Journal of Pure and Applied Algebra 215 (2011) 161–167

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

The Buchsbaum property of symbolic powers of Stanley–Reisner ideals of dimension 1

Nguyen Cong Minh [a,*], Yukio Nakamura [b]

[a] *Department of Mathematics, Hanoi National University of Education, 136 Xuân Thuy, Hanoi, Viet Nam*

[b] *Department of Mathematics, School of Science and Technology, Meiji University, 214-8571, Japan*

ARTICLE INFO

Article history:
Received 25 November 2009
Received in revised form 10 March 2010
Available online 4 May 2010
Communicated by A.V. Geramita

MSC:
Primary: 13D45
Secondary: 05C90; 05E99

ABSTRACT

Let S be a polynomial ring and I be the Stanley–Reisner ideal of a simplicial complex Δ. The purpose of this paper is investigating the Buchsbaum property of $S/I^{(r)}$ when Δ is pure dimension 1. We shall characterize the Buchsbaumness of $S/I^{(r)}$ in terms of the graphical property of Δ. That is closely related to the characterization of the Cohen–Macaulayness of $S/I^{(r)}$ due to the first author and N.V. Trung.

1. Introduction

Let Δ be a simplicial complex on a vertex set $[n] = \{1, 2, \ldots, n\}$. Let $S = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring of n variables over an infinite field k. The Stanley–Reisner ideal I is a square-free monomial ideal of S associated with Δ and the residue class ring S/I is called the Stanley–Reisner ring. In this article, we assume that Δ is pure and $\dim \Delta = 1$, which means that any maximal element of Δ consists of two elements. We investigate the Buchsbaum property of $S/I^{(r)}$, where $I^{(r)}$ is the r-th symbolic power of I. In our situation, the Stanley–Reisner ring S/I satisfies $\dim S/I = 2$ and $\operatorname{depth} S/I > 0$. It is also known that S/I is always a Buchsbaum ring, and that S/I is Cohen–Macaulay if and only if Δ is connected. In [3], the first author and Trung studied the Cohen–Macaulay property of S/I^r and $S/I^{(r)}$ for the Stanley–Reisner ideal I of a pure simplicial complex Δ of dimension 1. They give the complete characterization on their Cohen–Macaulayness in terms of a geometric property of Δ.

Corollary 3.6 ([3, Theorems 2.3 and 2.4])**.** *Let I be the Stanley–Reisner ideal of a pure simplicial complex Δ of dimension 1. Let $r > 0$ be an integer. Then the following statements hold true.*

(1) *$S/I^{(2)}$ is Cohen–Macaulay if and only if* $\operatorname{diam}(\Delta) \leq 2$.
(2) *Let $r > 2$. Then, $S/I^{(r)}$ is Cohen–Macaulay if and only if any pair of disjoint edges of Δ is contained in a cycle of length* 4.

For the definition of $\operatorname{diam}(\Delta)$ and the *cycle* of length 4, see Definition 3.3.

The characterization of the Buchsbaumness of $S/I^{(r)}$ uses slightly weaker conditions. The following statement is the main result of this paper.

Theorem 3.7. *Let I be the Stanley–Reisner ideal of a pure simplicial complex Δ of dimension 1. Let $r > 0$ be an integer. Then the following statements hold true.*

* Corresponding author.
E-mail addresses: ngcminh@gmail.com (N.C. Minh), ynakamu@math.meiji.ac.jp (Y. Nakamura).

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doi:10.1016/j.jpaa.2010.04.007

(1) $S/I^{(2)}$ *is Buchsbaum if and only if* Δ *is connected.*
(2) $S/I^{(3)}$ *is Buchsbaum if and only if* $\operatorname{diam}(\Delta) \le 2$.
(3) *Let* $r > 3$. *If* $S/I^{(r)}$ *is Buchsbaum, then it is Cohen–Macaulay.*

Moreover, we also can compute the smallest non-negative integer $k = k(r)$ such that $\mathfrak{m}^k H^1_{\mathfrak{m}}(S/I^{(r)}) = (0)$ when Δ is connected.

Theorem 3.8. *Let* $r > 1$ *be an integer. Assume* Δ *is connected and* $S/I^{(r)}$ *is not Cohen–Macaulay. Then*

$$k(r) = \operatorname{diam}(H^1_{\mathfrak{m}}(S/I^{(r)})) = \begin{cases} r-1 & \text{if } \operatorname{diam}(\Delta) > 2 \\ r-2 & \text{if } \operatorname{diam}(\Delta) \le 2. \end{cases}$$

Let us explain the organization of this paper. In Section 2, we give a brief outline of Takayama's formula. In the paper [5], he gave a generalization of Hochster's formula about the local cohomology modules of Stanley–Reisner rings. Takayama's formula is very useful in itself, but we need more detailed information obtained from Takayama's formula. Lemma 2.3 states the behavior of the multiplicative map in the formula of Takayama. In Section 3, we treat the symbolic power of Stanley–Reisner ideals of dimension 1. After giving some statements on local cohomology modules, we prove the main results.

