(t-r) f(r) \, dr, \quad \forall t \geq s$$

(6)

and in this case $L_M u := f$.

Lemma 2.3. Let $A$ be the generator of a $C$-semigroup and $\mathcal{M}$ be a closed subspace of $BUC(\mathbb{R}, X)$. Then the operator $L_M$ is well-defined single valued operator.

Proof. First we show that $L_M$ is a well-defined single valued operator on $\mathcal{M}$. To this purpose, we suppose that there are $u, f_1, f_2 \in \mathcal{M}$ such that $L_M u = f_1$, $L_M u = f_2$. By definition this means that Eq. (18) holds for $f = f_i$, $i = 1, 2$. We now show that $f_1 = f_2$. In fact, we have

$$Cu(t) = T(t-s)u(s) + \int_s^t T(t-r) f_1(r) \, dr$$

$$= T(t-s)u(s) + \int_s^t T(t-r) f_2(r) \, dr, \quad \forall t \geq s.$$  

This yields that

$$\int_s^t T(t-r) (f_1(r) - f_2(r)) \, dr = 0, \quad \forall t \geq s.$$  

(7)

Since $(T(t))_{t \geq 0}$ is strongly continuous, the integrand in the left-hand side of (7) is continuous with respect to $r$ ($r \leq t$). Thus,

$$0 = \lim_{s \to t^-} \frac{1}{t-s} \int_s^t T(t-r) (f_1(r) - f_2(r)) \, dr = T(t-t) (f_1(t) - f_2(t))$$

$$= C(f_1(t) - f_2(t)), \quad \forall t.$$  

Since $C$ is an injection, this yields that $f_1(t) = f_2(t), \forall t$. $\Box$

Definition 2.4. Let $\mathcal{Y}$ be either $C_0(\mathbb{R}, X)$ or $AP(X)$ and let $(T(t))_{t \geq 0}$ be a $C$-semigroup on $X$. Then the following family of bounded linear operators $(T^h)_{h \geq 0}$ on $\mathcal{Y}$ is called the evolution $C$-semigroup associated with $(T(t))_{t \geq 0}$:

$$(T^h v)(t) := T(h) v(t-h), \quad v \in \mathcal{Y}, \quad h \geq 0.$$  

(8)
Lemma 2.5. Under the above notation the evolution semigroup \((T^h)_h \geq 0\) is a \(\tilde{C}\)-semigroup, where \(\tilde{C}\) denotes the operator of multiplication by \(C\) in \(L(Y)\).

Proof. First, we can show that it is a strongly continuous family of bounded linear operators. This can be checked using the precompactness of the range of a function \(f \in Y\). Next, we have

\[
\frac{T^\alpha T^\beta f(t)}{h} = T(\alpha)T(\beta)f(t - (\alpha + \beta)) = CT(\alpha + \beta)f(t - (\alpha + \beta))
\]

This shows that \((T^h)_h \geq 0\) is a \(\tilde{C}\)-semigroup.

Consider the integral equation (6) and the operator \(L := L_{AP}(X)\).

Lemma 2.6. Let \((T^h)_h \geq 0\) be the evolution \(C\)-semigroup as above. Then its generator \(\mathcal{G}\) is defined as follows:

\[
D(\mathcal{G}) = D(\mathcal{L})
\]

\[
\mathcal{G}u = -\mathcal{L}u, \quad \forall u \in D(\mathcal{L}).
\]

Proof. By definition, if \(u \in D(\mathcal{L})\) we have

\[
\frac{T^h u - Cu}{h}(t) = \frac{1}{h}[T(h)u(t - h) - Cu(t)] = -\frac{1}{h} \int_{t-h}^{t} T(t - \xi)f(\xi)d\xi,
\]

\(\forall t \in \mathbb{R}, h \geq 0\).

