Topology and its Applications 158 (2011) 2343–2355

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Intrinsic approach spaces on domains

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ARTICLE INFO

To our dear friend and collaborator Eraldo Giuli on the occasion of his 70th birthday

MSC:
06A06
06b35
06F30
54A05
54B30
54E35

Keywords:
Domain
Continuous dcpo
Weightable quasi metric
Approach space
Scott topology
Quantification
Kernel condition

ABSTRACT

The paper is a contribution to quantifiability of domains. We show that every domain X, regardless of cardinality conditions for a domain bases, is quantifiable in the sense that there exists an approach structure on X (Lowen (1997) [9]), defined by means of a gauge of quasi metrics, inducing the Scott topology. We get weightability for free and in the case of an algebraic domain satisfying the Lawson condition (Lawson (1997) [8]), a quantifying approach space can be obtained with a weight satisfying the kernel condition.

1. Introduction

Motivated by central problems in theoretical computer science, mathematical structures have been created to model semantics of programming languages. These are the so called semantic domains and are mainly intended to define the meaning of a computer program. The models that are useful in this respect are continuous directed complete partial orders (domains) endowed with the Scott topology [5]. Quantitative domain theory has been created to be able to extract also quantitative information, such as measures of complexity in the sense of [4]. The mathematical structures that have been used in this respect are domains endowed with a weightable quasi metric structure or equivalently a partial metric structure inducing the Scott topology [10]. This has lead to the study of quantifiability of domains. Important results showing that all continuous domains with a countable basis are quantifiable have been obtained by Waszkiewicz [13], and later using different methods by Schellekens [12]. In this context weightable quasi metrics are constructed by taking some infinite sum $\sum \frac{1}{2^n}$ over some suitable subset of $\mathbb{N}$. The role of $(\frac{1}{2^n})_{n\in\mathbb{N}}$ could be replaced by some other suitable sequence, which means that, although the existence of such a quasi metric is important, numerical values computed with it are not canonically determined.

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[1] The author has a PhD fellowship of the research Foundation Flanders (FWO).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.01.025

The paper is a contribution to quantifiability of domains. We show that every domain X, regardless of cardinality conditions for a domain bases, is quantifiable in the sense that there exists an approach structure on X [9], inducing the Scott topology and which in general need not be topological, nor quasi metric. We get weightability for free as we show that such an approach space always has a subbase $\mathcal{H}$ for its gauge, consisting of weightable quasi metrics. In the case of an algebraic domain satisfying the Lawson condition [8], a quantifying approach space can be obtained with weights w_q corresponding to quasi metrics $q \in \mathcal{H}$ choosen in such a way that the set $\bigcap_{q\in\mathcal{H}} w_q^{-1}(0)$ coincides with the set of maximal elements of the domain. Ongoing research in collaboration with M. Schellekens, explores applications of quantifying approach spaces in the setting of measures for complexity. We also prove that there are structural advantages of working in the category of approach spaces as it is a topological construct [1], meaning that an arbitrary structured source has a unique initial lift. In particular this implies that the construct has products, subobjects, coproducts and quotients and that the set of objects on a fixed underlying set forms a complete lattice. We show that on the other hand the construct of weightable quasi metric spaces is not topological.

In order to fix notations we recall that X stands for an arbitrary set, 2^X stands for its powerset and $[0,\infty]$ is considered as a lattice-ordered semigroup with the usual order and addition denoted as $\leqslant$ and $+$. Supremum (maximum) and infimum (minimum) in $[0,\infty]$ will be denoted by sup (max) and inf (min) respectively. A pre quasi metric on X is a function $q : X \times X \to [0,\infty]$ which vanishes on the diagonal, if moreover q satisfies the triangular inequality it is called a quasi metric. So our terminology differs slightly from the usual one since in this paper all (pre) quasi metrics are allowed to take the value ∞ and the fact that both distances $q(x,y)$ and $q(y,x)$ are zero need not imply that the points x and y coincide. We denote by qMet the construct of all quasi metric spaces with non-expansive maps as morphisms and by Met the full subconstruct consisting of all metric spaces, i.e. those fulfilling the symmetry condition. The collection of all (quasi) metrics on a set X is denoted by (q)Met(X).

Quasi metric structures do not behave well with respect to the formation of initial structures, in particular products. The product in qMet of an infinite family of quasi metric spaces is not compatible with the topological product of the associated underlying topologies. As a remedy to this defect, the common supercategory App (the objects of which are called approach spaces) of Top and qMet was introduced [9]. The basic difference between approach spaces and metric spaces is that in the former, one specifies and axiomatizes point-set distances, where such a point-set distance, unlike the situation for quasi metric spaces, is not necessarily derivable from the point–point distances.

We recall some definitions and results on approach spaces that are used frequently in the sequel. For more information we refer to [9].

A *distance* on a set X is a function $\delta : X \times 2^X \to [0,\infty]$ with the following properties:

(D1) $\forall x \in X \; \delta(x,\{x\}) = 0$,
(D2) $\forall x \in X \; \delta(x,\emptyset) = \infty$,
(D3) $\forall x \in X \; \forall A, B \in 2^X \; \delta(x, A\cup B) = \min\{\delta(x,A), \delta(x,B)\}$,
(D4) $\forall x \in X \; \forall A \in 2^X \; \forall \epsilon \in [0,\infty] \; \delta(x,A) \leqslant \delta(x,A^{(\epsilon)}) + \epsilon$ with $A^{(\epsilon)} = \{x \mid \delta(x,A) \leqslant \epsilon\}$.

The morphisms between approach spaces are called contractions. A map $f : (X,\delta_X) \to (Y,\delta_Y)$ is a *contraction* if

$$\forall x \in X \; \forall A \subseteq X \quad \delta_Y\big(f(x), f(A)\big) \leqslant \delta_X(x,A)$$

The construct of approach spaces and contractions is denoted by App. Despite the naturalness of the distance function just defined, the following concepts play a bigger role in the sequel. A *gauge* $\mathcal{G}$ of quasi metrics on X is an ideal in the pointwise ordered lattice qMet(X) which is saturated in the following sense:

(G) A quasi metric e belongs to $\mathcal{G}$ whenever $\forall x \in X \; \forall \varepsilon > 0 \; \forall \omega < \infty \; \exists q \in \mathcal{G} \; \forall y \in X \; (\min\{e(x,y),\omega\} \leqslant q(x,y) + \varepsilon)$.

Any subset $\mathcal{H}$ of qMet(X) generates a gauge $\mathcal{G}$ in the following way:

$$e \in \mathcal{G} \quad \Leftrightarrow \quad \forall x \in X \; \forall \varepsilon > 0 \; \forall \omega < \infty \; \exists q_1 \cdots q_n \in \mathcal{H}$$
$$\forall y \in X \quad \big(\min\{e(x,y),\omega\} \leqslant \sup\nolimits_{i=1}^n q_i(x,y) + \varepsilon\big)$$

In this case we say that $\mathcal{H}$ is a *gauge subbasis* for $\mathcal{G}$. Recall that a subset $\mathcal{H}$ of qMet(X) is an *ideal basis* if it is directed (meaning that for any d, e in $\mathcal{H}$ there exists $c \in \mathcal{H}$ such that $d \leqslant c$ and $e \leqslant c$ for the pointwise order). If a subset $\mathcal{H}$ of qMet(X) is an ideal basis and generates $\mathcal{G}$ then $\mathcal{H}$ is called a *gauge basis*. In order to derive the gauge $\mathcal{G}$ from a gauge basis $\mathcal{H}$ we require a simplified version of the saturation operation, where the finite collection $q_1 \cdots q_n$ can be replaced by one single q.

