On the additivity of tunnel number of knots

Kanji Morimoto

Department of Mathematics, Takushoku University, Tatemachi, Hachioji, Tokyo 193, Japan

Received 16 January 1992
Revised 8 June 1992, 26 August 1992 and 12 October 1992

Abstract


Let $K_1$ and $K_2$ be nontrivial knots in the 3-sphere $S^3$. In this paper, we show that if the tunnel number of $K_1 \# K_2$ is two, then either both tunnel numbers of $K_1$ and $K_2$ are one, or one of $K_1$ and $K_2$ is a 2-bridge knot and the other's tunnel number is at most two.

Keywords: Tunnel number; Knots; Connected sum; Additivity.

AMS (MOS) Subj. Class.: 57M25.

Introduction

Let $K$ be a knot in the 3-sphere $S^3$, and $t(K)$ the tunnel number of $K$. Here, the tunnel number of $K$ is the minimum number of mutually disjoint arcs properly embedded in the exterior of $K$ whose complementary space is a handlebody. We call such arcs an unknotting tunnel system for $K$. In particular, we call it an unknotting tunnel for $K$ if the family of the arcs consists of a single arc. Concerning the additivity of tunnel number of knots under connected sum, so far by several people [6, 7, 9, etc.], it has been proved only that tunnel number one knots are prime. In this paper we show:

Theorem. Let $K_1$ and $K_2$ be nontrivial knots in $S^3$, and suppose $t(K_1 \# K_2) = 2$. Then:

1. if neither $K_1$ nor $K_2$ are 2-bridge knots, then $t(K_1) = t(K_2) = 1$ and at least one of $K_1$ and $K_2$ admits a (1, 1)-decomposition, or

2. if one of $K_1$ and $K_2$, say $K_1$, is a 2-bridge knot, then $t(K_2)$ is at most two and $K_2$ is prime.

Correspondence to: Professor K. Morimoto, Department of Mathematics, Takushoku University, Tatemachi, Hachioji, Tokyo 193, Japan.

0166-8641/93/$06.00 \copyright$ 1993 - Elsevier Science Publishers B.V. All rights reserved
Here, we say that a knot $K$ in $S^3$ admits a $(g, b)$-decomposition if there is a genus $g$ Heegaard splitting $(V_1, V_2)$ of $S^3$ such that $V_i \cap K$ is a $b$-string trivial arc system in $V_i$ ($i = 1, 2$) (cf. [4, 6]).

**Remark.** After the author had done the work in this paper, he proved in [5] that there are infinitely many tunnel number two knots $K$ such that the tunnel number of $K \# K'$ is equal to two again for any 2-bridge knot $K'$. This shows that the estimate of Theorem is the best possible.

**Corollary 1.** Every tunnel number two knot has at most two connected sum sum-mands.

**Proof.** Suppose $t(K_1 \# K_2 \# K_3) = 2$ for some nontrivial knots $K_1$, $K_2$ and $K_3$. Put $K_4 = K_2 \# K_3$. Then $t(K_1 \# K_4) = 2$. Then by Theorem, both $K_1$ and $K_4$ are prime because tunnel number one knots are prime. Since $K_4$ is not prime, we have a contradiction, and this completes the proof of the corollary. □

**Corollary 2.** Let $K_1$ and $K_2$ be tunnel number one knots. Then $K_1 \# K_2$ has tunnel number two if and only if at least one of $K_1$ and $K_2$ admits a $(1, 1)$-decomposition.

**Proof.** Suppose $K_1 \# K_2$ has tunnel number two. Then by Theorem and since 2-bridge knots admit $(1, 1)$-decompositions, at least one of $K_1$ and $K_2$ admits a $(1, 1)$-decomposition.

Conversely, suppose at least one of $K_1$ and $K_2$ admits a $(1, 1)$-decomposition. Then by tracing back the argument in the paragraphs previous to Lemma 2.1 of Section 2, we see that $K_1 \# K_2$ has tunnel number two. This completes the proof of the corollary. □

By the way, by a little observation, we have the following facts:

**Fact 0.1.** If $t(K) \leq t$ for a knot $K$, then $g(\Sigma_2(K)) \leq 2t + 1$, where $\Sigma_2(K)$ is the 2-fold branched covering space of $S^3$ along $K$ and $g(\cdot)$ denotes the Heegaard genus.

**Fact 0.2.** $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$ for any two knots $K_1$ and $K_2$.

**Fact 0.3.** If a knot $K$ admits a $(g, b)$-decomposition, then $t(K) \leq g + b - 1$.

The author does not know if there is a knot which realizes the upper equality in Fact 0.1, etc. But he expects that the following conjectures are true.

**Conjecture 1.** There is a tunnel number one knot $K_0$ such that $g(\Sigma_2(K_0)) = 3$. 
Conjecture 2. There are two tunnel number one knots $K_1$ and $K_2$ such that $t(K_1 \# K_2) = 3$.

Conjecture 3. There is a tunnel number one knot $K_3$ which admits no $(1, 1)$-decomposition.

Concerning the above conjectures, we have:

**Proposition 0.4.** (1) Conjecture 1 implies Conjecture 2. (2) Conjecture 2 implies Conjecture 3.

**Proof.** (1) Suppose $t(K_0 \# K_0) \leq 2$, then by Fact 0.1, $g(\Sigma_2(K_0 \# K_0)) \leq 2 \cdot 2 + 1 = 5$. On the other hand, since the Heegaard genus of closed 3-manifolds is additive under connected sum by [1], we have $g(\Sigma_2(K_0 \# K_0)) = g(\Sigma_1(K_0) \# \Sigma_1(K_0)) = g(\Sigma_2(K_0)) + g(\Sigma_2(K_0)) = 3 + 3 = 6$, a contradiction.

(2) Suppose both $K_1$ and $K_2$ admit $(1, 1)$-decompositions. Then by the definition of $(g, b)$-decomposition, we see that $K_1 \# K_2$ admits a $(2, 1)$-decomposition. Then by Fact 0.3, $t(K_1 \# K_2) \leq 2$, a contradiction. □

As a consequence of Theorem, in particular Corollary 2, we have:

**Corollary 3.** Conjecture 3 implies Conjecture 2.

**Proof.** Suppose $t(K_3 \# K_3) \leq 2$. Then since tunnel number one knots are prime, $t(K_3 \# K_3) = 2$. Hence by Corollary 2, $K_3$ admits a $(1, 1)$-decomposition, a contradiction. □

