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Faithful Flatness of Hopf Algebras

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INTRODUCTION

We work over a field k. Let $G = \operatorname{Sp} A$ be an affine k-group scheme represented by a commutative Hopf algebra A. Let B be a right coideal (and) subalgebra of A. The affine k-scheme $\operatorname{Sp} B$ with natural right G-action is isomorphic canonically with the dur k-sheaf of left cosets $H \setminus G$ for some closed subgroup scheme H, if and only if A is faithfully flat as a B-module [T3]. Thus the question of faithful flatness of Hopf algebras (including the non-commutative case) comes to our attention.

A commutative Hopf algebra is not necessarily faithfully flat over every right coideal subalgebra. We prove, however, in Section 3 of this paper the following:

THEOREM. *A commutative Hopf algebra is a flat module over every right coideal subalgebra.*

On the other hand, we give in Section 2 some necessary and sufficient conditions for a non-commutative Hopf algebra to be faithfully flat over a right coideal subalgebra, part of which is found in the following:

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THEOREM. *Let A be a Hopf algebra with bijective antipode, and $B \subset A$ a right coideal subalgebra. Then the following are equivalent with each other:*

(a) *A is flat as a left B-module and B is a simple object in the category M_B^A;*

(b) *A is faithfully flat as a left B-module;*

(c) *A is a projective generator as a left B-module.*

Note that a Hopf subalgebra is necessarily a right coideal subalgebra. Applying the two theorems, we obtain a simple proof of the following important theorem [T1, 3] due to M. Takeuchi: *a commutative Hopf algebra is a faithfully flat module, or more strongly a projective generator, over every Hopf subalgebra.*

1. PRELIMINARIES

We work over a fixed base field k. Unadorned $\otimes$ means $\otimes_k$.

Let A be a bialgebra with coproduct $\Delta: A \to A \otimes A$, counit $\varepsilon: A \to k$, and $B \subset A$ a right coideal subalgebra, that is, a subalgebra such that $\Delta(B) \subset B \otimes A$.

We denote by

$$\mathsf{M}_B^A \qquad (\text{resp., } {}_B\mathsf{M}^A)$$

the category consisting of right (resp., left) B-modules M with a right A-comodule structure $\rho: M \to M \otimes A$ such that

$$\rho(mb) = \sum m_{(0)}b_{(1)} \otimes m_{(1)}b_{(2)}$$
$$\left(\text{resp., } \rho(bm) = \sum b_{(1)}m_{(0)} \otimes b_{(2)}m_{(1)}\right)$$

for $m \in M$, $b \in B$. Here we write as usual

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \qquad (a \in A),$$
$$\rho(m) = \sum m_{(0)} \otimes m_{(1)} \qquad (m \in M).$$

Morphisms in M_B^A or ${}_B\mathsf{M}^A$ are B-linear and A-colinear maps, and these categories are abelian [T3, p. 454]. A sequence in M_B^A or ${}_B\mathsf{M}^A$ is exact, if and only if it is exact viewed in the category of k-vector spaces. Both A and B are contained in M_B^A or in ${}_B\mathsf{M}^A$ with obvious structures. We have a natural identification

$${}_B\mathsf{M}^A = \mathsf{M}_{B^{\mathrm{op}}}^{A^{\mathrm{op}}},$$

where $?^{\mathrm{op}}$ means the opposite algebra.

Write $\bar{A} = A/AB^+$, where $B^+ = B \cap \operatorname{Ker} \varepsilon$. This is a quotient left A-module coalgebra of A (that is, a quotient left A-module and quotient coalgebra). For $M \in \mathsf{M}_B^A$, write $\bar{M} = M/MB^+$ and denote by $m \mapsto \bar{m}$, $M \to \bar{M}$ the quotient map. (This notation is consistent with $\bar{A}$.) Then $\bar{M}$ has an induced right $\bar{A}$-comodule structure. Thus we have a natural functor

$$(1.1) \qquad \bar{?}: \mathsf{M}_B^A \to \mathsf{M}^{\bar{A}}, \qquad M \mapsto \bar{M},$$

where $\mathsf{M}^{\bar{A}}$ denotes the category of right $\bar{A}$-comodules. This is left adjoint to

$$(1.2) \qquad ? \,\square_{\bar{A}} A: \mathsf{M}^{\bar{A}} \to \mathsf{M}_B^A, \qquad V \mapsto V \,\square_{\bar{A}} A.$$