The third concept we need in this paper is a localization of the previous one. An *approach system* on X is a collection $\mathcal{A} = (\mathcal{A}(x))_{x\in X}$, where each $\mathcal{A}(x)$ is an ideal in the pointwise ordered lattice $[0,\infty]^X$, satisfying certain axioms for which we refer the reader to [9], one of which is the following saturation: φ belongs to $\mathcal{A}(x)$ whenever

$$\forall \epsilon > 0 \; \forall \omega < \infty \; \exists \varphi_\epsilon^\omega \in \mathcal{A}(x) \; \forall y \in X \quad \big(\min\{\varphi(y),\omega\} \leqslant \varphi_\epsilon^\omega(y) + \epsilon\big)$$

As was the case for gauges, if for $(\mathcal{B}(x))_{x\in X}$ each $\mathcal{B}(x)$ is only an ideal basis, then one can generate the approach system via saturation.

The foregoing concepts, the distance, the gauge and the approach system are equivalent in the sense that one uniquely determines the other. The following are some important formulas for going from one structure to another. None of these transitions occurs at any loss, the original structure can always be completely retrieved.

From a gauge one easily derives the distance via the formula

$$\delta(x, A) = \sup_{q\in\mathcal{G}} \inf_{a\in A} q(x, a)$$

The other way around given the distance δ one calculates the gauge by the following formula

$$\mathcal{G} = \left\{ q \in \mathrm{qMet(X)} \;\middle|\; \forall A \subseteq X\ \forall x \in X\ \inf_{a\in A} q(x, a) \leqslant \delta(x, A) \right\}$$

From a gauge (basis) $\mathcal{H}$ one easily describes an approach system $(\mathcal{A}(x))_{x\in X}$. A basis of the approach system is given by

$$\mathcal{B}(x) = \left\{ q(x, .) \;\middle|\; q \in \mathcal{H} \right\}$$

The other way around, given an approach system $(\mathcal{A}(x))_{x\in X}$ on X then

$$\mathcal{G} = \left\{ q \in \mathrm{qMet(X)} \;\middle|\; \forall x \in X\ q(x, .) \in \mathcal{A}(x) \right\}$$

defines the gauge. For a proof of these formulas we refer to [9]. On a given set X we will often denote a given approach space simply by X and then we will use its distance δ_X, its gauge $\mathcal{G}_X$ or its approach system $\mathcal{A}_X$ whenever appropriate.

Contractions between approach spaces can be equivalently described by means of gauges,

$$\forall q \in \mathcal{G}_Y \quad q \circ f \times f \in \mathcal{G}_X$$

Moreover initial structures have an easy description. Given a structured source $(f_i : X \to (X_i, \mathcal{G}_{X_i}))_{i\in I}$, the initial gauge on X is generated by the subbase $\{q_i \circ f_i \times f_i \mid i \in I,\ q_i \in \mathcal{G}_i\}$. The construct App constitutes a framework wherein other important constructs can be fully embedded. Top is embedded as a full concretely reflective and concretely coreflective subconstruct and qMet is embedded as a concretely coreflective subconstruct. The embedding of topological spaces is determined by associating with every topological space $(X, \mathcal{T})$ (with closure of A written as $\mathrm{cl}\, A$) the distance

$$\delta_{\mathcal{T}}(x, A) = \begin{cases} 0 & x \in \mathrm{cl}\, A \\ \infty & x \notin \mathrm{cl}\, A \end{cases}$$

The property of being $\{0, \infty\}$-valued actually characterizes approach spaces which are derived from topological spaces.

Every approach space (X, δ) has two natural topological spaces associated with it, the topological coreflection (which has to be thought of as the underlying topology in the same way as for metric spaces) and the topological reflection. In this paper we will mainly deal with the coreflection. It is the topological space $(X, \mathcal{T}_\delta)$ determined by the closure $x \in \mathrm{cl}\, A \Leftrightarrow \delta(x, A) = 0$.

When the approach space is defined through its gauge with gauge subbasis $\mathcal{H}$, the topology $\mathcal{T}_\delta$ can be calculated as the supremum in the lattice of topologies on X of the collection $\{\mathcal{T}_q \mid q \in \mathcal{H}\}$.

The embedding of quasi metric spaces is given in the usual way that one defines a distance between points and sets in a metric space, given q we put $\delta_q(x, A) = \inf_{a\in A} q(x, a)$. The quasi metric coreflection of an approach space is the quasi metric space (X, q_δ) determined by $q_\delta(x, y) = \delta(x, \{y\}) = \sup_{q\in\mathcal{G}} q(x, y)$.

Both topological spaces and quasi metric spaces can be viewed as special kinds of approach spaces via the full embedding functors

$$\mathsf{Top} \to \mathsf{App} \begin{cases} (X, \mathcal{T}) \mapsto (X, \delta_{\mathcal{T}}) \\ f \mapsto f \end{cases} \quad \text{and} \quad \mathsf{qMet} \to \mathsf{App} \begin{cases} (X, q) \mapsto (X, \delta_q) \\ f \mapsto f \end{cases}$$

Hence in the sequel, when useful or required, we will make no distinction between a topological space in any of its various classical forms (with open sets, closure, convergence, neighborhoods) or in its form as a topological (approach) space with a $\{0, \infty\}$-valued distance, and analogously for quasi metric spaces.

Just as an approach space which comes from a topological space can be recognized, so can a quasi metric (approach) space be recognized by the property that for any $x \in X$ and $A \subseteq X$ one has $\delta(x, A) = \inf_{a\in A} \delta(x, \{a\})$. Using the equivalent description via gauges, a quasi metric (approach) space is characterized by the fact that its gauge has a gauge basis consisting of a singleton.

An important object in App is the space $\mathbb{P} = ([0, \infty], \delta_{\mathbb{P}})$ where for $x \in [0, \infty]$ and $A \subseteq [0, \infty]$

$$\delta_{\mathbb{P}}(x, A) = \begin{cases} \max\{(x - \sup A), 0\} & A \neq \emptyset \\ \infty & A = \emptyset \end{cases}$$

A gauge basis for this object is given by $\{d_\alpha \mid \alpha \in [0,\infty[\}$ where $d_\alpha(x,y) = \max\{(\min\{x,\alpha\} - \min\{y,\alpha\}), 0\}$. This object is initially dense in App.

In this paper we will consider approach structures on a given domain. We refer the reader to [5] for terminology and basic results on domains. To fix notations recall that for a partially ordered set (poset) $(X,\leqslant)$ and elements x and y we write $x\#y$ if x and y have no common upperbound. A subset $D \subseteq X$ is *directed* if it is nonempty and any pair of elements of D has an upperbound in D. A poset in which every directed subset D has a supremum $(\sup D)$ is called a *directed complete poset* (*dcpo*). The set of maximal elements of X is denoted by $\mathrm{Max}(X)$.

We say that x is *way below* y if for all directed subsets $D \subseteq X$, $y \leqslant \sup D$ implies $x \leqslant a$ for some $a \in D$. We denote this by $x \ll y$. We say that x is a *compact element* if $x \ll x$. We use the notations $\Uparrow x := \{y \mid x \ll y\}$ and $\Downarrow x := \{y \mid y \ll x\}$. A subset $B \subseteq X$ is said to be a *basis* for X if for every element $x \in X$ the set $B \cap \Downarrow x$ is directed with supremum x. A poset is called a *domain* if it is a dcpo having a basis. If the class $K(X)$ of all compact elements is a domain basis then we call the domain *algebraic*. If a domain has a countable domain basis it is called an *ω-domain*.