1. Preliminaries

We work in the piecewise linear category. Put $K = K_1 \# K_2$, and let $N(K)$ be a regular neighborhood of $K$ in $S^3$ and $E(K) = \text{cl}(S^3 - N(K))$ an exterior of $K$. Let $\{\gamma_1, \gamma_2\}$ be an unknotting tunnel system for $K$ properly embedded in $E(K)$ and $N(\gamma_1 \cup \gamma_2)$ a regular neighborhood of $\gamma_1 \cup \gamma_2$ in $E(K)$. Put $V_1 = N(K) \cup N(\gamma_1 \cup \gamma_2)$ and $V_2 = \text{cl}(S^3 - V_1)$. Then both $V_1$ and $V_2$ are genus three handlebodies. Since $K$ is not prime, there is an 2-sphere $S$ in $S^3$ which gives a nontrivial connected sum of $K$. We may assume that $S \cap V_1$ consists of disks. Since $S$ intersects $K$ in two points, we can put $S \cap V_1 = D_1^* \cup D_2^* \cup D_3 \cup D_4 \cup \cdots \cup D_l$, where $D_i^*$ ($i = 1, 2$) is a nonseparating disk of $V_1$ intersecting $K$ in a point and $D_j \cap K = \emptyset$ ($j = 1, 2, \ldots, l$). Suppose $\#(S \cap V_1)$ is minimum among all 2-spheres which give nontrivial connected sums of $K$ and intersect $V_1$ in such disks, where $\#(\cdot)$ denotes the number of the components. Then, since $K$ is a core of a handle
of $V_i$ and $D_i^*$ is a nonseparating disk of $V_i$ ($i = 1, 2$), we have the following three cases:

Case I: $D_1^* \cup D_2^*$ splits $V_1$ into two solid tori (Fig. 1(I)).
Case II: $D_1^* \cup D_2^*$ does not separate $V_1$ (Fig. 1(II)) and
Case III: $D_1^* \cup D_2^*$ are mutually parallel (Fig. 1(III)).

Put $S_i = S \cap V_i$ ($i = 1, 2$). Then by the minimality of $\#(S_1)$, $S_2$ is incompressible in $V_2$. Let $(E_1, E_2, E_3)$ be a complete meridian disk system of $V_2$, and put $E = E_1 \cup E_2 \cup E_3$. Then by the incompressibility of $S_2$, we may assume that each component of $E \cap S_2$ is an arc. Let $\alpha$ be an outermost arc component of $E \cap S_2$ in $E$. If $\alpha$ cuts off a disk in $S_2$, then by using the disk, we can exchange $E$ for another complete meridian disk system $E'$ so that $\#(E' \cap S_2) < \#(E \cap S_2)$. Hence we may assume that $\alpha$ is essential in $S_2$. Let $\Delta$ be the disk cut off by $\alpha$ in $E$ such that $\Delta \cap S_2 = \alpha$. Then we can perform an isotopy through $\Delta$ which pushes a regular neighborhood of $\alpha$ in $S_2$ into $V_1$. According to Jaco [2, Ch. II], we call this an isotopy of type A at $\alpha$ through $\Delta$. Then as in [2, Ch. II], by exchanging complete meridian disk systems at each stage if necessary, we have a sequence of isotopies of type A at $\alpha_i$ through $\Delta_i$ ($i = 1, 2, \ldots, n$) such that each $\alpha_i$ is an essential arc.
properly embedded in $S_2^{i-1}$, where $S_2^0 = S_2$, $S_2^i = \text{cl}(S_2^{i-1} - N(\alpha_i))$ and $S_2^a$ consists of disks. Furthermore we may assume that each $\alpha_i$ is an essential arc properly embedded in $S_2$ and $\alpha_i \cap \alpha_j = \emptyset$ $(i \neq j)$. Put $\partial D_i^* - C_i^*$ $(i = 1, 2)$. Then each $\alpha_i$ is one of the following three types.

We say that $\alpha_i$ is of type I if $\alpha_i$ connects distinct components of $\partial S_2$, $\alpha_i$ is of type II if $\alpha_i$ meets a single component of $\partial S_2$ and does not separate $C_i^*$ and $C_2^*$, and $\alpha_i$ is of type III if $\alpha_i$ meets a single component of $\partial S_2 - (C_i^* \cup C_2^*)$ and separates $C_i^*$ and $C_2^*$ (see Fig. 2). Moreover we say that $\alpha_i$ is a $d$-arc if $\alpha_i$ is of type I and there exists a component $C$ of $\partial S_2 - (C_i^* \cup C_2^*)$ such that $\alpha_i$ meets $C$ and $\alpha_j$ does not meet $C$ for any $j < i$, and $\alpha_i$ is an $e$-arc if $\alpha_i$ connects $C_i^*$ and $C_2^*$.

Put $S^{(0)} = S$, and let $S^{(i)}$ be the image of $S^{(i-1)}$ after the isotopy of type A at $\alpha_i$ $(i = 1, 2, \ldots, n)$. Put $\text{cl}(\partial D_i - \alpha_i) = \beta_i$. Then, at each stage, $\beta_i$ is an arc in $\partial V_2 = \partial V_1$. By performing the isotopy of type A at $\alpha_i$, a band in $V_1$ whose core is $\beta_i$ is produced. We denote it by $b_i$.

Under the above terms and notations, in the following sections, we show that $l = 0$ or $1$ by using isotopy of type A argument. In Section 2, we consider Case I and show that $l = 0$, then we see that the conclusion (1) of Theorem holds. In Sections 3 and 4, we consider Cases II and III and show that $l = 1$, then we see that the conclusion (2) of Theorem holds. Here we note that any isotopy has to be fixing the knot $K$ setwise. Before going to the following sections, we prepare some facts.

**Fact 1.1.** If $l > 0$, then no $\alpha_i$ is a $d$-arc.

**Proof.** If there is a $d$-arc, then by the inverse operation of isotopy of type A introduced in [8], we can reduce the number $\#(S_1)$, a contradiction. $\square$
Fact 1.2. If $l > 0$, then no $\alpha_i$ is of type II.

Proof. If there is an arc $\alpha_i$ of type II, then we can find a $d$-arc in the planar surface in $S_2$ cut off by $\alpha_i$, a contradiction. □

Fact 1.3. If $l > 0$, then no $\alpha_i$ is an $e$-arc.

Proof. If there is an $e$-arc, then any arc is of type I or of type II. Then by Fact 1.2, $\alpha_1$ is of type I. Then it is a $d$-arc, a contradiction. □.

Fact 1.4. If $l > 0$, then $\alpha_1$ is of type III.

Proof. By Fact 1.2, $\alpha_1$ is of type I or of type III. If $\alpha_1$ is of type I, then it is a $d$-arc. Hence $\alpha_1$ is of type III. □

Throughout this paper, for an $m$-manifold $M$ ($m = 2$ or 3 respectively) and an $n$-manifold $N$ ($n = 1$ or 2 respectively) properly embedded in $M$, a component of $M - N$ means the closure of a component of $M - N$.

2. Case I

Suppose Case I occurs.