Here $\square_{\bar{A}}$ denotes the cotensor product [T2, p. 1526]. The structure of $V \,\square_{\bar{A}} A$ in M_B^A comes from that of A. A is called a *(faithfully) coflat* left $\bar{A}$-comodule, if $? \,\square_{\bar{A}} A$ is (faithfully) exact. The adjunctions Ξ, Θ determined by $\bar{?}, ? \,\square_{\bar{A}} A$ are given by

$$(1.3) \qquad \Xi_M: M \to \bar{M} \,\square_{\bar{A}} A, \qquad m \mapsto \sum \overline{m_{(0)}} \otimes m_{(1)},$$

$$(1.4) \qquad \Theta_V: \overline{V \,\square_{\bar{A}} A} \to V, \qquad \overline{\sum v_i \otimes a_i} \mapsto \sum v_i \varepsilon(a_i)$$

for $M \in \mathsf{M}_B^A$, $V \in \mathsf{M}^{\bar{A}}$. For details, refer to [T3, p. 455].

If A is a Hopf algebra, that is, a bialgebra with antipode, then for each $M \in \mathsf{M}_B^A$

$$(1.5) \qquad M \otimes_B A \simeq \bar{M} \otimes A, \qquad m \otimes a \mapsto \sum \overline{m_{(0)}} \otimes m_{(1)} a$$

is an isomorphism [T3, p. 456, line 4]. In particular,

$$(1.6) \qquad A \otimes_B A \simeq \bar{A} \otimes A, \qquad a \otimes a' \mapsto \sum \overline{a_{(1)}} \otimes a_{(2)} a'$$

is an isomorphism.

2. Generalities

2.1. Theorem. *Let A be a Hopf algebra with bijective antipode S, and $B \subset A$ a right coideal subalgebra. Write $\bar{A} = A/AB^+$. Then the following are equivalent with each other*:

(a) *A is flat as a left B-module and B is a simple object in M_B^A*;

(b) *A is faithfully flat as a left B-module*;

(c) *A is a projective generator as a left B-module*;

(d) *A is faithfully coflat as a left $\bar{A}$-comodule and* $B = \{a \in A \mid \sum \overline{a_{(1)}} \otimes a_{(2)} = \bar{1} \otimes a$ *in* $\bar{A} \otimes A\}$;

(e) *The functors* $\bar{?}, ? \,\square_{\bar{A}} A$ *defined in* (1.1)–(1.2) *are* (*mutually quasi-inverse*) *equivalences*;

(f) *The adjunctions* Ξ, Θ *defined in* (1.3)–(1.4) *are isomorphisms.*

If S is bijective, then A^{op} is a Hopf algebra by [DT, Prop. 7]. Hence by applying the result to A^{op}, one sees that the right versions of (a)–(c) are equivalent with each other.

Proof of (2.1). We prove the theorem as follows:

$$\begin{array}{ccccc} (a) & \leftarrow & (b) & \leftarrow & (c) \\ \downarrow & & & & \uparrow \\ (f) & \rightarrow & (e) & \rightarrow & (d) \end{array}$$

(c) $\Rightarrow$ (b), (f) $\Rightarrow$ (e). These are standard facts.

(e) $\Rightarrow$ (d). This implication as well as the converse is shown in [S1, Thm. 4.7]. For completeness we give the proof.

Suppose (e). Then $? \,\square_{\bar{A}} A$ is faithfully exact, which means A is a faithfully coflat left $\bar{A}$-comodule. On the other hand, the right-hand side of the equation in (d) is identified with $k \,\square_{\bar{A}} A$. One sees easily $B \subset k \,\square_{\bar{A}} A$. Apply $\bar{?}$; then $k = \bar{B} \subset \overline{k \,\square_{\bar{A}} A}$. Since $\overline{k \,\square_{\bar{A}} A} = k$ by (e), $\bar{B} = \overline{k \,\square_{\bar{A}} A}$. Hence the equation in (d) holds true.

(d) $\Rightarrow$ (c). Suppose (d). A is a faithfully coflat left comodule over its quotient coalgebra $\bar{A}$. Hence it follows by [S1, Prop. 1.3] that

$$\bar{A} \triangleleft\!\!\oplus A \qquad \text{as left } \bar{A}\text{-comodules,} \tag{2.2}$$

which means that $\bar{A}$ is as a left $\bar{A}$-comodule a direct summand of A. A comodule over a coalgebra is coflat, if and only if it is injective [T2, Prop. A.2.1]. Hence A is an injective left $\bar{A}$-comodule, or in other words

$$A \triangleleft\!\!\oplus \oplus \bar{A} \qquad \text{as left } \bar{A}\text{-comodules,} \tag{2.3}$$

where $\oplus \bar{A}$ denotes a direct sum of some copies of $\bar{A}$.