On a domain $(X,\leqslant)$ there are some intrinsic topologies. The *Scott topology* denoted by $\sigma(X)$, is the topology for which the opens are uppersets inaccessible for directed suprema. It has a basis $\{\Uparrow x \mid x \in X\}$. The specialization order of $\sigma(X)$ coincides with the original order. Another intrinsic topology on $(X,\leqslant)$ is called the *Lawson topology* denoted by $\lambda(X)$ and is the supremum in the lattice of topologies on X of $\sigma(X)$ and $\omega(X)$ where $\omega(X)$ is the topology generated by $\{X \setminus \uparrow x \mid x \in X\}$.

2. Weightable quasi metrics

We follow [10] and [7] and adapt the definition of a weightable quasi metric to our setting of extended quasi metrics.

2.1. Definition. A quasi metric space (X,q) is *weightable* if there exists a function $w : X \to [0,\infty]$ (called a weight), not identically ∞, and satisfying

$$q(x,y) + w(x) = q(y,x) + w(y)$$

whenever $x, y \in X$. We say that a weight w is *forcing* for q if $x \in X$ and $w(x) = \infty$ imply that the function $q(x,.)$ is identically zero on X. We let wqMet be the construct consisting of weightable quasi metric spaces with non-expansive maps.

Matthews showed [10] that weightable quasi metric spaces are in one to one correspondence with partial metric spaces. In [6] Künzi and Vajner studied topological spaces that can be induced by a weightable quasi metric and formulated necessary as well as sufficient conditions on the topology to ensure quasi metrizability by some weightable quasi metric. For applications of weightable quasi metrics to domain theory see for instance [14,12]. The following example is well known and will appear to be crucial in this paper.

2.2. Example. Consider $[0,\infty]$ endowed with the following quasi metric.

$$q_\sigma(x,y) = \max\{(y-x), 0\} \quad \text{for } x \text{ and } y \text{ not both equal to } \infty; \qquad q_\sigma(\infty,\infty) = 0$$

The function $w_\sigma : [0,\infty] \to [0,\infty]$ defined as $w_\sigma(x) = x$ is a weight for q_σ.

For a weightable quasi metric q we denote by $\mathcal{W}_q$ the collection of all its weights. The next result is also well-known [6].

2.3. Proposition. *If a quasi metric space (X,q) is weightable and w belongs to $\mathcal{W}_q$, then $w : (X,\mathcal{T}_q) \to ([0,\infty],\mathcal{T}_{q_\sigma})$ is continuous.* (*In other words, w is upper semicontinuous.*)

We establish some categorical properties of the construct wqMet. The first result is straightforward.

2.4. Proposition. *The coproduct in* qMet *of weightable quasi metric spaces is weightable.*

2.5. Proposition. *Let $f : X \to Y$ be an initial morphism in* qMet *with Y a weightable quasi metric space, then the initial structure on X is weightable.*

Proof. Let $Y = (Y,q)$ with q some weightable quasi metric, and let w be a weight for q. Then clearly the initial structure $q \circ f \times f$ has the weight function $w \circ f$. □

A similar result does not hold for arbitrary sources. Even a pointwise supremum of two weightable quasi metrics needs not be weightable. This proves that wqMet is not concretely reflective in qMet and will help us to conclude in 2.8 that wqMet is not a topological construct.

2.6. Example. We consider $X = \{x, y, z\}$ and the following quasi metrics on X:

$$q(x, y) = p(x, y) = 1; \qquad q(y, x) = p(y, x) = 2$$
$$q(x, z) = p(x, z) = 1; \qquad q(z, x) = 3; \qquad p(z, x) = 2$$
$$q(z, y) = p(z, y) = 1; \qquad q(y, z) = 0; \qquad p(y, z) = 1$$

Both quasi metrics q and p are weightable whereas their pointwise supremum is not weightable.

2.7. Proposition. wqMet *is finally dense in* qMet*.*

Proof. Let (X, d) be a quasi metric space with more than one point. For a fixed $x \in X$ define $X_x = (X, d_x)$ as follows: For $z, y \in X$

$$d_x(z, y) = \begin{cases} d(x, y) & z = x \\ \infty & z \neq x \end{cases}$$

Clearly X_x is weightable by the weight $w : X \to [0, \infty]$ defined by $w(x) = \infty$ and $w(y) = 0$ for every $y \neq x$.

Next form the coproduct $\sum_{x \in X} X_x$, which by 2.4 is weightable, and consider the identification

$$\varphi : \sum_{x \in X} X_x \to X : (z, x) \mapsto z$$

Clearly for the final pre quasi metric structure d_{fin} on X we have

$$d_{\mathrm{fin}}(z, y) = \inf_{x \in X} d_x(z, y) = d_z(z, y) = d(z, y)$$

and hence $\varphi : \sum_{x \in X} X_x \to (X, d)$ is final in qMet. □

2.8. Proposition. *The construct* wqMet *is not topological.*

Proof. Since wqMet is finally dense in qMet and by 2.6 not concretely reflective in it, it cannot be topological [1]. □

3. Weightable quasi metric spaces in App

As explained in the introduction a weightable quasi metric space can be considered as an approach space. We investigate how wqMet is embedded in App.

3.1. Proposition. *Let $f : X \to Y$ be an initial morphism in* App *with Y a weightable quasi metric space, then the approach space X too is a weightable quasi metric space.*

Proof. Suppose the gauge $\mathcal{G}_Y$ has a gauge basis $\{q\}$, with q some weightable quasi metric. Then clearly the initial gauge $\mathcal{G}_X$ has a gauge basis $\{q \circ f \times f\}$. Moreover it follows from 2.5 that the quasi metric $q \circ f \times f$ is weightable. □

The next proposition is crucial for the proof of our Theorem 3.4. It deals with the initially dense object $([0, \infty], \delta_{\mathbb{P}})$ of App, the distance and gauge of which were given explicitly in the introductory section.

3.2. Proposition. *The source*

$$\bigl(g_\beta : \bigl([0, \infty], \delta_{\mathbb{P}}\bigr) \to \bigl([0, \infty], q_\sigma\bigr)\bigr)_{\beta \in [0, \infty[}$$

with $g_\beta(y) = \max\{(\beta - y), 0\}$ for $y \in [0, \infty]$, is initial in App*.*

Proof. Let $\{d_\beta \mid \beta \in [0, \infty[\}$ be the gauge basis for $([0, \infty], \delta_{\mathbb{P}})$ as defined in the introduction. We prove that for $\beta \in [0, \infty[$ and $y, z \in [0, \infty]$,

$$q_\sigma \circ g_\beta \times g_\beta(y, z) = d_\beta(y, z)$$

Writing the explicit forms of both sides we get, for the left hand side

$$\begin{aligned} q_\sigma \circ g_\beta \times g_\beta(y, z) &= \max\bigl\{\bigl(g_\beta(z) - g_\beta(y)\bigr), 0\bigr\} \\ &= \max\bigl\{\max\bigl\{(\beta - z), 0\bigr\} - \max\bigl\{(\beta - y), 0\bigr\}, 0\bigr\} \end{aligned}$$

and for the right hand side

$$d_\beta(y,z) = \max\{\min\{y,\beta\} - \min\{z,\beta\}, 0\}$$

Observe that in each of the cases $y \leqslant z$ or $\beta < z < y$ both sides are zero. In case ($z < y$ and $z \leqslant \beta \leqslant y$) both sides equal $\beta - z$ and finally in case ($z < y$ and $y \leqslant \beta$) both sides equal $y - z$. These observations remain valid when y or z are ∞.