Suppose $l = 0$. Then $S_1 = D_1^\ast \cup D_2^\ast$ and $S_2$ is an essential (i.e., incompressible and not $\partial$-parallel) annulus in $V_2$. Since $\partial S_1(= \partial S_2)$ splits $\partial V_1(= \partial V_2)$ into two tori with two holes, $S_2$ is a separating annulus in $V_2$. Hence by the same argument as that of [4, Lemma 3.2], we can regard $S_2$ as a union of an essential separating disk, say $D$, and a band, say $b$. Since $D$ splits $V_2$ into a solid torus and a genus two handlebody, $b$ is contained in one of them. If $b$ is contained in the solid torus, then $\partial S_2$ splits $\partial V_2$ into an annulus and a genus two surface with two holes, a contradiction. Hence $b$ is contained in the genus two handlebody as illustrated in Fig. 3.
Fig. 3. Let $X_1$ and $X_2$ be the two components of $V_1 - S_1$, and $Y_1$ and $Y_2$ the two components of $V_2 - S_2$ indicated in Fig. 3.

We may assume that $Y_i \cap \partial V_i$ is identified with $X_i \cap \partial V_i$ ($i = 1, 2$). Put $B_i = X_1 \cup Y_i$ and $B_2 = X_2 \cup Y_2$. Then, since $R_1$ and $R_2$ are the two components of $S^3 - S_i$, $B_i$ is a 3-ball ($i = 1, 2$). Put $\delta_i = B_i \cap K (= X_i \cap K)$.

**Claim.** For $i = 1, 2$, $\delta_i$ is a trivial arc in $X_i$.

**Proof.** Since $K$ is a core of a handle of $V_1$, there is an annulus $A$ in $V_1$ such that a component of $\partial A$ is $K$ and the other component of $\partial A$ is in $\partial V_1$. Since $K$ intersects $D_i^+$ ($i = 1, 2$) in a point, by the standard innermost argument and the cut and paste argument, we can choose $A$ so that $A \cap D_i^+$ ($i = 1, 2$) consists of an arc which is essential in $A$. Put $A \cap X_i = R_i$ ($i = 1, 2$). Then $R_i$ is a disk in $X_i$ such that $\partial R_i = \delta_i \cup (\text{an arc in } \partial X_i)$. This shows that $\delta_i$ is a trivial arc in $X_i$ and completes the proof of the claim. \Box

Let $B_i'$ be a 3-ball and $\delta_i'$ a trivial arc properly embedded in $B_i'$ ($i = 1, 2$). Put $S^3_i = B_i \cup B_i'$ and $K_i = \delta_i \cup \delta_i'$ ($i = 1, 2$). Then $K_i$ is a knot in the 3-sphere $S^3_i$. In the following, we show that $K_1$ admits a $(1, 1)$-decomposition and that $K_2$ has tunnel number one.

We denote the images of $D_1^*, D_2^*$ and $S_2$ in $\partial X_1$ and $\partial Y_1$ by the same notations. Regard $B_1'$ as $D_1^* \times [0, 1]$ and $\delta_1'$ as $\{x_1\} \times [0, 1]$, where $D_1^*$ is a 2-disk, $I = [0, 1]$ and $x_1$ is a point in $\text{int}(D_1^*)$. Choose the gluing map $f$ of $\partial B_1'$ to $\partial D_1^*$ so that $f(D_1^* \times \{0\}) = \partial B_1'$, $f(D_1^* \times \{1\}) = D_2^*$ and $f(\partial D_1^* \times I) = S_2$. Put $W_1 = X_1$ and $W_2 = Y_1 \cup \partial D_1^* \times I$. Then $W_1$ is a solid torus. And since there is a nonseparating disk, say $\Delta_i$, in $Y_i$ such that $\Delta_i \cap S_2$ is an arc (see Fig. 3), $W_2$ is also a solid torus because $D_2^* \times I$ is a cancelling 2-handle for $Y_i$. Hence $(W_1, W_2)$ is a genus one Heegaard splitting of the 3-sphere $S^3_i (= B_i \cup B_i')$. Since $\delta_i$ is a trivial arc in $W_i = X_i$ by Claim and $\delta_i$ is a trivial arc in $W_2$ by the definition, $K_i (= \delta_i \cup \delta_i')$ admits a $(1, 1)$-decomposition.

Next we denote the images of $D_1^*, D_2^*$ and $S_2$ in $\partial X_2$ and $\partial Y_2$ by the same notations. Regard $B_2'$ as $D_2^* \times [0, 1]$ and $\delta_2'$ as $\{x_2\} \times [0, 1]$, where $D_2^*$ is a 2-disk and $x_2$ is a point in $\text{int}(D_2^*)$. Choose the gluing map $f$ of $\partial B_2'$ to $\partial D_2^*$ so that $f(D_2^* \times \{0\}) = D_1^*$, $f(D_2^* \times \{1\}) = D_2^*$ and $f(\partial D_2^* \times I) = S_2$. Put $W_1 = X_2 \cup_{f(D_2^* \times \{0\}) \cup (D_2^* \times \{1\})} (D_2^* \times I)$ and $W_2 = Y_2$. Then $W_2$ is a genus two handlebody. And since $D_2^* \times I$ is a 1-handle for $X_2$, $W_1$ is also a genus two handlebody. Hence $(W_1, W_2)$ is a genus two Heegaard splitting of the 3-sphere $S^3_i (= B_2 \cup B_2')$. Since $\delta_2$ is a trivial arc in $X_2$ by Claim and $\delta_2'$ is a trivial arc in $D_2^* \times I$ by the definition, $K_2 (= \delta_2 \cup \delta_2')$ is a core of a handle of $W_1$. Thus $K_2$ has tunnel number one.

In the rest of this section, we show $l = 0$. Then by the uniqueness of prime decomposition of knots [10], and since tunnel number one knots are prime, we have the conclusion (1) of Theorem. The next lemma is trivial but important.
Lemma 2.1. Let $U$ be a solid torus in $S^3$ and $c$ an essential loop in $\partial U$. If $c$ bounds a disk in $\partial (S^3 - U)$, then $U$ is an unknotted solid torus and $c$ is isotopic in $U$ to a core of $U$.

By Fact 1.4, by performing the isotopy of type A at $\alpha_1$, we have an annulus in $V_1$, say $A_1$.

Lemma 2.2. If $l > 0$, then no core of $A_1$ bounds a disk in $S^3 - K$.

Proof. Suppose a core of $A_1$, say $c$, bounds a disk, say $D$, in $S^3 - K$. Let $A$ be an annulus in $S^{(1)}$ such that $\partial A = C_1^* \cup c$. Then $A \cap K = \emptyset$ because $\alpha_1$ is of type III by Fact 1.4. Then by using the annulus $A$ and the disk $D$, we see that $C_1^*$ is contractible in $S^3 - K$ to a point. This is a contradiction because $C_1^*$ is a meridian of $N(K)$. □

Lemma 2.3. If $l > 0$, then $b_1$ is not contained in any 3-ball component of $V_1 - S_1$.