By the precompactness of the range of \(f \in AP(X)\), the right-hand side converges uniformly to \(-Cf(t)\). Hence, by the definition of the generator of \(C\)-semigroup, we have \(u \in D(\mathcal{G})\) and

\[
\mathcal{G}u = C^{-1}\lim_{h \downarrow 0} \frac{T^h u - Cu}{h} = -C^{-1}Cf = -f = -\mathcal{L}u.
\]

Conversely, let \(u \in D(\mathcal{G})\). Then by definition,

\[
\int_{0}^{h} T^\xi \mathcal{G} u d\xi(t) = [T^h u - Cu](t) = T(h)u(t - h) - Cu(t)
\]

\(\forall t \in \mathbb{R}, h \geq 0\).

\[
= \int_{0}^{t} T(t - \xi)f(\xi)d\xi.
\]
Now for every \( t \geq s \) letting \( s := t - h \), we have
\[
Cu(t) = T(t - s)u(s) - \int_s^t T(t - \xi)f(\xi)\,d\xi, \quad \forall t \geq s.
\]
This shows that \( u \in D(\mathcal{L}) \) and \( \mathcal{L}u = -\mathcal{G}u \). Finally, we have proved that \( D(\mathcal{L}) = D(\mathcal{G}) \) and \( \mathcal{G}u = -\mathcal{L}u \) for any \( u \in D(\mathcal{L}) \).

2.2. Evolution C-semigroups in invariant subspaces of \( AP(\mathbb{X}) \)

Let \( \Lambda \) be a closed subset of the real line. Then we denote by \( \Lambda AP(\mathbb{X}) \) the subspace of \( AP(\mathbb{X}) \) consisting of all functions \( u \in AP(\mathbb{X}) \) such that \( \text{Sp}(u) \subset \Lambda \). We consider the evolution \( C \)-semigroup \( (T_h)_{h \geq 0} \) on \( AP(\mathbb{X}) \). Obviously, \( \Lambda AP(\mathbb{X}) \) is invariant under \( (T_h)_{h \geq 0} \). For the sake of simplicity we also denote by \( (T_h)_{h \geq 0} \) its restriction to \( \Lambda AP(\mathbb{X}) \).

Theorem 2.7. Let \( \Sigma \) be a closed subset of the unit circle in the complex plane and let \( C \) be an injection in a Banach space \( \mathbb{X} \). Assume further that
\[
(\sigma(C) \times \Sigma) \cap \sigma(T(1)) = \emptyset.
\]
Then for every \( f \in \Lambda AP(\mathbb{X}) \) there exists a unique mild solution \( x_f \) of Eq. (4) such that \( e^{i\sigma(x_f)} \subset \Sigma \).

Proof. Let \( \Lambda := \{ \xi \in \mathbb{R} : e^{i\xi} \in \Sigma \} \). Obviously, \( \Lambda \) is a closed subset of the real line. Consider the evolution \( C \)-semigroup \( (T_A^h)_{h \geq 0} \) on \( \Lambda AP(\mathbb{X}) \). By Lemma 2.6, it is sufficient to prove the invertibility of the generator \( \mathcal{G} \) in the function space \( \Lambda AP(\mathbb{X}) \). By definition, we have
\[
T_h^b u = T(h)u(-h) = T(h)_A S(-h)u, \quad \forall u \in \Lambda AP(\mathbb{X}), \; t \in \mathbb{R}, \; h \geq 0.
\]
Now, by Lemma 1.6 it suffices to show that \( T^1 - \tilde{C} = \tilde{T}(1)S(-1) - \tilde{C} \) is invertible. In fact, by the Weak Spectral Mapping Theorem (see [10, p. 283]), we have
\[
\sigma(S_A(-1)) = e^{-\sigma(D)},
\]
where \( D \) is the (differential) operator \( d/dt \) which generates the group of translations \( (S_A(t))_{t \in \mathbb{R}} \). By Lemma 1.3, we have \( \sigma(D) = i\Lambda \) so that
\[
\sigma(S_A(-1)) = e^{-i\Lambda} = \Sigma^{-1}.
\]
Since \( \tilde{C} \), \( \tilde{T}(1) \), and \( S(-1) \) are commutative, by [28, Theorem 11.23] the following spectral estimate holds:
\[
\sigma(\tilde{C} - T^1) \subset \sigma(C) - \Sigma^{-1} \times \sigma(T(1)).
\]
By the assumption of the theorem, it is clear that \( 0 \notin \sigma(\tilde{C} - T^1) \), i.e., \( \tilde{C} - T^1 \) is invertible. This completes the proof of the theorem. \( \square \)
Corollary 2.8. Let $C$ be an injection on a Banach space $X$ and let $A$ generate a $C$-semigroup $(T(t))_{t \geq 0}$. Then for any almost periodic function $f$ with $(\sigma(C) \times e^{\text{sp}(f)}) \cap \sigma(T(1)) = \emptyset$ there exists a unique almost periodic mild solution $x_f$ to Eq. (4) such that $e^{\text{sp}(x_f)} \subset e^{\text{sp}(f)}$.