It follows that the approach gauges generated by $\{q_\sigma \circ g_\beta \times g_\beta \mid \beta \in [0,\infty[\}$ and $\{d_\beta \mid \beta \in [0,\infty[\}$ coincide. The first collection generates the initial approach structure determined by the given source, the second collection generates $\delta_{\mathbb{P}}$. So we can conclude that $\delta_{\mathbb{P}}$ is the initial approach structure. □

Remark that our next result in 3.3 also follows from Theorem 3.1 in [2] on local metrically generated theories. Here we obtain it as an immediate corollary of 3.2.

3.3. Proposition. *The object* $([0,\infty], q_\sigma)$ *is initially dense in* App.

Proof. Using the fact that $([0,\infty], \delta_{\mathbb{P}})$ is initially dense in App, the result follows from transitivity. □

3.4. Theorem. App *is the concretely reflective hull of* wqMet *(of* $\{q_\sigma\}$*).*

Moreover for every approach space X, we have:

1. *X is a subspace of a product of weightable quasi metric spaces in* App *or equivalently X belongs to the epireflective hull of* wqMet *(of* $\{q_\sigma\}$*) in* App.
2. *The gauge* $\mathcal{G}_X$ *has a gauge subbasis consisting of weightable quasi metrics.*
3. *X is the supremum in* App *of all weightable quasi metric spaces that are coarser.*

Proof. Since 3.3 clearly implies that App is the concretely reflective hull of $\{q_\sigma\}$, it suffices to prove the equivalence with the other assertions.

1. Since indiscrete (quasi) metric spaces are weightable, the epireflective hull coincides with the concrete reflective hull.

2. Let the source $(f_i : X \to Y_i)_{i\in I}$ be initial in App with each Y_i a weightable quasi metric space. Suppose each $\mathcal{G}_i$ has a gauge basis $\{q_i\}$ with q_i weightable. As pointed out in 3.1 the quasi metric $q_i \circ f_i \times f_i$ is weightable. Moreover

$$\{q_i \circ f_i \times f_i \mid i \in I\}$$

is a subbase for the gauge $\mathcal{G}_X$.

3. Since by 2 X has a gauge $\mathcal{G}_X$ with a gauge subbase $\mathcal{H}_X$ consisting of weightable quasi metrics, for each $q \in \mathcal{H}_X$ we can consider X_q the quasi metric space on X with gauge basis $\{q\}$. Then the source $(id : X \to X_q)_{q\in\mathcal{H}}$ is initial. It follows that X is the supremum of all quasi metric spaces coarser than X. □

3.5. Corollary. *If* (X,d) *is a quasi metric space, then for every* $x \in X$*, for every* $\epsilon > 0$ *and* $\omega < \infty$ *there exists a weightable quasi metric* q_x *on X such that*

$$\forall y \in X \quad \min\{d(x,y), \omega\} \leqslant q_x(x,y) + \epsilon$$

Proof. This follows immediately by applying 3.4 to the approach space (X, δ_d). □

3.6. Definition. If (X,δ) is an approach space and $\mathcal{H}$ is a gauge subbasis consisting of weightable quasi metrics, then an element $(w_q)_{q\in\mathcal{H}} \in \prod_{q\in\mathcal{H}} \mathcal{W}_q$ is called a *weight* for $\mathcal{H}$ and $\bigcap_{q\in\mathcal{H}} w_q^{-1}(0)$ is called its *kernel*. We say the weight $(w_q)_{q\in\mathcal{H}}$ is *forcing* for $\mathcal{H}$ if every w_q with $q \in \mathcal{H}$ is forcing for q.

4. Approach spaces on domains

In this section we investigate the special situation where X carries a domain structure and its associated Scott topology, in particular we study the impact of having an approach structure inducing the Scott topology.

4.1. Definition. Let $(X, \leqslant)$ be a domain, $\sigma(X)$ the associated Scott topology and δ an approach structure on X. We say that (X,δ) *quantifies the domain* if:

1. Its topological coreflection coincides with the Scott topology, i.e. $\mathcal{T}_\delta = \sigma(X)$.

If moreover

2. (X,δ) has a gauge subbase $\mathcal{H}$ of weightable quasi metrics and a weight $(w_q)_{q\in\mathcal{H}}$ such that its kernel $\bigcap_{q\in\mathcal{H}} w_q^{-1}(0) = \mathrm{Max}(X)$.

Then we say that the quantifying approach space (X,δ) *satisfies the kernel condition*.

Of course using the terminology of [14], whenever q is a weightable quasi metric on a domain X, with weight w_q, (satisfying the kernel condition i.e. $w_q^{-1}(0) = \mathrm{Max}(X)$) and inducing the Scott topology i.e. $\mathcal{T}_q = \sigma(X)$, then (X,δ_q) quantifies the domain (and satisfies the kernel condition). In this case w_q is a weight associated with $\mathcal{H} = \{q\}$.

Next we come back to Example 2.2 now taking into account that $[0,\infty]$ carries a domain structure.

4.2. Example. We endow $[0,\infty]$ with the opposite order $x \preccurlyeq y \Leftrightarrow y \leqslant x$. The Scott topology associated to the domain $[0,\infty]^{op} = ([0,\infty],\preccurlyeq)$ has

$$\big\{[0,b[\ \big|\ b<\infty\big\} \cup \big\{[0,\infty]\big\} \cup \{\emptyset\}$$

as a basis. It coincides with the topology $\mathcal{T}_{q_\sigma}$ induced by the quasi metric q_σ described in 2.2 (which explains our choice of notation).

4.3. Definition. Given an approach space (X,δ) with gauge $\mathcal{G}$, one defines the *specialization preorder* as follows

$$x \leqslant y \quad \Leftrightarrow \quad \big(q(x,y) = 0 \text{ whenever } q \in \mathcal{G}\big)$$

Remark that as the quasi metric coreflection (X,q_δ) of an approach space (X,δ) is given by $q_\delta(x,y) = \delta(x,\{y\}) = \sup_{q\in\mathcal{G}} q(x,y)$ the following expressions are equivalent

$$\big(q(x,y) = 0 \text{ whenever } q \in \mathcal{G}\big) \quad \Leftrightarrow \quad q_\delta(x,y) = 0$$

Moreover, since the topology $\mathcal{T}_\delta$ of the topological coreflection of (X,δ) is the supremum of the topologies $\{\mathcal{T}_q \mid q \in \mathcal{G}\}$ we also have

$$\big(q(x,y) = 0 \text{ whenever } q \in \mathcal{G}\big) \quad \Leftrightarrow \quad x \in \mathrm{cl}_{\mathcal{T}_\delta}\{y\}$$

So the specialization preorder of (X,δ) defined in 4.3 coincides with the specialization preorders determined by the quasi metric or topological coreflections. From these observations we have the following.

4.4. Proposition. *Let $(X,\leqslant)$ be a domain, δ a quantifying approach structure on X with $\mathcal{H}$ a gauge subbasis consisting of weightable quasi metrics and $(w_q)_{q\in\mathcal{H}}$ a weight associated with $\mathcal{H}$.*

1. *The specialization preorder induced by δ coincides with the original order.*
2. *Each w_q for $q \in \mathcal{H}$ is Scott continuous and hence monotone to $[0,\infty]^{op}$.*
3. *If $(w_q)_{q\in\mathcal{H}}$ is forcing for $\mathcal{H}$ then it is strictly monotone in the sense that ($x \leqslant y$ and $w_q(x) = w_q(y)$ whenever $q \in \mathcal{H}$) implies $x = y$.*
4. $\bigcap_{q\in\mathcal{H}} w_q^{-1}(0) \subseteq \mathrm{Max}(X)$.

Proof. 1. In view of the assumption that $\mathcal{T}_\delta = \sigma(X)$, the specialization preorder induced by δ coincides with the specialization order induced by $\sigma(X)$.