2. Preliminaries

We begin with the notation for a simplicial complex. A simplicial complex Δ on a finite set $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. Notice that, for convenience in the later discussions, we do not assume the condition that $\{i\} \in \Delta$ for $i = 1, 2, \dots, n$. We put $\dim F = |F| - 1$, where $|F|$ means the cardinality of F, and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$, which is called the dimension of Δ. When we assume a linear order on $[n]$, say $<$, Δ is called an oriented simplicial complex. In such a case, we define $F = \{i_1, \dots, i_r\}$ for $F \in \Delta$ with the order sequence $i_1 < \cdots < i_r$. Let Δ be an oriented simplicial complex with $\dim \Delta = d$. We denote by $\mathcal{C}(\Delta)$ the augmented oriented chain complex of Δ:

$$\mathcal{C}(\Delta)_\bullet : 0 \to C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to C_{-1} \to 0$$

where

$$C_k = \bigoplus_{\substack{F\in\Delta \\ \dim F = k}} \mathbb{Z}F \quad \text{and} \quad \partial F = \sum_{j=0}^{k} (-1)^j F_j$$

for all $F \in \Delta$. Here we define $F_j = \{i_0, \dots, \hat{i_j}, \dots, i_k\}$ for $F = \{i_0, \dots, i_k\}$. For any field k, we define the i-th reduced simplicial homology group $\tilde{H}_i(\Delta; k)$ of Δ to be the i-th homology group of the complex $\mathcal{C}(\Delta)_\bullet \otimes k$. Further we define the i-th reduced simplicial cohomology group $\tilde{H}^i(\Delta; k)$ of Δ to be the i-th cohomology group of the dual chain complex $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta)_\bullet, k)$ for all i. We note that

$$\dim_k \tilde{H}_i(\Delta; k) = \dim_k \tilde{H}^i(\Delta; k) \quad \text{for all } i,$$
$$\tilde{H}_{-1}(\Delta; k) \cong \tilde{H}^{-1}(\Delta; k) \cong \begin{cases} k & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{otherwise,} \end{cases}$$

and if $\Delta = \emptyset$, we define $\dim \Delta = -1$ and $\tilde{H}_i(\Delta; k) = \tilde{H}^i(\Delta; k) = 0$ for all i.

Next, we give notation for the Cech complex. Let $S = k[x_1, x_2, \dots, x_n]$ be a polynomial ring of n variables over a field k and I an ideal generated by monomials of S. Let $\mathfrak{m} = (x_1, x_2, \dots, x_n)$. We fix an orientation on $[n]$ and the subset $F = \{i_1, \dots, i_r\} \in [n]$ is considered together with the order sequence $i_1 < \cdots < i_r$. We put $x_F = \prod_{i\in F} x_i$. The Cech complex of $R = S/I$ with respect to $\mathfrak{m}$ is defined as follows:

$$C^\bullet : 0 \to C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}^n \to 0, \quad \text{where } C^t = \bigoplus_{|F|=t} R[1/x_F].$$

For an element $a/1 \in R[1/x_F]$ in C^t, the differential $\partial : C^t \to C^{t+1}$ is defined by

$$\partial(a/1) = \sum_{j\in[n]\setminus F} (-1)^{F(j)} a/1 \in \bigoplus_{j\in[n]\setminus F} R[1/x_{F\cup\{j\}}] \subseteq C^{t+1},$$

where $F(j) = |\{i_p \in F \mid i_p < j\}|$. Because this Cech complex can be considered as a complex of $\mathbb{Z}^n$-graded S-modules and $\mathbb{Z}^n$-graded homomorphisms, the cohomology modules also have the structure of $\mathbb{Z}^n$-graded S-modules. Hence for $\mathbf{a} \in \mathbb{Z}^n$, we have

$$H^i_{\mathfrak{m}}(R)_{\mathbf{a}} = H^i(C^\bullet)_{\mathbf{a}} \cong H^i(C^\bullet_{\mathbf{a}})$$

for each i, where $H^i_{\mathfrak{m}}(R)$ is the i-th local cohomology module of R with respect to $\mathfrak{m}$ and $H^i_{\mathfrak{m}}(R)_{\mathbf{a}}$ is its $\mathbf{a}$-homogeneous component.

For a monomial ideal I of S, we denote by $G(I)$ the set of minimal generators of I. For $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, let $G_{\mathbf{a}}$ be the subset $\{i | a_i < 0\}$ of $[n]$. With this notation, according to [1, Lemma 5.3.6] or [5, Lemma 1], we have the following lemma.