Proof. It suffices to take $\Sigma := e^{\text{sp}(f)}$ in the above theorem. \qed

2.3. The resonant case

In this subsection we consider the existence of almost periodic mild solutions to Eq. (4) in the case where the condition $(\sigma(C) \times \Sigma) \cap \sigma(T(1)) = \emptyset$ fails to be satisfied.

To proceed we need the following lemma.

Lemma 2.9. Let $C$ be a bounded injection, $(T(t))_{t \geq 0}$ be a $C$-semigroup with generator $A$ and let $f$ be a continuous function on the real line. Then a continuous function $u$ is a solution of Eq. (5) if and only if both of the following assertions hold:

\begin{align} 
\int_0^t Cu(\xi) \, d\xi & \in D(A), \quad \forall t \in \mathbb{R}, \quad (13) \\
Cu(t) - Cu(0) & = A \int_0^t Cu(s) \, ds + \int_0^t Cf(s) \, ds, \quad \forall t \in \mathbb{R}. \quad (14)
\end{align}

Proof. Let us fix any real $s \in \mathbb{R}$. For any given $x_0 \in X$ define the function

$$w(t) = T(t-s)x_0 + \int_s^t T(t-\xi)f(\xi) \, d\xi, \quad t \geq s.$$ 

We now show the following.

Claim. $w$ satisfies $\int_s^t w(\xi) \, d\xi \in D(A), \forall t \geq s$ and

$$w(t) - w(s) = A \int_s^t w(\xi) \, d\xi + C \int_s^t f(\xi) \, d\xi, \quad \forall t \geq s. \quad (15)$$

Indeed, from the following fact in $C$-semigroup theory:

$$T(t)x = Cx = A \int_0^t T(\xi)x \, d\xi, \quad \forall x \in X,$$

we see that for all $\eta \leq t$,
\[ A \int_0^{t-\eta} T(\xi) x d\xi = T(t-\eta)x - Cx, \quad \forall x \in X. \]  

(16)

By a change of order of integration, we get

\[ \int_s^t d\xi \int_s^\xi T(\xi - \eta) f(\eta) d\eta = \int_s^t d\eta \int_\eta^t T(\xi - \eta) f(\eta) d\xi, \quad \forall t \geq s. \]

By (16) and the closedness of \( A \), we have

\[ \int_s^t d\xi \int_s^\xi T(\xi - \eta) f(\eta) d\eta \in D(A) \]

and

\[ A \int_s^t d\xi \int_s^\xi T(\xi - \eta) f(\eta) d\eta = \int_s^t d\eta A \int_\eta^t T(\xi - \eta) f(\eta) d\xi = \int_s^t T(t-\eta)f(\eta)d\eta - C \int_s^t f(\eta)d\eta. \]

(17)

Therefore, by (16) and (17), we have \( \int_s^t w(\xi) d\xi \in D(A) \) and

\[ A \int_s^t w(\xi) d\xi = T(t-s)x_0 - Cx_0 + A \int_s^t \int_s^\xi T(\xi - \eta) f(\eta) d\eta d\xi \]

\[ = T(t-s)x_0 - Cx_0 + \int_s^t T(t-\eta)f(\eta)d\eta - C \int_0^s f(\eta)d\eta \]

\[ = w(t) - w(s) - C \int_s^t f(\xi) d\xi. \]

Therefore, \( w(t) \) satisfies Eq. (15) for all \( t \geq s \).

We are now in a position to prove the lemma.