2. By 2.3 a weight w for some quasi metric q is continuous considered as map $(X,\mathcal{T}_q) \to ([0,\infty],\mathcal{T}_{q_\sigma})$. The rest follows from $\mathcal{T}_q \leqslant \sigma(X)$ and the fact that $\mathcal{T}_{q_\sigma}$ is the Scott topology, 4.2.

3. Since $q \in \mathcal{H}$ is a quasi metric with weight w_q, from $q(x,y) + w_q(x) = q(y,x) + w_q(y)$ for $x,y \in X$ and $w_q(x) = w_q(y) < \infty$ we can deduce $q(x,y) = q(y,x)$. Since moreover w_q is forcing for q the same conclusion also follows in case $w_q(x) = w_q(y) = \infty$. Assuming $w_q(x) = w_q(y)$ for all $q \in \mathcal{H}$ and applying 1 we have

$$x \leqslant y \quad \Leftrightarrow \quad \big(q(x,y) = 0\ \forall q \in \mathcal{H}\big) \quad \Leftrightarrow \quad \big(q(y,x) = 0\ \forall q \in \mathcal{H}\big) \quad \Leftrightarrow \quad y \leqslant x.$$

Since the domain order is antisymmetric we are done.

4. Let $x \in \bigcap_{q\in\mathcal{H}} w_q^{-1}(0)$ and $x \leqslant y$. Applying 2 we have that $w_q(y) = 0$ whenever $q \in \mathcal{H}$ and as in 3 we can conclude that $x = y$. $\square$

5. Quantification of algebraic domains

In this section we prove that on any algebraic domain there are intrinsic quantifying approach spaces. We start by an example of one particular algebraic domain which will later be shown to be universal for all algebraic ones. The example generalizes the well-known construction of Plotkin [11] which he developed in the setting of ω-domains.

5.1. Example. (The domain $\mathbb{T}^\gamma$.) For any cardinal, γ, we define the following algebraic domain

$$\mathbb{T}^\gamma = \{u = (u_0, u_1) \mid u_0 \subseteq \gamma,\ u_1 \subseteq \gamma \text{ and } u_0 \cap u_1 = \emptyset\}$$

with the order defined by

$$u \leqslant v \quad \Leftrightarrow \quad u_0 \subseteq v_0 \quad \text{and} \quad u_1 \subseteq v_1$$

$\mathbb{T}^\gamma$ has a least element $(\emptyset, \emptyset)$. The maximal elements are of the form $u = (u_0, u_1) \in \mathbb{T}^\gamma$ with

$$u_0 \cup u_1 = \gamma$$

The way-below relation is defined by

$$u \ll v \quad \Leftrightarrow \quad (u_0 \text{ and } u_1 \text{ finite, and } u \leqslant v)$$

The compact elements are the finite elements, i.e. the elements $u = (u_0, u_1) \in \mathbb{T}^\gamma$ such that u_0 and u_1 are finite sets. The compact elements form a domain basis for $\mathbb{T}^\gamma$. Hence it is an algebraic domain.

5.2. Proposition. *The domain $\mathbb{T}^\gamma$ has a quantifying approach structure $\delta_{\mathbb{T}^\gamma}$ satisfying the kernel condition.*

Proof. 1. The approach space we are looking for will be defined by means of a suitable gauge basis. For some finite subset $K \subseteq \gamma$, let $q_K : \mathbb{T}^\gamma \times \mathbb{T}^\gamma \to [0, \infty]$ be defined as follows

$$q_K(x, y) = \bigl|K \cap \bigl[(x_0 \setminus y_0) \cup (x_1 \setminus y_1)\bigr]\bigr|,$$

where $|\,.\,|$ stands for the cardinality of the set. Clearly for points $x, y, z \in \mathbb{T}^\gamma$ we have $x_i \setminus y_i \subseteq x_i \setminus z_i \cup z_i \setminus y_i$ for $i = 1$ and $i = 2$, so q_K is a quasi metric. It is weightable by the function

$$w_K : \mathbb{T}^\gamma \to [0, \infty] \quad \text{with} \quad w_K(x) = \bigl|K \setminus (x_0 \cup x_1)\bigr|.$$

Consider $\mathcal{H} = \{q_K \mid K \text{ some finite subset of } \gamma\}$. Observe that $\mathcal{H}$ is an ideal basis since $q_K \leqslant q_{K \cup K'}$ and $q_{K'} \leqslant q_{K \cup K'}$ The gauge $\mathcal{G}_{\mathbb{T}^\gamma}$ generated by $\mathcal{H}$ via saturation, defines an approach space.

2. In order to compare the topological coreflection of the approach space, which is the supremum of the topologies $\{\mathcal{T}_{q_K} \mid K \text{ some finite subset of } \gamma\}$, with the Scott topology, we prove the following equality with respect to open balls with $\epsilon < 1$:

$$B_{q_K}(x, \epsilon) = \bigcap \{\Uparrow t \mid t \ll x,\ t_0 \cup t_1 \subseteq K\}$$

One inclusion follows from the fact that $y \in B_{q_K}(x, \epsilon)$ implies that points of K that belong to x_i also are in y_i. So whenever $t \ll x$, $t_0 \cup t_1 \subseteq K$ clearly $t \ll y$. For the other inclusion assume that $t \ll y$, whenever $t \ll x$ and $t_0 \cup t_1 \subseteq K$. Let $a \in K \cap x_0$ and put $t_0 = \{a\}$ and $t_1 = x_1 \cap (K \setminus \{a\})$. Then we have $t \ll x$ and $t_0 \cup t_1 \subseteq K$, and hence $t \ll y$. It follows that $a \in y_0$. Similarly we deduce that points in K that belong to x_1 also are in y_1.

3. Applying 4.4 we already know that $\bigcap \{w_{q_K}^{-1}(0) \mid K \text{ some finite subset of } \gamma\} \subseteq \mathrm{Max}(\mathbb{T}^\gamma)$. Conversely if x is maximal and K is an arbitrary finite subset of γ then the fact that $x_0 \cup x_1 = \gamma$ implies that $w_{q_K}(x) = 0$. □

As we will see next the approach space constructed in the previous proposition is neither topological nor quasi metric. We start by calculating its quasi metric coreflection.

5.3. Proposition. *With the notations of the previous proposition, the quasi metric coreflection of the approach space $(\mathbb{T}^\gamma, \delta_{\mathbb{T}^\gamma})$ is given by*

$$q_{\delta_{\mathbb{T}^\gamma}}(x, y) = \begin{cases} |(x_0 \setminus y_0) \cup (x_1 \setminus y_1)| & \text{if the set involved is finite} \\ \infty & \text{otherwise} \end{cases}$$

Moreover $(\mathbb{T}^\gamma, \delta_{\mathbb{T}^\gamma})$ is not a quasi metric approach space, even in case $\gamma = \omega$.