By [T3, p. 457, line 18], the equation in (d) implies that

$$A \otimes B \to A \,\square_{\bar{A}} A, \qquad a \otimes b \mapsto \sum a_{(1)} \otimes a_{(2)} b \tag{2.4}$$

is an isomorphism. Denote by S^{-} the composite-inverse of S, and compose (2.4) with the isomorphism

$$A \otimes S^{-}(B) \to A \otimes B, \qquad a \otimes S^{-}(b) \mapsto \sum a S^{-}(b_{(2)}) \otimes b_{(1)},$$

which has inverse $\Sigma a b_{(2)} \otimes S^{-}(b_{(1)}) \leftarrow a \otimes b$. Then one has the isomorphism

$$(2.5) \quad A \otimes S^{-}(B) \simeq A \square_{\bar{A}} A, \quad a \otimes S^{-}(b) \mapsto \sum a_{(1)} S^{-}(b) \otimes a_{(2)}.$$

Since $S^{-}(b) \otimes 1 \in A \square_{\bar{A}} A$ for $b \in B, S^{-}(b)$ is invariant under the right $\bar{A}$-coaction, so that the right $S^{-}(B)$-action and the right $\bar{A}$-coaction on A commute with each other. Hence, by applying $A \square_{\bar{A}} ?$ to (2.2) and (2.3), we have

$$(2.6) \quad A \mathrel{<\!\!\!\oplus} A \square_{\bar{A}} A, \quad A \square_{\bar{A}} A \mathrel{<\!\!\!\oplus} \oplus A \quad \text{as right } S^{-}(B)\text{-modules.}$$

It follows from (2.5) and (2.6) that A is a projective generator as a right $S^{-}(B)$-module. By twisting by S, (c) follows.

(b) $\Rightarrow$ (a). It is proved in [M, Lemma 2.2] that, if $? \otimes_B A$ is a faithful functor, B is simple in M_B^A (that is, there exist no non-zero proper right ideals of B which are simultaneously right coideals of A). Hence (b) implies (a).

For (a) $\Rightarrow$ (f), we prove the following:

2.7. LEMMA. *Let A be a Hopf algebra, and $B \subset A$ a right coideal subalgebra. Suppose that B is a simple object in ${}_B\mathsf{M}^A$. Then, for every $0 \neq M \in {}_B\mathsf{M}^A$, B can be embedded as a left B-module into a direct sum $\oplus M$ of some copies of M.*

Proof. Regard $M \otimes A$ as an object in ${}_B\mathsf{M}^A$ with the structures

$$b(m \otimes a) = bm \otimes a, \qquad m \otimes a \mapsto \sum m_{(0)} \otimes a_{(1)} \otimes m_{(1)} a_{(2)},$$

where $b \in B$, $m \otimes a \in M \otimes A$. For any $0 \neq m \in M$,

$$b \mapsto \sum b m_{(0)} \otimes S(m_{(1)}), \qquad B \to M \otimes A$$

is a morphism in ${}_B\mathsf{M}^A$, where S is the antipode of A. This is an injection by simplicity of B. Thus the lemma is established. ∎

Proof of (2.1) (*continued*). (a) $\Rightarrow$ (f). Suppose (a). Since A is left B-flat, it follows by the proof of [T3, Thm. 1] that $\Theta_V \colon \overline{V \square_{\bar{A}} A} \to V$ ($V \in \mathsf{M}^{\bar{A}}$) is an isomorphism. In fact, the composite

$$(V \square_{\bar{A}} A) \otimes_B A \underset{(1.5)}{\simeq} \overline{(V \square_{\bar{A}} A)} \otimes A \underset{\Theta_V \otimes A}{\to} V \otimes A$$

is identified with the isomorphism [T3, p. 456, line 8]

$$(2.8) \qquad \begin{aligned}(V \,\square_{\bar{A}}\, A) \otimes_B A &\simeq V \,\square_{\bar{A}}(A \otimes_B A) \\ &\simeq V \,\square_{\bar{A}}(\bar{A} \otimes A) \simeq V \otimes A,\end{aligned}$$