Proof. Suppose $b_1$ is contained in a 3-ball component of $V_1 - S_1$, say $B$. Then we can consider that $A_1$ is an annulus in $\partial B$. If $B \cap K = \emptyset$, then a core of $A_1$ bounds a disk in $B(\subset S^3 - K)$. This contradicts Lemma 2.2. Hence $B \cap K \neq \emptyset$. Since $K$ is decomposed by the 2-sphere $S$ into two components, the two components cannot be contained in the same component of $V_1 - S_1$. Hence $B \cap K$ is a single arc. Moreover, by Lemma 2.2, we can find a disk, say $D$, in $B$ such that $D \cap K$ is a point and $D \cap A_1 = \partial D$ is a core of $A_1(\subset S^{(1)})$. Let $P_1$ and $P_2$ be the two components of $S^{(1)} - \partial D$ containing $D_1^*$ and $D_2^*$ respectively. Put $Q_i = P_i \cup D$ $(i = 1, 2)$. Since $S^{(1)}$ gives a nontrivial connected sum of $K$, we may assume that $Q_1$ gives a nontrivial connected sum of $K$. Then, since $D_i^*$ is contained in $Q_2$, $Q_1 \cap V_1$ consists of at most $l + 1$ disks. This contradicts the minimality of $\#(S_1)$. □

We note that the above three lemmas remain valid in Cases II and III too.
Lemma 2.4. \( l = 0 \).

Proof. Suppose \( l > 0 \) and \( b_1 \) is attached to \( D_1 \). If \( D_1 \) is a nonseparating disk in \( V_1 \), then \( b_1 \) is contained in the 3-ball in \( V_1 \) cut off by \( D_1^* \cup D_2^* \cup D_1 \). This contradicts Lemma 2.3. Hence \( D_1 \) is a separating disk and \( b_1 \) is contained in the solid torus in \( V_1 \) cut off by \( D_1 \). Note here that \( D_1 \cap K = \emptyset \). Let \( U \) be the solid torus in \( V_1 \) cut off by \( A_1 \) indicated in Fig. 4.

Let \( c \) be a core of \( A_1 \), then by Lemma 2.2, \( c \) is an essential loop in \( \partial U \). Since \( c \) is a loop in \( S^{(1)} \), \( c \) splits \( S^{(1)} \) into two disks. Then by using one of the two disks and the fact that \( S^{(1)} \cap U = A_1 \subset \partial U \), we see that \( c \) bounds a disk in \( \text{cl}(S^3 - U) \). Hence by Lemma 2.1, \( c \) is isotopic in \( U \) to a core of \( U \). This shows that \( b_1 \) wraps a handle of \( V_1 \) exactly once. Hence \( A_1 \) is isotopic rel. \( \partial A_1 \) to the annulus \( \text{cl}(\partial U - A_1) \). Thus we can reduce the number \( \#(S_1) \) by pushing back \( A_1 \) into \( V_2 \). This contradiction completes the proof of the lemma and the proof of Case I. \( \square \)

3. Case II

Suppose Case II occurs.

If \( l = 0 \), then we can find a loop in \( V_1 \) which intersects \( D_1^* \cup D_2^* \) in a single point. This shows that \( S \) is a nonseparating 2-sphere in \( S^3 \). Hence \( l > 0 \).

Suppose \( l = 1 \), and \( D_1 \) is a nonseparating disk in \( V_1 \) such that \( D_1^* \cup D_2^* \cup D_1 \) splits \( V_1 \) into a 3-ball and a solid torus. By Lemma 2.3, \( b_1 \) is contained in the solid torus component of \( V_1 - S_1 \) as illustrated in Fig. 5, and let \( A_1 \) be the annulus as a union of \( D_1 \) and \( b_1 \). Since \( V_2 \cap S^{(1)} \) consists of two annuli, we can put \( V_2 \cap S^{(1)} = F_1 \cup F_2 \). Moreover we can regard \( F_i \) as a union of an essential disk in \( V_2 \), say \( G_i \), and a band in \( V_2 \), say \( h_i \) \((i = 1, 2) \). In addition, suppose \( G_1 \) and \( G_2 \) are mutually parallel nonseparating disks and that \( h_1 \) and \( h_2 \) are not mutually parallel bands as illustrated in Fig. 6.

Let \( X_1 \) and \( X_2 \) be the two components of \( V_1 - (D_1^* \cup D_2^* \cup A_1) \) indicated in Fig. 5, and \( Y_1 \) and \( Y_2 \) the two components of \( V_2 - (F_1 \cup F_2) \) indicated in Fig. 6. If \( Y_1 \cap \partial V_2 \) is identified with \( X_2 \cap \partial V_1 \), then the band \( b_2 \), which is produced by the
isotopy of type A at $\alpha_2$ through $\Delta$ (indicated in Fig. 6), does not run over $b_1$. Hence we can push back $b_1$ into $V_2$, leaving $b_2$ in $V_1$. Then since $\alpha_2$ is of type I and is not an e-arc, $b_2$ connects $D_1$ and one of $D_1^+$ and $D_2^+$. Hence we can reduce the number $\#(S_i)$, a contradiction. Thus, for $i = 1, 2$, $Y_i \cap \partial V_2$ is identified with $X_i \cap \partial V_1$.

Put $B_1 = X_1 \cup Y_1$ and $B_2 = X_2 \cup Y_2$. Then, since $B_1$ and $B_2$ are the two components of $S^3 - S^{(1)}$, $B_i$ is a 3-ball ($i = 1, 2$). Put $\delta_i = B_i \cap K = X_i \cap K$. Then by the argument in the proof of Claim in Section 2, we see that $\delta_i$ is a trivial arc in $X_i$ ($i = 1, 2$). Let $B_i'$ be a 3-ball and $\delta_i'$ a trivial arc properly embedded in $B_i'$ ($i = 1, 2$). Put $S_i^3 = B_i \cup B_i'$ and $K_i = \delta_i \cup \delta_i'$ ($i = 1, 2$), then $K_i$ is a knot in the 3-sphere $S_i^3$. In the following, we show that $K_1$ has a 2-bridge decomposition and that $K_2$ has tunnel number at most two.