Proof. Let x and y be fixed elements in the domain. We have

$$q_{\delta_{\mathbb{T}^\gamma}}(x,y) = \delta_{\mathbb{T}^\gamma}\big(x,\{y\}\big) = \sup\big\{q_K(x,y) \bigm| K \text{ some finite subset of } \gamma\big\}$$
$$= \sup\big\{\big|K \cap \big[(x_0 \setminus y_0) \cup (x_1 \setminus y_1)\big]\big| \bigm| K \text{ some finite subset of } \gamma\big\}$$

So the first assertion follows. In order to prove the second assertion, assume $\gamma = \omega$. Remark that $(\mathbb{T}^\omega, \delta_{\mathbb{T}^\omega})$ being a quasi metric approach space, would imply that for all $x \in \mathbb{T}^\omega$ and for all $A \subseteq \mathbb{T}^\omega$ we would have that

$$\delta_{\mathbb{T}^\omega}(x,A) = \sup_{K \text{ finite}} \inf_{y\in A} q_K(x,y) \tag{1}$$
$$= \inf_{y\in A} \delta\big(x,\{y\}\big) \tag{2}$$
$$= \inf_{y\in A} \sup_{K \text{ finite}} q_K(x,y) \tag{3}$$

Put $x = (2\mathbb{N}, 2\mathbb{N}+1)$ and put $A = \{y \mid y_0 \text{ and } y_1 \text{ finite}\}$. We first compute Eq. (1): Let $K \subseteq \omega$ be a finite set and let $y \in \mathbb{T}^\omega$ be a finite element, then

$$q_K(x,y) = \big|K \cap \big[(x_0 \setminus y_0) \cup (x_1 \setminus y_1)\big]\big|$$

and thus

$$\inf_{y\in A} q_K(x,y) \leqslant \inf_{\{y\in A \mid y_0 \subseteq 2\mathbb{N} \;\&\; y_1 \subseteq 2\mathbb{N}+1\}} \big|K \cap \big[(x_0 \setminus y_0) \cup (x_1 \setminus y_1)\big]\big| = 0$$

Computing Eq. (3), for any y finite we get $\sup_{K \text{ finite}} q_K(x,y) = \infty$. □

5.4. Remark. The coreflection described in the previous proposition yields a bicomplete quasi metric.

The symmetrization of the quasi metric $q_{\delta_{\mathbb{T}^\gamma}}$ defined in 5.3 is

$$q^*_{\delta_{\mathbb{T}^\gamma}}(x,y) = \begin{cases} \max\{|(x_0 \setminus y_0) \cup (x_1 \setminus y_1)|, |(y_0 \setminus x_0) \cup (y_1 \setminus x_1)|\} & \text{both sets finite} \\ \infty & \text{otherwise} \end{cases}$$

It is clear that a Cauchy sequence in $\mathbb{T}^\gamma$ eventually becomes constant.

We will show that the approach space $(\mathbb{T}^\gamma, \delta_{\mathbb{T}^\gamma})$ is not topological, meaning that the distance is not $\{0,\infty\}$-valued. This can be established by investigating the quasi metric coreflection. Indeed in case the distance would be $\{0,\infty\}$-valued, so would be the quasi metric coreflection.

5.5. Proposition. *$(\mathbb{T}^\gamma, \delta_{\mathbb{T}^\gamma})$ is not a topological approach space, even in case $\gamma = \omega$.*

Proof. Again consider the quasi metric coreflection as calculated in 5.3. For $x = (\{0,\ldots,n\},\{n+1\})$ and $y = (\{n+2\},\{n+1\})$, we have

$$q_{\delta_{\mathbb{T}^\gamma}}(x,y) = \big|(x_0 \setminus y_0) \cup (x_1 \setminus y_1)\big| = n$$

and thus all values $n \in \mathbb{N}$ are obtained. □

We proceed in proving that every algebraic domain has a quantifying approach structure and that under the Lawson condition the quantifying approach structure also satisfies the kernel condition.

So in the sequel $(X, \leqslant)$ is an algebraic domain with Scott topology $\sigma(X)$. We use a technique inspired by the one developed by Waszkiewicz in [14] for ω-algebraic domains based on a theorem of Plotkin in [11]. The proof of the next result is quite similar to the ω case given in [14].

5.6. Proposition. *For every algebraic domain $(X, \leqslant)$ the space $(X, \sigma(X))$ can be topologically embedded in some space $(\mathbb{T}^\gamma, \sigma(\mathbb{T}^\gamma))$, for a suitable cardinal γ.*

Proof. Suppose the cardinality of the set of compact elements in the given domain is $|K(X)| = \gamma$, so this set can be labeled as $K(X) = \{b_\alpha \mid \alpha < \gamma\}$. Consider the map

$$\eta : X \to \mathbb{T}^\gamma : x \mapsto \big(\{\alpha \mid b_\alpha \ll x\}, \{\alpha \mid \exists \rho \in \gamma\colon\ b_\rho \ll x \;\&\; b_\rho \# b_\alpha\}\big)$$

We make the following observations:

1. η is properly defined: For all $\alpha, \rho \in \gamma$ with $b_\alpha \ll x$ and $b_\rho \ll x$ we have that b_α and b_ρ are compatible.
2. η is injective: Let $\eta(x) = \eta(y)$, then $\Downarrow x \cap K(X) = \Downarrow y \cap K(X)$ and thus by continuity of $(X, \leqslant)$ we have that $x = y$.
3. η is order preserving: Let $x, y \in (X, \leqslant)$ such that $x \leqslant y$, then clearly $\eta(x) \leqslant \eta(y)$ in $\mathbb{T}^\gamma$.
4. η is Scott continuous: It suffices to observe that for $x \in X$ we have

$$\eta(x) = \sup\{\eta(b_\beta) \mid b_\beta \ll x\}$$

5. $\eta : (X, \sigma(X)) \to (\mathbb{T}^\gamma, \sigma(\mathbb{T}^\gamma))$ is initial in TOP and therefore also an order embedding.
It is sufficient to prove that $\eta : (X, \sigma(X)) \to (\eta(X), \sigma(\mathbb{T}^\gamma)|_{\eta(X)})$ is open. Let $U = \Uparrow b_\rho = \uparrow b_\rho$, with $\rho \in \gamma$ be basic open in the Scott topology. Define

$$T = \{u = (u_0, u_1) \in \mathbb{T}^\gamma \mid \exists \alpha \in \gamma \; b_\rho \leqslant b_\alpha \text{ and } \alpha \in u_0\}$$

This is clearly an upper set and inaccessible for directed suprema and hence Scott open in $\mathbb{T}^\gamma$. We prove that $\eta(U) = \eta(X) \cap T$. Since for any $x \in X$ we have that $b_\rho \leqslant x$ implies $\rho \in (\eta(x))_0$, the inclusion $\eta(U) \subseteq T$ is clear.
For the converse let $x \in X$ such that $\eta(x) \in T$. Then there is some $\alpha \in \gamma$ with $b_\rho \leqslant b_\alpha$ and $\alpha \in (\eta(x))_0$. This implies $b_\alpha \ll x$ and so $x \in U$.
That η also is an order embedding follows from the following standard argument. If $x \nleqslant y$ then $X \setminus \downarrow y$ is Scott open for the initial structure, so there exists a Scott open subset U in $\mathbb{T}^\gamma$ such that $x \in \eta^{-1}(U) \subseteq X \setminus \downarrow y$. Since U is an upperset we can conclude that $\eta(x) \nleqslant \eta(y)$. □

5.7. Remark. Although there is a topological embedding of any algebraic domain into some $\mathbb{T}^\gamma$, there is an example of an algebraic domain X such that there does not exist any cardinal γ with $\gamma > \omega$ such that X is a Scott continuous retract of $\mathbb{T}^\gamma$ [15].