Lemma 2.1 ([1,5]). *Let $F \subseteq [n]$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. Then the following conditions are equivalent.*

(1) $R[1/x_F]_{\mathbf{a}} \neq 0$.
(2) *The following two conditions hold true.*
  (a) $G_{\mathbf{a}} \subseteq F$.
  (b) *For $\mathbf{x}^{\mathbf{b}} = \prod_{j=1}^n x_j^{b_j} \in G(I)$, there exists $j \in [n] \setminus F$ such that $b_j > a_j$.*

Next, we establish an isomorphism between the chain complex of S/I and the dual chain complex associated with a simplicial complex. Let I be a monomial ideal of S and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. We define two kinds of simplicial complexes $\tilde{\Delta}_{\mathbf{a}}(I)$ and $\Delta_{\mathbf{a}}(I)$ as follows:

$$\tilde{\Delta}_{\mathbf{a}}(I) \ni F \;\Leftrightarrow\; \begin{cases} (1)\ F \supseteq G_{\mathbf{a}} \\ (2)\ \forall \mathbf{x}^{\mathbf{b}} = \prod_{j=1}^{n} x_j^{b_j} \in G(I), \exists j \in [n] \setminus F \text{ such that } b_j > a_j \end{cases}$$

and

$$\Delta_{\mathbf{a}}(I) = \{F \setminus G_{\mathbf{a}} \mid F \in \tilde{\Delta}_{\mathbf{a}}(I)\}.$$

By Lemma 2.1, for $F \subseteq [n]$, we have

$$R[1/x_F]_{\mathbf{a}} = \begin{cases} k & F \in \tilde{\Delta}_{\mathbf{a}}(I) \\ 0 & \text{otherwise.} \end{cases}$$

Then each homogeneous piece of a component of $C^\bullet$ can be written as follows:

$$C^t_{\mathbf{a}} = \sum_{\substack{F \in \tilde{\Delta}_{\mathbf{a}}(I) \\ \dim F = t-1}} R[1/x_F]_{\mathbf{a}} = \sum_{\substack{F \in \tilde{\Delta}_{\mathbf{a}}(I) \\ \dim F = t-1}} k \cdot b_F$$

for $t \geq 0$ and $\mathbf{a} \in \mathbb{Z}^n$, where $\{b_F\}$ means the basis as a k-vector space. On the other hand, for a simplicial complex Δ, $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta)_t, k)$ is k-vector space having basis $\{\varphi_F | F \in \Delta, \dim F = t\}$, where

$$\varphi_F(G) = \begin{cases} 1 & F = G \\ 0 & \text{otherwise.} \end{cases}$$

By [1, Lemma 5.3.7] or [5, Lemma 2], we have the isomorphism of complexes.

Lemma 2.2 ([1,5]). *Let $\mathbf{a} \in \mathbb{Z}^n$ and $j = |G_{\mathbf{a}}|$. Then there exists an isomorphism of complexes*

$$C^\bullet_{\mathbf{a}}[j+1] \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}}(I))_\bullet, k)$$

which is induced by the assignment $b_F \mapsto \varphi_{F \setminus G_{\mathbf{a}}}$. In particular, for $t \geq -1$ we have an isomorphism of homology modules

$$H^{t+j+1}_{\mathfrak{m}}(R)_{\mathbf{a}} = H^{t+j+1}(C^\bullet_{\mathbf{a}}) \cong \widetilde{H}^t(\Delta_{\mathbf{a}}(I), k).$$

Let $\mathbf{b} \in \mathbb{N}^n$ and take the monomial $\mathbf{x}^{\mathbf{b}} = \prod_{j=1}^n x_j^{b_j} \in R$. The multiplication map $R \ni f \mapsto \mathbf{x}^{\mathbf{b}} f \in R$ induces a chain map $C^\bullet \xrightarrow{\mathbf{x}^{\mathbf{b}}} C^\bullet$. Taking their $\mathbb{Z}^n$-graded homogeneous piece, we have a chain map $\eta : C^\bullet_{\mathbf{a}} \to C^\bullet_{\mathbf{a}+b}$ for each $\mathbf{a} \in \mathbb{N}^n$. Notice that the homomorphism between the homology modules induced by η coincides with the homomorphism $H^t_{\mathfrak{m}}(R)_{\mathbf{a}} \xrightarrow{\mathbf{x}^{\mathbf{b}}} H^t_{\mathfrak{m}}(R)_{\mathbf{a}+b}$.