We denote the images of $D_i^+$, $D_i^-$, $A_i$, $F_1$ and $F_2$ in $\partial X_1$ and $\partial Y_1$ by the same notations. Let $D_2$ be a 2-disk and $a_1$ and $a_2$ two points in $\text{int}(D_2)$, and let $N(a_i)$ be a regular neighborhood of $a_i$ ($i = 1, 2$) in $\text{int}(D_2)$ with $N(a_1) \cap N(a_2) = \emptyset$. Put $D_i^+ = \text{cl}(D_2 - N(a_1))$ and $D_i^- = \text{cl}(D_2 - (N(a_1) \cup N(a_2)))$. Since there is a nonseparating disk in $X_1$, say $\delta_0$, such that $\delta_0 \cap \delta_1 = \emptyset$ and $\delta_0 \cap \partial A$ is an arc and since there are two nonseparating disks in $Y_1$, say $\Delta_1$ and $\Delta_2$, such that $\Delta_i \cap \partial A$ is an arc ($i = 1, 2$), we see that $(X_1, \delta_1, A_1)$ is homeomorphic to $(D_2^+ \times I, (a_1) \times I, \partial N(a_2) \times I)$ and that $(Y_1, F_1, F_2)$ is homeomorphic to $(D_2^- \times I, \partial N(a_1) \times I, \partial N(a_2) \times I)$, where $I = [0, 1]$ (see Fig. 7). Regard $B_i'$ as $D_i^+ \times [0, 3]$ and $\delta_i'$ as $(x_1) \times [0, 3]$, where $D_i^+$ is a 2-disk and $x_1$ is a point in $\text{int}(D_i^+)$.

Choose the glueing map $f$ of $\partial B_i'$ to $\partial B_1$ so that $f(D_i^+ \times (0)) = D_i^+$, $f(\partial D_i^+ \times [0, 1]) = F_1$, $f(\partial D_i^+ \times [1, 2]) = A_i$, $f(\partial D_i^+ \times [2, 3]) = F_2$ and $f(D_i^- \times [3]) = D_i^-$. Put $W_1 = X_1 \cup_{f_1(D_2^+ \times [1, 2])} (D_2^+ \times [1, 2])$ and $W_2 = Y_1 \cup_{f_2(D_2^- \times [1, 2])} (D_2^- \times [1, 2])$. Then $(W_1, W_2)$ is a genus zero Heegaard splitting of the 3-sphere $S_i^3 = B_1 \cup B_1'$ which gives a 2-bridge decomposition of $K_i = \delta_i \cup \delta_i'$. Hence $K_i$ has a 2-bridge decomposition. We note here that the above argument is due to Kobayashi [3] (cf. [3, Fig. 7, p. 18]).

Next we denote the images of $D_i^+$, $D_i^-$, $A_i$, $F_1$ and $F_2$ in $\partial X_2$ and $\partial Y_2$ by the same notations. Let $e$ be a trivial arc properly embedded in $X_2$ which is obtained
by pushing an essential arc properly embedded in $A_1$ into $X_2$, and let $N(\varepsilon)$ be a regular neighborhood of $\varepsilon$ in $X_2$. Then $\text{cl}(X_2 - N(\varepsilon))$ is a genus two handlebody and $Y_2 \cup N(\varepsilon)$ is a genus three handlebody. By the definition of $\varepsilon$, there is a nonseparating disk, say $D$, properly embedded in $\text{cl}(X_2 - N(\varepsilon))$ such that $D \cap A_1 = \partial D \cap A_1$ is an essential arc properly embedded in $A_1$. Since $\delta_2$ is a trivial arc in $X_2$ and $\delta_2 \cap D = \emptyset$, there is a disk $R$ in $\text{cl}(X_2 - N(\varepsilon))$ such that $\partial R = \delta_2$ and $\partial R = \delta_2 \cup (\text{an arc in } \partial \text{cl}(X_2 - N(\varepsilon)))$.

**Claim.** We can choose $R$ so that $R \cap A_1 = \emptyset$.

**Proof.** Suppose $R \cap A_1(= \partial R \cap A_1) \neq \emptyset$. Then we may assume that $R \cap A_1$ consists of essential arcs in $A_1$. Put $D \cap A_1 = e_0$ and $R \cap A_1 = e_1 \cup e_2 \cup \cdots \cup e_k$, and suppose that these arcs lie in $A_1$ in this order. Then $e_0 \cup e_1$ cuts off a disk in $A_1$, say $G_1$, such that $G_1 \cap \{e_0 \cup e_1\} = e_0 \cup e_1$. Then since $R \cup G_1 \cup D$ is a disk, by pushing it off slightly, we get a new disk $R_1$ in $\text{cl}(X_2 - N(\varepsilon))$ such that $R_1 \cap D = \emptyset$, $\partial R_1 = \delta_2 \cup (\text{an arc in } \partial \text{cl}(X_2 - N(\varepsilon)))$ and $R_1 \cap A_1 = e_2 \cup e_3 \cup \cdots \cup e_k$. Hence by repeating these operations, we get a disk $R_k$ in $\text{cl}(X_2 - N(\varepsilon))$ such that $R_k \cap D = \emptyset$, $\partial R_k = \delta_2 \cup (\text{and arc in } \partial \text{cl}(X_2 - N(\varepsilon)))$ and $R_k \cap A_1 = \emptyset$. This completes the proof of the claim. □

By Claim, and since the regular neighborhood of $A_1 \cup D$ in $\text{cl}(X_2 - N(\varepsilon))$ is a solid torus, we can consider that $\text{cl}(X_2 - N(\varepsilon))$ is a disk sum of two solid tori such
that one of them contains $\delta_2$ as a trivial arc and the other contains $A_1$ in the boundary as a core of it (see Fig. 8).

Regard $B_2$ as $D_2 \times [0, 3]$ and $\delta_2$ as $\{x_2\} \times [0, 3]$, where $D_2$ is a 2-disk and $x_2$ is a point in int$(D_2^2)$. Choose the glueing map $f$ of $\partial B_2$ to $\partial B_1$ so that $f(D_2^2 \times \{0\}) = D_1^*$, $f(\partial D_2^2 \times [0, 1]) = F_1$, $f(\partial D_2^2 \times [1, 2]) = A_1$, $f(\partial D_2^2 \times [2, 3]) = F_2$ and $f(D_2^2 \times \{3\}) = D_2^*$. Put $W_1 = \text{cl}((X_2 - N(e)) \cup \eta(D_2^2 \times \{0\}) \cup \eta(D_2^2 \times [1, 2]) \cup \eta(D_2^2 \times \{3\})) \cup (D_2^2 \times \{0, 3\})$ and $W_2 = Y_2 \cup N(e)$. Then $(W_1, W_2)$ is a genus three Heegaard splitting of the 3-sphere $S^3(= B_2 \cup B_1)$. Moreover by Claim and Fig. 8, $K_2(= \delta_2 \cup \delta_2')$ is a core of a handle of $W_1$. Hence $K_2$ has tunnel number at most two.