5.8. Theorem. *Every algebraic domain has a quantifying approach structure.*

Proof. The proof is based on the embedding described in Proposition 5.6. So with the same notations as before consider

$$\eta : (X, \sigma(X)) \to (\mathbb{T}^\gamma, \sigma(\mathbb{T}^\gamma)) : x \mapsto (\{\alpha \mid b_\alpha \ll x\}, \{\alpha \mid \exists \rho \in \gamma \; b_\rho \ll x \,\&\, b_\rho \# b_\alpha\})$$

which is initial in Top. We endow $\mathbb{T}^\gamma$, with the quantifying approach structure $\delta_{\mathbb{T}^\gamma}$ described in 5.2 of which the gauge has a basis

$$\mathcal{H} = \{q_K \mid K \text{ some finite subset of } \gamma\}$$

Let δ_X be the approach structure on X which makes the source

$$\eta : (X, \delta_X) \to (\mathbb{T}^\gamma, \delta_{\mathbb{T}^\gamma})$$

initial in App. By initiality the approach structure δ_X has a gauge basis

$$\mathcal{H} \circ \eta \times \eta = \{q_K \circ \eta \times \eta \mid K \text{ some finite subset of } \gamma\}$$

and a weight

$$\mathcal{W} \circ \eta = (w_K \circ \eta)_{K \subseteq \gamma, \text{ finite}}$$

Apply the coreflector to Top, then we obtain that the source

$$\eta : (X, \mathcal{T}_{\delta_X}) \to (\mathbb{T}^\gamma, \mathcal{T}_{\delta_{\mathbb{T}^\gamma}})$$

is initial in Top. As was shown in 5.2 $\mathcal{T}_{\delta_{\mathbb{T}^\gamma}} = \sigma(\mathbb{T}^\gamma)$ and so in view of 5.6 we finally can conclude that $\mathcal{T}_{\delta_X} = \sigma(X)$. □

Next we investigate whether the quantifying approach structure on X satisfies the kernel condition. In this respect we need the Lawson condition [8] which makes use of the Lawson topology.

5.9. Definition. A domain satisfies the *Lawson condition* (L) if the Lawson and Scott topologies agree on the set of maximal elements, i.e.

$$\lambda(X)|_{\mathrm{Max}(X)} = \sigma(X)|_{\mathrm{Max}(X)}$$

We will use the following characterization of maximal elements given in [14].

5.10. Proposition. *In an algebraic domain that satisfies* (L), *the following are equivalent*:

1. $x \in \mathrm{Max}(X)$;
2. $\forall b \in K(X)\ b \ll x$ *or* $\exists c \in K(X)\ c \ll x$ *and* $c\#b$.

5.11. Theorem. *An algebraic domain satisfying* (L) *has a quantifying approach structure satisfying the kernel condition.*

Proof. So with the same notations as before consider

$$\eta : X \to \mathbb{T}^\gamma : x \mapsto \big(\{\alpha \mid b_\alpha \ll x\}, \{\alpha \mid \exists \rho \in \gamma\ b_\rho \ll x \,\&\, b_\rho \# b_\alpha\}\big)$$

The condition (L) ensures that

$$x \in \mathrm{Max}(X) \quad \Leftrightarrow \quad \eta(x) \in \mathrm{Max}\big(\mathbb{T}^\gamma\big)$$

So we have

$$x \in \mathrm{Max}(X) \quad \Leftrightarrow \quad \big(\forall K \text{ finite subset of } \gamma\ \ w_K \circ \eta(x) = 0\big)$$

As we already know from 5.8 the collection $\mathcal{W} \circ \eta = \{w_K \circ \eta \mid K$ some finite subset of $\gamma\}$ is a weight for (X, δ), we are done. □

5.12. Remark. As in 5.3 we can calculate the quasi metric coreflection (X, q_{δ_X}) of the approach space (X, δ_X) defined in 5.8. Using the gauge basis $\mathcal{H} \circ \eta \times \eta$ we get that

$$\begin{aligned} q_{\delta_X}(x, y) &= \sup\big\{q_K\big(\eta(x), \eta(y)\big) \bigm| K \text{ some finite subset of } \gamma\big\} \\ &= \begin{cases} |((\eta(x))_0 \setminus (\eta(y))_0) \cup ((\eta(x))_1 \setminus (\eta(y))_1)| & \text{if the set involved is finite} \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

It is clear that for a Cauchy sequence $(x_n)_n$ for the symmetrization $q^*_{\delta_X}$ the sequence $(\eta(x_n))_n$ eventually becomes constant. Since η is injective the sequence $(x_n)_n$ too becomes constant. Hence q_{δ_X} is bicomplete.

6. Quantification of arbitrary domains

In this section we turn to arbitrary domains and we develop another technique for the construction of a quantifying approach structure.

6.1. Proposition. *Let* $(X, \leqslant)$ *be a domain,* $B \subseteq X$ *a domain basis and* $\sigma(X)$ *the Scott topology. Then the collection*

$$\mathcal{H}^B := \big\{q^B_K \bigm| K \textit{ some finite subset of } B\big\}$$

with $q^B_K(x, y) = |\{b \in K \mid x \in \Uparrow b,\ y \notin \Uparrow b\}|$ *for* $x \in X$, $y \in X$ *is a gauge basis for a quantifying approach space* (X, δ^B) *with weight*

$$\big(w^B_K\big)_{K \subseteq B,\ \text{finite}}$$

where $w^B_K(x) = |\{b \in K \mid x \notin \Uparrow b\}|$.

Proof. It is clear that each of the functions $q^B_K : X \times X \to [0, \infty]$ defined above for finite $K \subseteq B$, satisfies the triangular inequality and has weight w^B_K. Moreover for K and K' finite subsets of B we have $q^B_K \leqslant q^B_{K \cup K'}$ and $q^B_{K'} \leqslant q^B_{K \cup K'}$, so $\mathcal{H}^B = \{q^B_K \mid K \subseteq B,\ \text{finite}\}$ is a gauge basis. Let (X, δ^B) be the approach space generated by $\mathcal{H}^B$.

Finally we prove that $\sigma(X)$ is the topological coreflection, meaning that it is the supremum of the topologies $\{\mathcal{T}_{q^B_K} \mid K$ some finite subset of $B\}$. For $x \in X$ we have

$$B_{q^B_K}(x, 1) = \bigcap\{\Uparrow b \mid b \in K,\ x \in \Uparrow b\}$$

so they are Scott open. Conversely, given a Scott basic open set $\Uparrow b$ we let $K = \{b\}$. Then clearly $B_{q^B_{\{b\}}}(x, 1) = \Uparrow b$. □

In general the approach space constructed in 6.1 is not quasi metric nor topological. This is illustrated by the following example.

6.2. Example. On the domain $X = \mathbb{N} \cup \{\infty\}$ with the usual order, the domain basis X generates an approach space δ^X with gauge basis

$$\mathcal{H}^X = \{q_K \mid K \text{ some finite subset of } X\}$$

with $q_K(m,n) = |\{b \in K \mid b \leqslant m,\ n < b\}|$. The quasi metric coreflection is given by $q(m,n) = \max\{(m-n), 0\}$ for m and n in $\mathbb{N} \cup \{\infty\}$. And since it is clearly not $\{0,\infty\}$-valued, neither is δ^X, and so the approach space is not topological. In order to show that it is neither quasi metric, as in 5.3, we show that for some $x \in X$ and $A \subseteq X$ we have

$$\sup_{K \text{ finite}} \inf_{y \in A} q_K(x,y) < \inf_{y \in A} \sup_{K \text{ finite}} q_K(x,y)$$

For $x = \infty$ and $A = \mathbb{N}$ the left hand side equals 0 whereas on the right hand side we obtain $\inf_{n \in A} \max\{(\infty - n), 0\} = \infty$.

Remark that in general, even assuming the Lawson condition, the gauge basis constructed in 6.1 does not satisfy the kernel condition.