Lemma 2.3. *Let I be a monomial ideal of S and $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$. For any integers $j \geq 0$, we have the following commutative diagram:*

$$\begin{array}{ccc} H^j_{\mathfrak{m}}(S/I)_{\mathbf{a}} & \xrightarrow{\mathbf{x}^{\mathbf{b}}} & H^j_{\mathfrak{m}}(S/I)_{\mathbf{a}+\mathbf{b}} \\ \downarrow & & \downarrow \\ \widetilde{H}^{j-1}(\Delta_{\mathbf{a}}(I); k) & \longrightarrow & \widetilde{H}^{j-1}(\Delta_{\mathbf{a}+\mathbf{b}}(I); k) \end{array}$$

where the vertical maps are isomorphisms as in Lemma 2.2 and the bottom map is induced from the natural embedding $\Delta_{\mathbf{a}+\mathbf{b}}(I) \subseteq \Delta_{\mathbf{a}}(I)$ of simplicial complexes.

Proof. Let $\rho : \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}}(I))_\bullet, k) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}+\mathbf{b}}(I))_\bullet, k)$ be the homomorphism induced from the natural embedding $\Delta_{\mathbf{a}+\mathbf{b}}(I) \subseteq \Delta_{\mathbf{a}}(I)$. Then, for $F \in \Delta_{\mathbf{a}}(I)$ with $\dim F = t$, we have

$$\rho(\varphi_F) = \begin{cases} \varphi_F & \text{if } F \in \Delta_{\mathbf{a}+\mathbf{b}}(I) \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, passing through isomorphisms

$$C^{t+1}_{\mathbf{a}} \cong \sum_{\substack{F\in\tilde{\Delta}_{\mathbf{a}}(I)\\ \dim F = t}} k \cdot b_F \quad \text{and} \quad C^{t+1}_{\mathbf{a}+\mathbf{b}} \cong \sum_{\substack{F\in\tilde{\Delta}_{\mathbf{a}+\mathbf{b}}(I)\\ \dim F = t}} k \cdot b_F,$$

$\eta : C^{t+1}_{\mathbf{a}} \longrightarrow C^{t+1}_{\mathbf{a}+\mathbf{b}}$ can be written as

$$\eta(b_F) = \begin{cases} b_F & \text{if } F \in \Delta_{\mathbf{a}+\mathbf{b}}(I) \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have the following commutative diagram:

$$\begin{array}{ccc} C^{t+1}_{\mathbf{a}} & \xrightarrow{\eta} & C^{t+1}_{\mathbf{a}+\mathbf{b}} \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}}(I))_t, k) & \xrightarrow{\rho} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}+\mathbf{b}}(I))_t, k) \end{array}$$

for all $t \geq -1$, where we may assign b_F to φ_F for the vertical maps as in Lemma 2.2, since $|G_{\mathbf{a}}| = 0$. This diagram yields the commutative diagram of complexes with isomorphic vertical maps:

$$\begin{array}{ccc} C^{\bullet}_{\mathbf{a}}[1] & \xrightarrow{\eta} & C^{\bullet}_{\mathbf{a}+\mathbf{b}}[1] \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}}(I))_\bullet, k) & \xrightarrow{\rho} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(\Delta_{\mathbf{a}+\mathbf{b}}(I))_\bullet, k) \end{array}$$

and we get the required diagrams of homology modules. □

3. The Buchsbaum property

Let Δ be a simplicial complex of $[n]$, which is pure, and $\dim \Delta = 1$, i.e., each facet of Δ consists of two elements. Let I be the Stanley–Reisner ideal of Δ in the polynomial ring $S = k[x_1, x_2, \ldots, x_n]$, which is the ideal generated by the square-free monomial $x_{i_1}x_{i_2}\ldots x_{i_r}$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. In this section, we discuss the Buchsbaum property of the residue ring $S/I^{(r)}$, where $I^{(r)}$ is the r-th symbolic power of I. For $1 \leq i < j \leq n$, we put $P_{ij} = (x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)$. Then

$$I = \bigcap_{\{i,j\}\in\Delta} P_{ij} \quad \text{and} \quad I^{(r)} = \bigcap_{\{i,j\}\in\Delta} P_{ij}^r.$$

Now, a facet F of Δ can be written as $F = \{i, j\}$. For simplicity, we may write $F = ij$ instead of $F = \{i, j\}$.

These kinds of ideals have been studied by several authors. Here we pick up important results stated in [2,3], which will be applied several times in our argument.