In the rest of this section, we show that $I = 1$, that $D_1$ and $V_2 \cap S^{(1)}$ satisfy the above conditions and that the knot $K_2$ is prime. Then by the uniqueness of prime decomposition of knots, we have the conclusion (2) of Theorem. In the following proof, put $\mathcal{D}^* = D_1^* \cup D_2^*$ and $\mathcal{D} = \{D_i\}_{i=1}^l$.

**Lemma 3.1.** There is no separating disk in $\mathcal{D}$.

**Proof.** Suppose there is a separating disk in $\mathcal{D}$, say $D_1$. Then $D_1$ splits $V_1$ into a solid torus and a genus two handlebody containing $\mathcal{D}^*$. If the solid torus contains a nonseparating disk in $\mathcal{D}$, then each component of $V_1 - S_1$ is a 3-ball. This contradicts Lemma 2.3. Hence we may assume that the solid torus intersects $S_1$ in only $D_1$ and that $b_1$ is contained in the solid torus. Then by the argument in the proof of Lemma 2.4 (cf. Fig. 4), we have a contradiction. □

**Lemma 3.2.** There is no disk $D_i$ in $\mathcal{D}$, such that $\{D_1^*, D_2^*, D_i\}$ is a complete meridian disk system of $V_i$.

**Proof.** If there is such a disk in $\mathcal{D}$, then each component of $V_1 - S_1$ is a 3-ball. This contradicts Lemma 2.3. □

**Lemma 3.3.** We may assume that $\mathcal{D}$ consists of one parallel class.

**Proof.** Suppose $\mathcal{D}$ has more than one parallel classes. Then by Lemmas 3.1 and 3.2, it has exactly two parallel classes as illustrated in Fig. 9.
Tunnel number of knots

Put $\mathscr{D}_1 = \{D_i\}_{i=1}^{r+1}$ and $\mathscr{D}_2 = \{D_i\}_{i=r+1}^{r+1}$ as in Fig. 9. Then we may assume that $b_1$ is attached to $D_1$ and is contained in the solid torus (not containing $K$) in $V_1$ cut off by $D_1 \cup D_{r+1}$. Let $U$ be the solid torus in $V_1$ cut off by $A_1 \cup D_{r+1}$. Then by Lemma 2.1, there is a meridian disk of $U$, say $D$, intersecting $A_1$ in an arc.

Continue isotopies of type A at $\alpha_i$ ($i = 1, 2, \ldots$). Suppose $\alpha_k$ is of type I and $\alpha_j$ is of type III for all $j \leq k$, and let $A_j$ be the annulus in $V_1$ produced by the isotopy of type A at $\alpha_j$ ($1 \leq j \leq k$). Since $\alpha_k$ is not a $d$-arc, $b_{k+1}$ connects two annuli or a disk in $\mathscr{D}^*$ and an annulus. If $b_{k+1}$ connects two annuli, say $A_s$ and $A_t$ ($s < t$), then $b_{k+1}$ is contained in the region between $A_s$ and $A_t$. Then since $b_{k+1}$ does not run over $b_i$ ($i \leq k$), by pushing back the bands $\{b_i\}_{i=1}^{k}$ into $V_2$ leaving $D_{k+1}$ in $V_1$, we can change the order of $\{\alpha_i\}_{i=1}^{r}$ so that $\alpha_i (= \alpha_{k+1}$ in the old order) is a $d$-arc. Then by Fact 1.1, we can reduce the number $\#(S_1)$, a contradiction. Thus $b_{k+1}$ connects a disk in $\mathscr{D}^*$ and an annulus. This shows that, at the stage that the isotopy of type A at $\alpha_k$ has just performed, every disk in one of $\mathscr{D}_1$ and $\mathscr{D}_2$, say $\mathscr{D}_1$, has been attached to a band (see Fig. 10).

By pushing back the bands being attached to the disks in $\mathscr{D}_1$ into $V_2$ leaving the other bands in $V_1$, we can put $V_1 \cap S = \mathscr{D}^* \cup \{A_i\}_{i=1}^{r} \cup \mathscr{D}_2$. Since $A_i$ intersects $D$ in a single arc and $A_1, A_2, \ldots, A_r$ are all mutually parallel, $\{A_i\}_{i=1}^{r} \cap D$ consists of mutually parallel $r$-arcs each of which cuts off a disk $Q_i$ such that $Q_1 \subset Q_2 \subset \cdots \subset Q_r$ (see Fig. 11).

Perform the isotopies of type A from $V_1$ to $V_2$ through the disks $\{Q_i\}_{i=1}^{r}$. Then, since in Fig. 12 the disk $D_0$ is parallel to the disk $D_0'$, we see that $S$ is isotopic to a 2-sphere which intersects $V_1$ in $\mathscr{D}^* \cup \{l$ parallel disks$\}$. This completes the proof of the lemma. $\Box$

Fig. 9.

Fig. 10.
Lemma 3.4. \( l = 1 \), \( S_1 = D_1^* \cup D_2^* \cup D_1 \) and \( V_2 \cap S^{(1)} = F_1 \cup F_2 \), where \( D_1 \) is a nonseparating disk in \( V_1 \), and \( F_1 \) and \( F_2 \) are nonseparating annuli in \( V_2 \) such that \( F_i \) is a union of a nonseparating disk \( G_i \) and a band \( h_i \) \((i = 1, 2)\), \( G_1 \) and \( G_2 \) are mutually parallel and \( h_1 \) and \( h_2 \) are not mutually parallel as illustrated in Fig. 6.

Proof. By Lemma 3.3 and its proof, we can put \( V_1 \cap S^{(1)} = \varnothing \cup \{ A_i \}_{i=1}^{l} \), where \( A_1, A_2, \ldots, A_l \) are all mutually parallel nonseparating annuli as illustrated in Fig. 13.
Since $V_2 \cap S^{(l)}$ consists of $l + 1$ annuli, we can put $V_2 \cap S^{(l)} = \{F_i\}_{i=1}^{l+1}$. Then we can regard $F_i$ as a union of an essential disk in $V_2$, say $G_i$, and a band in $V_2$, say $h_i$ $(i = 1, 2, \ldots, l + 1)$. For $i = 1, 2$, we may assume that $C_i^* = \partial F_i$. Since, for $i = 1, 2$, $\partial F_i$ is identified with $C_i^* \cup (a$ component of $\partial A_i)_{1-l_1}$ and there is a loop in $\partial V_1$ which intersects $C_i^*$ in a point and intersects no component of $\partial A_i_{1-l_1}$. $F_i$ is a nonseparating annulus in $V_2$. Hence $G_i$ is a nonseparating disk in $V_2$ $(i = 1, 2)$. Moreover we may assume that $h_2$ does not run over $h_1$. If another band runs over $h_1$, then we can perform an isotopy of type A from $V_2$ to $V_1$ which produces a band in $V_1$ connecting two annuli in $\{A_i\}_{i=1}^{l+1}$. Then by the argument in the proof of Lemma 3.3, we have a $d$-arc and can reduce the number $\#(S_1)$, a contradiction. Hence we see that no band runs over $h_1$. For $G_1$ and $G_2$, we have the following three cases.