**Sufficiency.** Suppose that \( u \) satisfies the conditions (13) and (14). Then, we will show that it is a solution to Eq. (5). In fact, it follows from (13) and (14) that \( \int_s^t Cu(\xi) d\xi \in D(A) \) and

\[ Cu(t) - Cu(s) = A \int_s^t Cu(\xi) d\xi + C \int_s^t f(\xi) d\xi, \quad \forall t \geq s. \]  

(18)
On the other hand, under the above notation, for \( x_0 := u(s) \) our previous argument shows that \( w \) satisfies Eq. \((15)\) for all \( t \geq s \). Define \( g(t) = w(t) - Cu(t), \forall t \geq s \). Obviously, from \((18)\) and \((15)\) we obtain
\[
g(t) = A \int_s^t g(\xi) \, d\xi, \quad \forall t \geq s.
\]
Since \( A \) generates a \( C \)-semigroup, the Cauchy problem
\[
\begin{aligned}
dx(t)/dt &= Ax(t), & t > s, \\
Cx(s) &= 0 \in D(A)
\end{aligned}
\]
has a unique solution zero. Hence, \( w(t) - Cu(t) = g(t) = 0, \forall t \geq s \), i.e., \( u(t) \) satisfies Eq. \((5)\) for all \( t \geq s \).

**Necessity.** Now suppose that \( u \) satisfies Eq. \((5)\). For any fixed \( s \in \mathbb{R} \) define
\[
w(t) := Cu(t) = T(t-s)u(s) + \int_s^t T(t-\xi) f(\xi) \, d\xi, \quad t \geq s.
\]
By the above claim, \( \int_s^t w(\xi) \, d\xi \in D(A), \forall t \geq s \) and
\[
w(t) - w(s) = A \int_s^t w(\xi) \, d\xi + C \int_s^t f(\xi) \, d\xi, \quad \forall t \geq s.
\]
Since \( s \) is arbitrary in \( \mathbb{R} \), it follows in particular that the necessity holds.

From this lemma, by the injectiveness of the operator \( C \), it follows in particular that every classical solution is a mild one.

Below we denote \( \sigma_i(A) := \{ \xi \in \mathbb{R} : i\xi \in \sigma(A) \} \).

**Corollary 2.10.** Let all conditions of the above lemma be satisfied and let \( f \) be a bounded uniformly continuous function on \( \mathbb{R} \). Moreover, assume that \( u \) be a bounded uniformly continuous mild solution of Eq. \((4)\). Then the following estimates hold:
\[
sp(u) \subset \sigma_i(A) \cup sp(f). \tag{19}
\]

**Proof.** The estimate \((19)\) can be proved as in [2]. For the reader’s convenience, we give it here. Assume that there is \( \xi_0 \in \mathbb{R} \) such that \( i\xi_0 \in \rho(A) \). Taking the Laplace transforms of both sides of \((14)\), we can easily show that (for \( \text{Re} \lambda > 0 \) and \( \lambda \) close to \( i\xi_0 \))
\[
\widehat{Cu}(\lambda) = R(\lambda, A)Cu(0) + R(\lambda, A)\widehat{C}f(\lambda). \tag{20}
\]
Therefore, if \( \xi_0 \notin sp(f) \), the function \( \widehat{Cu}(\lambda) \) has an analytic extension around \( i\xi \). Therefore, \( sp(Cu(\cdot)) \subset \sigma_i(A) \cup sp(Cf(\cdot)) \). Observe that by the definition of Beurling spectrum of a bounded uniformly continuous function, if \( C \) is a bounded linear injective operator, then \( sp(Cu(\cdot)) = sp(u(\cdot)) \). And hence we have \((19)\).
Definition 2.11. Let $A$ and $B$ be two subsets of a topological space. Then we say that $A$ and $B$ are completely disjoint if the closures $\bar{A}$ and $\bar{B}$ are disjoint.

Theorem 2.12. Let $C$ be an injection, $(T(t))_{t \geq 0}$ be a $C$-semigroup with generator $A$, and assume that $f \in BUC(\mathbb{R}, \mathbb{X})$ have precompact range. Moreover, assume that Eq. (4) has a bounded uniformly continuous mild solution $u$. Then the following assertions hold true:

(i) If $e^{t\sigma_i(A)} \setminus e^{t\text{sp}(f)}$ and $e^{t\text{sp}(f)}$ are completely disjoint, then Eq. (4) has a bounded uniformly continuous mild solution $w$ such that $e^{t\text{sp}(w)} \subset e^{t\text{sp}(f)}$.