Theorem 3.1. *$S/I^{(r)}$ and $\Delta_{\mathbf{a}}(I^{(r)})$ satisfy the following properties.*

(1) $\dim S/I^{(r)} = 2$ *and* $\operatorname{depth} S/I^{(r)} > 0$.
(2) *Let* $\mathbf{a} \in \mathbb{Z}^n$. *If* $G_{\mathbf{a}} \neq \emptyset$, *then* $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} = 0$.
(3) *Let* $\mathbf{a} \in \mathbb{N}^n$. *Then* $\Delta_{\mathbf{a}}(I^{(r)})$ *is a subcomplex of* Δ *of pure dimension 1.*
(4) *Let* $\mathbf{a} \in \mathbb{N}^n$. *Then* $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \cong \tilde{H}_0(\Delta_{\mathbf{a}}(I^{(r)}), k)$.
(5) $\tilde{H}_0(\Delta_{\mathbf{a}}(I^{(r)}), k) = 0$ *if and only if* $\Delta_{\mathbf{a}}(I^{(r)})$ *is connected.*
(6) *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. *For* $i, j \in [n]$, *we put* $\sigma^{\mathbf{a}}_{ij} = |\mathbf{a}| - (a_i + a_j)$, *where* $|\mathbf{a}| = \sum_{k=1}^n a_k$. *Then we have the following equivalent conditions:*
   (a) $ij \in \Delta_{\mathbf{a}}(I^{(r)})$.
   (b) $\sigma^{\mathbf{a}}_{ij} < r$ *and* $ij \in \Delta$.

Proof. (1) Obvious.
(2) See [2, Lemma 1.3].
(3) See [3, Lemma 1.3].
(4) Apply Lemma 2.2.

(5) By the definition of the reduced homology group.
(6) See [3, Lemma 2.1]. □

Lemma 3.2. (1) $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = (0)$ *for* $j > 2r - 2$.
(2) *Let* $0 \le j < r$. *Then* $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = (0)$ *if and only if* Δ *is connected.*

Proof. (1) Take $\mathbf{a} \in \mathbb{N}^n$ such that $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \neq (0)$. Then there exist at least two connected components in $\Delta_{\mathbf{a}}(I^{(r)})$, since $\Delta_{\mathbf{a}}(I^{(r)})$ is not connected. We may say that $12, 34 \in \Delta_{\mathbf{a}}(I^{(r)})$ belong to different components. From the inequalities $\sigma_{12} < r$ and $\sigma_{34} < r$, we obtain that $a_3 + a_4 \le \sigma_{12} \le r - 1$, whence it follows that

$$|\mathbf{a}| = \sigma_{34} + (a_3 + a_4) \le 2r - 2.$$

This implies the first statement.
(2) Let $0 \le j < r$. Take $\mathbf{a} \in \mathbb{N}^n$ with $|\mathbf{a}| = j$. Then one can check that $\Delta = \Delta_{\mathbf{a}}(I^{(r)})$. In fact, take $pq \in \Delta$. Then

$$\sigma_{pq} \le |\mathbf{a}| = j < r.$$

Thus it follows that $\Delta \subseteq \Delta_{\mathbf{a}}(I^{(r)})$. The opposite inclusion is obvious. □

For the proof of the next lemma, we need the following definition.

Definition 3.3. (1) For $i, j \in [n]$, we define

$$\operatorname{dist}(i,j) = \min \left\{ k \;\middle|\; \begin{array}{c} \exists x_0, x_1, \ldots, x_k \in [n] \text{ such that} \\ x_0 = i, x_k = j, x_l x_{l+1} \in \Delta \text{ for } 0 \le l \le k-1 \end{array} \right\}$$

and $\operatorname{dist}(i,j) = \infty$ if there is no path connecting them. Further, we put $\operatorname{diam}(\Delta) = \max_{i,j\in[n]} \operatorname{dist}(i,j)$ and call it the diameter of Δ.
(2) Let $i \in [n]$. A star of Δ is a subcomplex of Δ defined as follows:

$$\operatorname{Star}_\Delta(i) = \{F \in \Delta \mid F \cup \{i\} \in \Delta\}.$$

(3) For a simplicial complex Δ, the subset

$$\{x_i x_{i+1} \in \Delta \mid i = 1, \ldots, k\,,\ x_1 = x_{k+1}\}$$

of Δ is called a cycle of length k.

We will give some properties of $H^1_{\mathfrak{m}}(S/I^{(r)})$ as follows.

Lemma 3.4. *Let* $r > 1$. *The following conditions are equivalent.*

(1) $[H^1_{\mathfrak{m}}(S/I^{(r)})]_r = (0)$.
(2) $\operatorname{diam}(\Delta) \le 2$.