where the first isomorphism in (2.8) is given since A is left B-flat, and the second is induced from (1.6). Hence Θ_V is an isomorphism. Let $0 \neq M \in \mathsf{M}_B^A$. By applying (2.7) to A^{op}, it follows that there is a right B-linear injection $B \hookrightarrow \oplus M$. Applying the exact functor $? \otimes_B A$, one has $A \hookrightarrow \oplus (M \otimes_B A)$. This implies that the functor $\overline{?}$ is faithfully exact, since one has the isomorphism $M \otimes_B A \simeq \overline{M} \otimes A$ given in (1.5). To show that Ξ_M is an isomorphism, apply $\overline{?}$. Then one has a right $\bar{A}$-colinear map

$$\overline{\Xi_M}\colon \overline{M} \to \overline{M \,\square_{\bar{A}}\, A}, \qquad \bar{m} \mapsto \overline{\sum m_{(0)} \otimes m_{(1)}},$$

which satisfies $\Theta_{\overline{M}} \circ \overline{\Xi_M} = \mathrm{id}$. Since $\Theta_{\overline{M}}$ is an isomorphism as shown above, $\overline{\Xi_M}$ is, too. Therefore Ξ_M is an isomorphism, since $\overline{?}$ is faithfully exact.

Thus we have completed the proof of (2.1). ▮

2.9. Corollary. *Let A be a Hopf algebra, and $B \subset A$ a Hopf subalgebra. Suppose that the antipodes of A and B are both bijective. Then the following are equivalent with each other*:

(a) *A is flat as a left B-module*;

(b) *A is faithfully flat as a left B-module*;

(c) *A is a projective generator as a left B-module*;

(d) *Every non-zero object in ${}_B\mathsf{M}^A$ is a projective generator as a left B-module*;

(a°)–(d°) *The right versions of* (a)–(d).

Proof. (a) ⇔ (b) ⇔ (c). Since B is a Hopf algebra, B is simple in M_B^B by [Sw, Thm. 4.1.1], hence so in M_B^A. Therefore (a) ⇔ (b) ⇔ (c) follows by (2.1).

(d) ⇒ (c). Trivial.

(c) ⇒ (d). Suppose that A is left B-projective, and let $0 \neq M \in {}_B\mathsf{M}^A$. Then it follows by applying [D, Thm. 4] to Hopf algebras $B^{\mathrm{op}} \subset A^{\mathrm{op}}$ that M is left B-projective. Since B is simple in ${}_B\mathsf{M}^A$, it follows from the proof of (2.7) that there is an injection $B \hookrightarrow M \otimes A$ in ${}_B\mathsf{M}^A$. This has a left B-linear retraction, since the cokernel is left B-projective. Thus M is a left B-generator.

(a°) ⇔ (b°) ⇔ (c°) ⇔ (d°). Apply (a) ⇔ (b) ⇔ (c) ⇔ (d) to $B^{\mathrm{op}} \subset A^{\mathrm{op}}$.

(a) ⇔ (a°). Twist by S or S^{-}. ▮

There is some overlap in (2.9) with the result [S2, Cor. 1.8] due to H.-J. Schneider.

3. Commutative Hopf Algebras

Throughout this section, we let A be a commutative Hopf algebra with antipode S, and $B \subset A$ a right coideal subalgebra. Note that S is bijective, in fact $S \circ S = \mathrm{id}$ [Sw, Prop. 4.0.1].

3.1. Lemma. *Let $M \in \mathsf{M}_B^A$ such that M is finitely generated as a B-module. Then the localization M_{B^+} by $B^+ = B \cap \operatorname{Ker} \varepsilon$ is free as a B_{B^+}-module.*

Proof. Set $r = \dim \overline{M}$, the k-dimension of $\overline{M} = M/MB^+$. Then by the isomorphism $M \otimes_B A \simeq \overline{M} \otimes A$ given in (1.5), $M \otimes_B A$ is a free A-module of rank r. Take a B-linear map

$$f: \oplus^r B \to M,$$

which is an isomorphism modulo B^+. Here $\oplus^r B$ denotes the direct sum of r copies of B. Then

$$f_{B^+}: \oplus^r B_{B^+} \to M_{B^+}$$

is a B_{B^+}-linear surjection by the Nakayama lemma, since this is an isomorphism modulo the unique maximal ideal $B^+B_{B^+}$. Apply $? \otimes_{B_{B^+}} A_{B^+}$ to f_{B^+}. Then one has an A_{B^+}-linear surjection

$$\oplus^r A_{B^+} \to (M \otimes_B A)_{B^+} \simeq \oplus^r A_{B^+}. \tag{3.2}$$