6.3. Example. We consider the domain of partial functions on the naturals. A partial function $f : X \to Y$ between sets X and Y is a function $f : A \to Y$ defined on a subset $A \subseteq X$. We write $\mathrm{dom}(f) = A$ for the domain of a partial map $f : X \to Y$. The set X of partial mappings from $\mathbb{N}$ to $\mathbb{N}$ is ordered by the extension order $f \leqslant g \Leftrightarrow \mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $g|_{\mathrm{dom}(f)} = f$. X is a domain with the way below relation characterized by $f \ll g \Leftrightarrow (f \leqslant g$ and $\mathrm{dom}(f)$ is finite). The set of compact elements $K(X) = \{f \in X \mid \mathrm{dom}(f) \text{ is finite}\}$ forms a countable domain basis. Applying our construction 6.1 to this example yields an approach space $(X, \delta^{K(X)})$ with gauge basis $\mathcal{H}^{K(X)} = \{q_K \mid K \text{ some finite subset of } K(X)\}$, where q_K is defined by $q_K(f,g) = |\{b \in K \mid b \leqslant f,\ b \nleqslant g\}|$ and has weight $w_K(f) = |\{b \in K \mid b \nleqslant f\}|$. For this domain $\mathrm{Max}(X)$ consists of those functions $f \in X$ with $\mathrm{dom}(f) = \mathbb{N}$. It can be seen that X satisfies the Lawson condition but the gauge basis $\mathcal{H}^{K(X)}$ does not satisfy the kernel condition.

Next we investigate whether the approach spaces constructed in 6.1 via different domain bases coincide. In order to obtain refined results this study has to be pursued locally. So we use approach systems rather than gauges. Using the notations of 6.1, for a given domain basis B let $(\mathcal{B}^B(x))_{x \in X}$ be the basis for the approach system associated with the gauge basis $\mathcal{H}^B$. More explicitly we have

$$\mathcal{B}^B(x) = \{q(x,.) \mid q \in \mathcal{H}^B\} = \{q_K^B(x,.) \mid K \text{ some finite subset of } B\}$$

and let $(\mathcal{A}^B(x))_{x \in X}$ be the generated approach system.

For $x \in X$ and $K \subseteq X$ finite, we denote $K_x = K \cap \mathord{\downarrow\!\!\downarrow} x$. Using these notations we have the following results.

6.4. Proposition. *If K_x contains only compact elements then we have $q_K^X(x,.) \in \mathcal{A}^B(x)$.*

Proof. If K_x contains only compact elements then also $K_x \subseteq B$. With $L = K_x$ we then have $q_K^X(x,.) = q_L^B(x,.)$. □

It follows from 6.4 that in domains like the powerset of $\mathbb{N}$, $\mathbb{T}^{\mathcal{Y}}$ (Example 5.1) or the domain of partial functions on the naturals (Example 6.3), where the relation $z \ll y$ implies that z is compact, the constructed approach space will not depend on the domain basis.

6.5. Proposition. *Let $x \in X$, $K \subseteq X$ finite, and for every $k \in K_x$, put $B_k = \{b \in B \mid k \ll b \ll x\}$. If for some choice $\varphi \in \prod_{k \in K_x} B_k$: $|\varphi(K_x)| = |K_x|$ then we have $q_K^X(x,.) \in \mathcal{A}^B(x)$.*

Proof. Let $y \in X$ be arbitrary. For φ chosen with $|\varphi(K_x)| = |K_x|$ we have that $\varphi : K_x \to \varphi(K_x)$ is bijective, so we also have a bijection between

$$\{k \mid k \in K,\ k \in \mathord{\downarrow\!\!\downarrow} x,\ k \notin \mathord{\downarrow\!\!\downarrow} y\} \quad \text{and} \quad \{\varphi(k) \mid k \in K,\ k \in \mathord{\downarrow\!\!\downarrow} x,\ k \notin \mathord{\downarrow\!\!\downarrow} y\}$$

Moreover since $k \ll \varphi(k) \ll x$ whenever $k \in K_x$, for $L = \varphi(K_x)$ we have

$$\{\varphi(k) \mid k \in K,\ k \in \mathord{\downarrow\!\!\downarrow} x,\ k \notin \mathord{\downarrow\!\!\downarrow} y\} \subseteq \{l \mid l \in L,\ l \in \mathord{\downarrow\!\!\downarrow} x,\ l \notin \mathord{\downarrow\!\!\downarrow} y\}$$

So the conclusion $q_K^X(x,y) \leqslant q_L^B(x,y)$ follows. □

6.6. Theorem. *If $x \in X$ is not compact then for two domain bases B and D the approach systems $(\mathcal{A}^B(x))_{x \in X}$ and $(\mathcal{A}^D(x))_{x \in X}$ coincide.*

Proof. Let $x \in X$ and suppose there is a domain basis B such that $\mathcal{A}^B(x) \neq \mathcal{A}^X(x)$. Since we clearly have $\mathcal{A}^B(x) \subseteq \mathcal{A}^X(x)$, this means that there exists $K \subseteq X$ with $q_K(x,.) \notin \mathcal{A}^B(x)$. So for every $L \subseteq B$ there exists $y \in X$ with

$$q_L^B(x,y) < q_K^X(x,y)$$

From 6.5 and using the same notations as before, we have $|\varphi(K_x)| < |K_x|$ for every $\varphi \in \prod_{k \in K_x} B_k$. Let

$$m = \max\left\{ \left|\varphi(K_x)\right| \;\middle|\; \varphi \in \prod_{k \in K_x} B_k \right\}$$

and take φ_0 such that $|\varphi_0(K_x)| = m$. Since $B \cap \downarrow x$ is directed there exists $b_0 \in B$, $b_0 \ll x$, such that $\varphi_0(k) \leqslant b_0$ whenever $k \in K_x$.

1. Either $b_0 = x$, and then the conclusion $x \ll x$ follows.
2. Or $b_0 \neq x$, and then in view of the particular choice for φ_0 there exists $k_0 \in K_x$ such that $b_0 = \varphi_0(k_0)$. In this case, for $b \in B$ with $b \ll x$ arbitrary, again using the fact that $B \cap \downarrow x$ is directed we choose $b' \in B$ satisfying $\varphi_0(k_0) \leqslant b' \ll x$ and $b \leqslant b' \ll x$.
   (a) Either one of the chosen $b' = x$, and then the conclusion $x \ll x$ follows.
   (b) Or, in view of the special choice of φ_0, for all chosen b' we have $b' = \varphi_0(k_0)$. This in particular implies $b \leqslant \varphi_0(k_0)$. Taking into account that $\sup \downarrow x \cap B = x$ we have that $\varphi_0(k_0) = x$, and the conclusion $x \ll x$ follows. □

As a corollary we obtain that in all domains where the set of compact elements is empty, as is the case in the formal ball model constructed from a complete metric space [3], the approach spaces constructed from different domain bases coincide. Also in Example 4.2 on $[0,\infty]$ with the opposite order, where ∞ is the only compact element, the previous results imply that the approach spaces constructed from different domain bases coincide.

However in compact elements the approach systems associated with different bases can differ, as shown by the following example.

6.7. Example. Let $X = [0,\infty] \cup \{c,d\}$ where c and d are new points added to the extended real line. We take the usual order on $[0,\infty]$ and we define $\infty < c < d$. This is a complete chain and therefore a domain. The only compact elements are 0, c and d. Clearly $B = X \setminus \{\infty\}$ is a domain basis for X. We investigate the approach structures in the point d associated with the domain basis X and B respectively. For $K = \{\infty, c, d\}$ we have

$$q_K^X(d,\infty) = \left|\{k \in K \mid k \in \downarrow d,\ k \notin \downarrow \infty\}\right| = 3$$

For every finite subset $L \subseteq B$ we have $\infty \notin L$ and so

$$q_L^B(d,\infty) = \left|\{l \in L \mid l \in \downarrow d,\ l \notin \downarrow \infty\}\right| = 2$$

So it is clear that $q_K^X(d,.)$ does not belong to $\mathcal{A}^B(d)$.

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