Proof. (1) ⇒ (2): Take $p, q \in [n]$. Let $\mathbf{a} = (r-1)\mathbf{e}_p + \mathbf{e}_q$, where $\mathbf{e}_p$ denotes the p-th unit vector. Then $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} = 0$ since $|\mathbf{a}| = r$. On the other hand, one can check that $\Delta_{\mathbf{a}}(I^{(r)}) = \operatorname{Star}_\Delta(p) \cup \operatorname{Star}_\Delta(q)$. In fact,

$$\begin{aligned} ij \in \Delta_{\mathbf{a}}(I^{(r)}) &\iff \sigma^{\mathbf{a}}_{ij} < r \text{ and } ij \in \Delta \\ &\iff \{i,j\} \cap \{p,q\} \neq \emptyset \text{ and } ij \in \Delta \\ &\iff ij \in \operatorname{Star}_\Delta(p) \text{ or } ij \in \operatorname{Star}_\Delta(q). \end{aligned}$$

Now $\operatorname{Star}_\Delta(p) \cup \operatorname{Star}_\Delta(q)$ is connected; thus $\operatorname{dist}(p,q) \le 2$, which implies that $\operatorname{diam}(\Delta) \le 2$.
(2) ⇒ (1): Take $\mathbf{a} \in \mathbb{N}^n$ with $|\mathbf{a}| = r$. We put

$$\mathbf{a} = u_1 \mathbf{e}_{\alpha_1} + u_2 \mathbf{e}_{\alpha_2} + \cdots + u_t \mathbf{e}_{\alpha_t},$$

where $u_i > 0$, $u_1 + u_2 + \cdots + u_t = r$ and $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_t \le n$. Then

$$\begin{aligned} ij \in \Delta_{\mathbf{a}}(I^{(r)}) &\iff \sigma^{\mathbf{a}}_{ij} < r \text{ and } ij \in \Delta \\ &\iff \{i,j\} \cap \{\alpha_1, \alpha_2, \ldots, \alpha_t\} \neq \emptyset \text{ and } ij \in \Delta \\ &\iff ij \in \operatorname{Star}_\Delta(\alpha_1) \cup \operatorname{Star}_\Delta(\alpha_2) \cup \cdots \cup \operatorname{Star}_\Delta(\alpha_t). \end{aligned}$$

From the assumption, it follows that $\operatorname{dist}(\alpha_i, \alpha_{i+1}) \le 2$. Thus $\operatorname{Star}_\Delta(\alpha_i) \cup \operatorname{Star}_\Delta(\alpha_{i+1})$ is connected for all $1 \le i < t$, whence we obtain that $\Delta_{\mathbf{a}}(I^{(r)})$ is connected and $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} = (0)$. □

Lemma 3.5. *Let* $r > 2$ *and* $r + 1 \le j \le 2r - 2$. *The following conditions are equivalent.*

(1) $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j = (0)$.
(2) *Any pair of disjoint edges of* Δ *is contained in a cycle of length* 4.

Proof. (1) ⇒ (2): We assume the contrary of (2) and take a pair of disjoint edges, say $12, 34 \in \Delta$, which is not contained in a cycle of length 4. Besides, we may assume $13, 14 \notin \Delta$. Let

$$\mathbf{a} = \begin{cases} (r-1)\mathbf{e}_1 + \frac{j-r+2}{2}\mathbf{e}_3 + \frac{j-r}{2}\mathbf{e}_4 & \text{if } j-r \text{ is even,} \\ (r-1)\mathbf{e}_1 + \frac{j-r+1}{2}\mathbf{e}_3 + \frac{j-r+1}{2}\mathbf{e}_4 & \text{if } j-r \text{ is odd.} \end{cases}$$

Then one can check that $\Delta_{\mathbf{a}}(I^{(r)}) = \mathrm{Star}_\Delta(1) \cup \{3, 4\}$, since $1 \le j - r \le r - 2$. Hence $\Delta_{\mathbf{a}}(I^{(r)})$ is not connected, which contradicts (1).
(2) ⇒ (1): If $[H^1_{\mathfrak{m}}(S/I^{(r)})]_j \neq (0)$ then there exists $\mathbf{a} \in \mathbb{N}^n$ such that $|\mathbf{a}| = j$ and $\Delta_{\mathbf{a}}(I^{(r)})$ is not connected. We may assume that two edges $12, 34 \in \Delta_{\mathbf{a}}(I^{(r)})$ belong to different components. From the assumption of (2), we may assume $13, 24 \in \Delta$. Then

$$r + r \le \sigma^{\mathbf{a}}_{13} + \sigma^{\mathbf{a}}_{24} = \sigma^{\mathbf{a}}_{12} + \sigma^{\mathbf{a}}_{34} < r + r,$$

which is a contradiction. □

From the above lemmas, we can obtain the characterization of the Cohen–Macaulayness of $S/I^{(r)}$ in terms of Δ given by the first author and N.V. Trung.