Case A: $G_1$ and $G_2$ are mutually parallel (cf. Fig. 1(III)).

Since $C_i^*$ is not parallel to any loop in $C_i^* \cup \{\partial A_i\}_{1-l_1}$, $F_1$ and $F_2$ are not mutually parallel. Hence $F_1$ and $F_2$ are in the position illustrated in Fig. 6. If $Y_i$ (indicated in Fig. 6) contains another annulus, then since the annulus is parallel to $F_1$ or $F_2$, $C_i^*$ $(i = 1 \text{ or } 2)$ is parallel to a loop in $\partial A_i_{1-l_1}$, a contradiction. Hence $Y_i \cap S^{(l)} = F_1 \cup F_2$. By the way, there are exactly two 2-spheres with four holes in the components of $\partial V_1 - \partial (V_1 \cap S^{(l)})$, one of which is bounded by $\partial (D_i^* \cup D_{i+1}^* \cup A_i)$ and the other is bounded by $\partial (D_i^* \cup D_{i+1}^* \cup A_i)$. Thus $Y_i \cap \partial V_1$ (= a 2-sphere with four holes) is identified with one of them, and in both cases we have $l = 1$ because $D_i^* \cup D_{i+1}^* \cup F_1 \cup F_2 \cup A_i$ $(i = 1 \text{ or } l)$ is a 2-sphere. Hence in Case A, we have the required conclusion.

Case B: $G_1 \cup G_2$ splits $V_2$ into two solid tori (cf. Fig. 1(I)).

Let $X_1$ and $X_2$ be the two components of $V_2 - (G_1 \cup G_2)$, and we may assume that $h_2$ is in $X_2$. Suppose $h_1$ is in $X_2$ too. If the genus three handlebody in $V_2$ cut off by $F_1 \cup F_2$ contains no component of $\{F_i\}_{i=1}^{l+1}$, then $\partial V_2 - \partial (V_1 \cap S^{(l)})$ has a component which is a torus with four holes. This is a contradiction because each component of $\partial V_1 - \partial (V_1 \cap S^{(l)})$ is a planar surface. Hence the genus three handlebody contains a component of $\{F_i\}_{i=1}^{l+1}$. Then by performing an isotopy of type A
from $V_2$ to $V_1$ leaving $h_1$ and $h_2$ in $V_2$, we have a band in $V_1$ connecting two annuli in $\{A_i\}_{i=1}^l$. Then by the argument in the proof of Lemma 3.3, we have a $d$-arc and can reduce the number $#(S_i)$, a contradiction. Hence $h_1$ is not in $X_2$ and meets $G_1$ in $X_1$ as illustrated in Fig. 14. Note that in general $h_1$ runs over $h_2$.

By noting that no band runs over $h_1$ and by the argument in the proof of Lemma 2.4, we see that $X_2$ contains no separating annuli in $\{Z\}$. Then by applying Lemma 2.1, we see that there is a nonseparating disk of $I'$ in $X_2$ which intersects any annulus in $X_2$ in an arc (cf. Fig. 10). Then, since in Fig. 15 the annulus $F_2$ is isotopic to the annulus $F_2'$, we can regard $F_2$ as a union of a nonseparating disk $G_2$ and a band such that $G_2$ is parallel to $G_1$. Hence this case is reduced to Case A.

**Case C: $G_1 \cup G_2$ does not separate $V_2$ (cf. Fig. 1(II)).**

**Claim 1. There is no separating disk in $\{G_i\}_{i=1}^{l+1}$.**

This is proved similarly to Lemma 3.1.

**Claim 2. There is no disk $G_i$ in $\{G_i\}_{i=3}^{l+1}$ such that $(G_1, G_2, G_i)$ is a complete meridian disk system of $V_2$.**

This is proved similarly to Lemma 3.2.

**Claim 3. There is no disk $G_i$ in $\{G_i\}_{i=3}^{l+1}$ such that $G_i$ is parallel to $G_1$ or $G_2$.**

**Proof.** Suppose there is such a disk, say $G_k$. If $h_1$ or $h_2$ runs over $h_k$, then, since $G_k$ is parallel to $G_1$ or $G_2$, $F_k$ is parallel to $F_1$ or $F_2$. This contradicts that $C_i^*(i = 1, 2)$ is not parallel to any component of $\partial A_i$. Hence neither $h_1$ nor $h_2$ runs over $h_k$. Then we can push $h_k$ into $V_1$ leaving $h_1$ and $h_2$ in $V_2$. Thus by the argument in the proof of Lemma 3.3, we have a $d$-arc and can reduce the number $#(S_i)$. This contradiction completes the proof of the claim.

By Claims 1, 2 and 3, $\{G_i\}_{i=3}^{l+1}$ has at most two parallel classes like $\mathcal{D}$ illustrated in Fig. 9. Then there is a loop in $V_2$ intersecting $V_2 \cap \partial S^{(l)}$ in $l$ points. On the other hand, there is a loop in $V_1$ intersecting $V_1 \cap S^{(l)}$ in $l + 1$ points. Hence we have a
Finally to complete the proof of Case II, we have to show that \( K_2 \) is prime.

Let \( S' \) be a 2-sphere which gives a nontrivial connected sum of \( K \) and is disjoint from \( S \). Put \( V_1 - S_1 = X_1 \cup X_2 \), where \( X_1 \) is a 3-ball and \( X_2 \) is a solid torus. Since 2-bridge knots are prime, we may assume that \( S' \cap X_1 = \emptyset \). Then we can put \( S' \cap X_2 = P_1^* \cup P_2^* \cup P_1 \cup \cdots \cup P_m \), where \( P_1^*, P_2^*, P_1, \ldots, P_m \) are disks such that \( P_i^* \) intersects \( K \) in a point \( (i = 1, 2) \) and \( P_j \cap K = \emptyset \) \((j = 1, 2, \ldots, m)\). In addition, we assume that \( \#(V_1 \cap S') \) has been minimized in its isotopy class rel. \( K \) among all 2-spheres which are disjoint from \( S \) and intersect \( V_1 \) in such disks.

If \( P_1^* \cup P_2^* \) splits \( V_1 \) into two solid tori, then by the argument in Section 2, we have the conclusion (1) of Theorem. If \( P_1^* \) and \( P_2^* \) are mutually parallel, then by exchanging \( S \) for \( S' \), this case is reduced to Case III. Hence we consider here only the case when \( P_1^* \cup P_2^* \) does not separate \( V_1 \). Put \( \mathcal{P}^* = P_1^* \cup P_2^* \) and \( \mathcal{S} = \{ P_i \}_{i=1}^m \).