(ii) If $\sigma_i(A)$ is bounded, and $\sigma_i(A) \setminus \text{sp}(f)$ and $\text{sp}(f)$ are completely disjoint, then Eq. (4) has a bounded uniformly continuous mild solution $w$ such that $\text{sp}(w) \subset \text{sp}(f)$.

Proof. (i) First notice that if we denote by $F \subset BUC(\mathbb{R}, \mathbb{X})$ the set of all functions in $BUC(\mathbb{R}, \mathbb{X})$ at which the evolution $C$-semigroup $(T^h)_{h \geq 0}$ is strongly continuous, then $F$ is a closed subspace of $BUC(\mathbb{R}, \mathbb{X})$. In particular, if $f$ has precompact range, then $f \in F$. Moreover, under the assumptions of the theorem, $f \in F$. And by a simple computation we can show that the mild solution $u \in F$. Actually, Lemma 2.6 can be extended to the function space $F$. Now, denote

$$\Sigma := e^{t\sigma_i(A)} \cup e^{t\text{sp}(f)}, \quad \Sigma_1 := e^{t\sigma_i(A)} \setminus e^{t\text{sp}(f)}, \quad \text{and} \quad \Sigma_2 := e^{t\text{sp}(f)}.$$

By [24, Theorem 1] there is a projection $P : \Sigma(X) \rightarrow \Sigma_2(X)$, where $\Sigma(X) := \{v \in F : e^{t\text{sp}(v)} \subset \Sigma\}$ and $\Sigma_2(X) := \{v \in F : e^{t\text{sp}(v)} \subset \Sigma_2\}$. Moreover, $P$ commutes with the evolution $C$-semigroup associated with $(T(t))_{t \geq 0}$. Since $u$ is a bounded uniformly continuous mild solution of Eq. (4) by Corollary 2.10, we have $u \in \Sigma(X)$. Therefore, by Lemma 2.6,

$$-\tilde{T}f = -P\tilde{T}f = P \lim_{h \searrow 0} \frac{T^h u - \tilde{T}u}{h} = \lim_{h \searrow 0} \frac{PT^h u - P\tilde{T}u}{h} = \lim_{h \searrow 0} \frac{T^h Pu - \tilde{T}Pu}{h}.$$

Hence, $Pu$ is a mild solution to Eq. (4) with desired spectrum.

(ii) In this case we let $A := \sigma_i(A) \cup \text{sp}(f)$, $A_1 := \sigma_i(A) \setminus \text{sp}(f)$, and $A_2 := \text{sp}(f)$. Then, there is a continuous projection $Q : A(X) \rightarrow A_2(X)$ which commutes with $(T^h)_{h \geq 0}$. By a similar argument as above, we get the mild solution $w := Qu$ with desired spectrum. □

Corollary 2.13. Let all assumptions of Theorem 2.12 be satisfied. Moreover, assume that $\mathbb{X}$ does not contain $c_0$ and $f$ is in AP($\mathbb{X}$). Then the following assertions hold:

(i) If $e^{t\sigma_i(A)} \setminus e^{t\text{sp}(f)}$ and $e^{t\text{sp}(f)}$ are completely disjoint, and $e^{t\text{sp}(f)}$ is countable, then Eq. (4) has an almost periodic mild solution $w$ such that $e^{t\text{sp}(w)} \subset e^{t\text{sp}(f)}$.

(ii) If $\sigma_i(A)$ is bounded, the sets $\sigma_i(A) \setminus \text{sp}(f)$ and $\text{sp}(f)$ are completely disjoint, then Eq. (4) has an almost periodic mild solution $w$ such that $\text{sp}(w) \subset \text{sp}(f)$.

Proof. By Theorem 2.12, we can find mild solutions $w$ with desired spectrum. Next, the almost periodicity of the mild solution $w$ follows from well-known criteria for almost periodicity of vector-valued functions (see, e.g., [2,3,16]). □

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References