Corollary 3.6 *([3, Theorems 2.3 and 2.4]). Let I be the Stanley–Reisner ideal of a pure simplicial complex Δ of dimension 1. Let $r > 0$ be an integer. Then the following statements hold true.*

(1) *$S/I^{(2)}$ is Cohen–Macaulay if and only if* $\mathrm{diam}(\Delta) \le 2$.
(2) *Let $r > 2$. Then, $S/I^{(r)}$ is Cohen–Macaulay if and only if any pair of disjoint edges of Δ is contained in a cycle of length* 4.

Now, we come to state the main result of this paper.

Theorem 3.7. *The following statements hold true.*

(1) *$S/I^{(2)}$ is Buchsbaum if and only if Δ is connected.*
(2) *$S/I^{(3)}$ is Buchsbaum if and only if* $\mathrm{diam}(\Delta) \le 2$.
(3) *Let $r > 3$. $S/I^{(r)}$ is Cohen–Macaulay if and only if $S/I^{(r)}$ is Buchsbaum.*

Proof. (1) By Theorem 3.1(6), one can check that $\Delta = \Delta_{\mathbf{0}}(I^{(2)}) = \Delta_{\mathbf{e}_1}(I^{(2)})$. Thus, by Lemma 2.3, we have the isomorphism

$$H^1_{\mathfrak{m}}(S/I^{(2)})_{\mathbf{0}} \xrightarrow{x_1} H^1_{\mathfrak{m}}(S/I^{(2)})_{\mathbf{e}_1}.$$

If Δ is not connected, then $H^1_{\mathfrak{m}}(S/I^{(2)})_{\mathbf{0}} \neq (0)$. This implies that $\mathfrak{m} \cdot H^1_{\mathfrak{m}}(S/I^{(2)}) \neq (0)$. Hence $S/I^{(2)}$ is not Buchsbaum.
Conversely, we suppose that Δ is connected. Then by Lemma 3.2,

$$H^1_{\mathfrak{m}}(S/I^{(2)}) = [H^1_{\mathfrak{m}}(S/I^{(2)})]_2.$$

Hence, $\mathfrak{m} \cdot H^1_{\mathfrak{m}}(S/I^{(2)}) = (0)$, which implies that $S/I^{(2)}$ is Buchsbaum (see [4, I. Corollary 3.6]).
(2) Suppose that $\mathrm{diam}(\Delta) \ge 3$. We may assume that $\{1\}, \{2\} \in \Delta$ and $\mathrm{dist}(1, 2) \ge 3$. Let $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2$. Then, one can check that $\Delta_{\mathbf{a}}(I^{(3)}) = \mathrm{Star}_\Delta(1) \cup \mathrm{Star}_\Delta(2)$, by Theorem 3.1(6). Hence, it follows that as $\mathrm{Star}_\Delta(1) \cup \mathrm{Star}_\Delta(2)$ is not connected since $\mathrm{dist}(1, 2) \ge 3$, then so is $\Delta_{\mathbf{a}}(I^{(3)})$ and we have $H^1_{\mathfrak{m}}(S/I^{(3)})_{\mathbf{a}} \neq 0$. By Theorem 3.1(6) again, one can check that $\Delta_{\mathbf{a}}(I^{(3)}) = \Delta_{\mathbf{a}+\mathbf{e}_1}(I^{(3)})$. Hence $H^1_{\mathfrak{m}}(S/I^{(3)})_{\mathbf{a}} \xrightarrow{x_1} H^1_{\mathfrak{m}}(S/I^{(3)})_{\mathbf{a}+\mathbf{e}_1}$ is an isomorphism, which implies that $\mathfrak{m} \cdot H^1_{\mathfrak{m}}(S/I^{(2)}) \neq (0)$ and $S/I^{(2)}$ is not Buchsbaum.
Conversely, we assume that $\mathrm{diam}(\Delta) \le 2$. Then $H^1_{\mathfrak{m}}(S/I^{(3)}) = [H^1_{\mathfrak{m}}(S/I^{(3)})]_4$ by Lemmas 3.2 and 3.4. Hence we have $\mathfrak{m} \cdot H^1_{\mathfrak{m}}(S/I^{(3)}) = (0)$ and $S/I^{(3)}$ is Buchsbaum.
(3) Let $r > 3$ and assume that $S/I^{(r)}$ is not Cohen–Macaulay. We may assume that $12, 34 \in \Delta$ are not contained in any cycle of length 4. Without loss of generality, we may assume that $13, 14 \notin \Delta$. Let $\mathbf{a} = (r-1)\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4$. Then one can check that $\Delta_{\mathbf{a}}(I^{(r)}) = \mathrm{Star}_\Delta(1) \cup \{34\} = \Delta_{\mathbf{a}+\mathbf{e}_3}(I^{(r)})$ by Theorem 3.1(6). Hence $H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \xrightarrow{x_3} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}+\mathbf{e}_3}$ is an isomorphism. This implies that $S/I^{(r)}$ is not Buchsbaum. □