One sees this is an isomorphism by counting ranks. (See [B2, Corollaire, p. 111].) Since

$$\oplus^r B_{B^+} \hookrightarrow \oplus^r A_{B^+} \underset{(3.2)}{\simeq} (M \otimes_B A)_{B^+}$$

factors through f_{B^+}, f_{B^+} is a B_{B^+}-linear isomorphism. ∎

3.3. Corollary. *For each $M \in \mathsf{M}_B^A$, M_{B^+} is a flat as a B-module. In particular, A_{B^+} is B-flat.*

Proof. M is written in the form of a directed union $\cup M_\alpha$ of subobjects $\{M_\alpha\}$ of M, where each M_α is a finitely generated B-module [T1, Cor. 2.3]. In fact, write $M = \cup V_\alpha$, a directed union of finite dimensional right A-subcomodules $\{V_\alpha\}$ [Sw, Cor. 2.1.4], and set $M_\alpha = V_\alpha B$. By (3.1), each $(M_\alpha)_{B^+}$ is B_{B^+}-free, hence B-flat. Therefore $M_{B^+} = \cup(M_\alpha)_{B^+}$ is B-flat. ▮

3.4. THEOREM. *A commutative Hopf algebra is a flat module over every right coideal subalgebra.*

Proof. First, we suppose A is finitely generated as an algebra. We may suppose k is algebraically closed. Then any maximal ideal P in A has codimension 1 by the Hilbert Nullstellensatz [B3, Chap. 5, Sect. 3, Prop.1]. Let p: $A \to A/P = k$ be the quotient map. Since A_{B^+} is B-flat by (3.3), A_{A^+} is B-flat ($A^+ = \operatorname{Ker}\varepsilon$). Define an algebra endomorphism T_p: $A \to A$ by

$$T_p(a) = \sum a_{(1)} p(a_{(2)}) \qquad (a \in A)$$

(cf. *the left translation operator* T_g in [W, p. 92]). Then T_p is an automorphism with composite-inverse $T_{p \circ S}$, and takes B to B, and P to A^+. Therefore A_P is B-flat. By the localization property [B2, Chap. 2, Sect. 3, Prop. 15], A is B-flat.

Next, we suppose B is finitely generated as an algebra. One can write $A = \cup A_\alpha$, a directed union of finitely generated Hopf subalgebras $\{A_\alpha\}$ containing B. Each A_α is B-flat, as shown above. Hence A is B-flat.

Finally, we consider the general case. One can write $B = \cup B_\alpha$, a directed union of finitely generated right coideal subalgebras $\{B_\alpha\}$. Since $? \otimes_B A \simeq \varinjlim(? \otimes_{B_\alpha} A)$ [B1, Sect. 6, Prop. 12], A is B-flat. ▮

3.5. COROLLARY. *Let A be a commutative Hopf algebra, and $B \subset A$ a right coideal subalgebra. Then the following are equivalent with each other*:

(a) *B is a simple object in M_B^A;*

(b) *A is faithfully flat as a B-module;*

(c) *A is a projective generator as a B-module;*

(d) *Every non-zero object in M_B^A is a projective generator as a B-module.*

Proof. (a) ⇔ (b) ⇔ (c). This holds by (2.1) and (3.4).

(d) ⇒ (c). Trivial.

(c) ⇒ (d). Suppose (c) and let $0 \neq M \in \mathsf{M}_B^A$. Since $B \oplus\!\!\!\!< A$ as B-modules, it follows by (1.5) that

$$M \oplus\!\!\!\!< M \otimes_B A \simeq \overline{M} \otimes A \qquad \text{as } B\text{-modules}.$$

Since A is B-projective, M is, too. By the same way as that in the proof of (2.9), it is shown that M is a B-generator. ▮

3.6. *Remark.* In (3.5), suppose that B is a Hopf subalgebra. Then B satisfies (a) by [Sw, Thm. 4.1.1]. Thus we obtain a simple proof of the following important theorem due to M. Takeuchi:

THEOREM [T1, THM. 3.1; T3, THM. 5]. *A commutative Hopf algebra is a faithfully flat module, or more strongly a projective generator, over every Hopf subalgebra.*

This theorem is a contribution to the algebraic theory of quotients of affine group schemes. In fact, as mentioned in [T1, Thm. 5.2(ii)], the weaker assertion "faithfully flat" is an algebraic counterpart of an old result [DG, III, Sect. 3, 7.2a] on such schemes.

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