Since the module $H^1_{\mathfrak{m}}(S/I^{(r)})$ is of finite length, there exists a smallest non-negative integer $k(r)$ such that $\mathfrak{m}^{k(r)} H^1_{\mathfrak{m}}(S/I^{(r)}) = (0)$. We will compute $k(r)$ when Δ is connected. In fact, if Δ is connected then

$$k(r) = \mathrm{diam}(H^1_{\mathfrak{m}}(S/I^{(r)})).$$

Here, $\mathrm{diam}(M)$, the diameter of $\mathbb{Z}$-graded module M of finite length, is the integer

$$\mathrm{diam}(M) = \max\{n | M_n \neq 0\} - \min\{n | M_n \neq 0\} + 1$$

when $M \neq (0)$ and $\mathrm{diam}(M) = 0$ when $M = (0)$. It is clear that $k(r) \le \mathrm{diam}(H^1_{\mathfrak{m}}(S/I^{(r)}))$.

Theorem 3.8. *Let $r > 1$ be an integer. Assume Δ is connected and $S/I^{(r)}$ is not Cohen–Macaulay. Then*

$$k(r) = \operatorname{diam}(H^1_{\mathfrak{m}}(S/I^{(r)})) = \begin{cases} r-1 & \textit{if } \operatorname{diam}(\Delta) > 2 \\ r-2 & \textit{if } \operatorname{diam}(\Delta) \le 2. \end{cases}$$

Proof. By our assumption, using Lemmas 3.2, 3.4 and 3.5, we have

$$t = \operatorname{diam}(H^1_{\mathfrak{m}}(S/I^{(r)})) = \begin{cases} r-1 & \text{if } \operatorname{diam}(\Delta) > 2 \\ r-2 & \text{if } \operatorname{diam}(\Delta) \le 2. \end{cases}$$

We need only check that $\mathfrak{m}^{t-1}H^1_{\mathfrak{m}}(S/I^{(r)}) \neq (0)$.

Case 1: $\operatorname{diam}(\Delta) > 2$. It is clear if $r = 2$ by Theorem 3.7(1). Assume $r \ge 3$. Since $\operatorname{diam}(\Delta) > 2$, there exists $1 \le i < j \le n$ such that $\operatorname{dist}(i, j) \ge 3$. Hence $\operatorname{Star}_\Delta(i) \cup \operatorname{Star}_\Delta(j)$ is not connected. Put

$$\mathbf{a} = (r-1)\mathbf{e}_i + \mathbf{e}_j \quad \text{and} \quad \mathbf{b} = (r-2)\mathbf{e}_j.$$

Then one can check that

$$\Delta_{\mathbf{a}}(I^{(r)}) = \Delta_{\mathbf{a}+\mathbf{b}}(I^{(r)}) = \operatorname{Star}_\Delta(i) \cup \operatorname{Star}_\Delta(j),$$

by Theorem 3.1(6). Hence

$$0 \neq H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \xrightarrow{x^{\mathbf{b}}} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}+\mathbf{b}}$$

is isomorphic, which implies that $\mathfrak{m}^{r-2}H^1_{\mathfrak{m}}(S/I^{(2)}) \neq (0)$. Therefore $k(r) = t$.

Case 2: $\operatorname{diam}(\Delta) \le 2$. Since $S/I^{(r)}$ is not Cohen–Macaulay and Corollary 3.6(2), we have $r \ge 3$ and there exists a pair of disjoint edges of Δ, say 12, 34, which is not contained in any cycle of length 4. Assume that $13, 14 \notin \Delta$. Using Theorem 3.7(2), we only check the assertion when $r \ge 4$. Put

$$\mathbf{a} = (r-1)\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4 \quad \text{and} \quad \mathbf{b} = (r-3)\mathbf{e}_3.$$

Then one can check that

$$\Delta_{\mathbf{a}}(I^{(r)}) = \Delta_{\mathbf{a}+\mathbf{b}}(I^{(r)}) = \operatorname{Star}_\Delta(1) \cup \{34\},$$

which is not connected, by Theorem 3.1(6). Hence

$$0 \neq H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}} \xrightarrow{x^{\mathbf{b}}} H^1_{\mathfrak{m}}(S/I^{(r)})_{\mathbf{a}+\mathbf{b}}$$

is isomorphic, which implies that $\mathfrak{m}^{r-3}H^1_{\mathfrak{m}}(S/I^{(2)}) \neq (0)$. The assertion is completely proved. □

Acknowledgements

Throughout this research, Professor Ngo Viet Trung at the Institute of Mathematics Hanoi gave the authors valuable advice. This was very important for obtaining the conclusion of the main results. The authors are grateful for his advice. The first author is supported by JSPS Ronpaku (dissertation Ph.D.) Program.

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