Suppose \( m = 0 \). Then by the argument in the case of \( l = 0 \) in Case II, we have a contradiction. Hence \( m > 0 \).

By the argument in the proof of Lemma 3.1, we see that \( \mathcal{S} \) has no separating disk in \( V_1 \). By Lemma 2.3, no disk in \( \mathcal{P}^* \cup \mathcal{S} \) is a meridian disk of \( X_2 \). Hence by the argument in the proof of Lemma 3.3, we may assume that \( \mathcal{S} \) consists of exactly one parallel class which is parallel to \( D_1 \). Then by the argument in the proof of Lemma 3.4, we have \( m = 1 \).

Suppose one of \( P_1^* \) and \( P_2^* \), say \( P_1^* \), is not parallel to \( D_1^* \) or \( D_2^* \). Since \( P_1^* \) is not a meridian disk of \( X_2 \), \( \partial P_1^* \) bounds a disk in \( \partial X_2 \) containing the image of exactly one of \( D_1^* \) and \( D_2^* \) and the image of \( D_1 \). Let \( X_3 \) be the 3-ball in \( X_2 \) cut off by \( P_1^* \). Then since \( P_1^* \) is contained in \( X_3 \), by Lemma 2.3 we have a contradiction. Thus \( \mathcal{P}^* \) is parallel to \( \mathcal{S} \). Hence \( P_1^* \cup P_2^* \cup P_1 \) is parallel to \( D_1^* \cup D_2^* \cup D_1 \).

By performing an isotopy of type A from \( V_2 \) to \( V_1 \), we see that \( S \) and \( S' \) are isotopic rel. \( K \) to \( S^{(1)} \) and \( S'' \) respectively such that \( S^{(1)} \cap V_1 = D_1^* \cup D_2^* \cup A_1 \) and \( S'' \cap V_1 = P_1^* \cup P_2^* \cup B_1 \), where \( A_1 \) and \( B_1 \) are mutually parallel annuli in \( V_1 \) as illustrated in Fig. 5. Then by Lemma 3.4, we can put \( S^{(1)} \cap V_2 = F_1 \cup F_2 \) and \( S'' \cap V_2 = Q_1 \cup Q_2 \), where \( F_i \) and \( Q_i \) are nonseparating annuli in \( V_2 \) \((i = 1, 2)\). Put \( F_i = G_i \cup h_i \) and \( Q_i = R_i \cup t_i \) \((i = 1, 2)\), where \( G_i \) and \( R_i \) are nonseparating disks in \( V_2 \) and \( h_i \) and \( t_i \) are bands in \( V_2 \). Moreover by Lemma 3.4, \( G_1 \) and \( G_2 \) \((R_1 \) and \( R_2 \) respectively) are mutually parallel and \( h_1 \) and \( h_2 \) \((t_1 \) and \( t_2 \) respectively) are not mutually parallel.

Suppose \( G_1 \) and \( R_1 \) are not mutually parallel. Then, since \( t_1 \) and \( t_2 \) do not run over each other, neither \( h_1 \) nor \( h_2 \) run over \( t_1 \) or \( t_2 \). Hence by pushing \( t_1 \) into \( V_1 \) leaving the other bands in \( V_2 \), we have a band, say \( t \), in \( V_1 \) connecting \( B_1 \) and one of \( P_1^* \) and \( P_2^* \). Then by performing an isotopy of type A from \( V_1 \) to \( V_2 \), leaving \( t \) in \( V_1 \), we see that \( S'' \) is isotopic rel. \( K \) to \( S'' \) such that \( S'' \cap V_1 \) consists of two nonseparating disks each of which intersects \( K \) in a point. This contradicts the
minimality of \(#(V_1 \cap S')\). Hence \(G_1\) and \(R_1\) are mutually parallel. Then \(F_1 \cup F_2\) and \(Q_1 \cup Q_2\) are mutually parallel. This shows that \(S'\) is isotopic rel. \(K\) to \(S\), and completes the proof of Case II.

4. Case III

Suppose Case III occurs.

If \(l = 0\), then \(S_2\) is an annulus. Since \(\partial S_1\) splits \(\partial V_1\) into an annulus, say \(A\), and the other, \(S_2\) is a separating annulus in \(V_2\). Then \(S_2\) is a union of a separating disk and a band in \(V_2\). Since the disk splits \(V_2\) into a solid torus and a genus two handlebody, the band is contained in one of them. If the band is contained in the genus two handlebody, then \(\partial S_2\) splits \(\partial V_2\) into two tori with two holes, a contradiction. Hence the band is contained in the solid torus. Then by the argument in the proof of Lemma 2.4, \(S_2\) is parallel rel. \(\partial S_2\) to the annulus \(A\) in \(\partial V_2\). This shows that \(S\) gives a trivial connected sum of \(K\), a contradiction. Hence \(l > 0\).

Suppose \(l = 1\), and \(D_1\) is a separating disk in \(V_1\) which splits \(V_1\) into a solid torus containing \(K\) and a genus two handlebody. Since \(b_1\) is not contained in a 3-ball component of \(V_1 - S_1\), \(b_1\) is contained in the genus two handlebody as in Fig. 16, and let \(A_1\) be the annulus as a union of \(D_1\) and \(b_1\).

Since \(V_2 \cap S^{(1)}\) consists of two annuli, we can put \(V_2 \cap S^{(1)} = F_1 \cup F_2\). In addition, suppose \(F_1\) and \(F_2\) are mutually nonparallel nonseparating annuli as illustrated in Fig. 6. Note that in this case a component of \(\partial F_1\) is parallel in \(\partial V_2\) to a component of \(\partial F_2\).

Let \(X_1\), \(X_2\), and \(X_3\) be the three components of \(V_1 - (D_1^* \cup D_2^* \cup A_1)\) indicated in Fig. 16 and \(Y_1\) and \(Y_2\) the two components of \(V_2 - (F_1 \cup F_2)\) indicated in Fig. 6. Then \(Y_1 \cap \partial V_2\) is identified with \(X_1 \cap \partial V_1\) and \(Y_2 \cap \partial V_2\) is identified with \((X_2 \cup X_3) \cap \partial V_1\).

Put \(B_1 = X_1 \cup Y_1\) and \(B_2 = (X_2 \cup X_3) \cup Y_2\). Then, since \(B_1\) and \(B_2\) are the two components of \(S^3 - S^{(1)}\), \(B_i\) is a 3-ball (\(i = 1, 2\)). Put \(\delta_i = B_i \cap K(= X_i \cap K)\). Then by the argument in the proof of Claim in Section 2, we see that \(\delta_i\) is a trivial arc in

---

**Fig. 16.**
Tunnel number of knots

55

X, (i = 1, 2). Let B' be a 3-ball and δ' be a trivial arc properly embedded in B' (i = 1, 2). Put S^3_i = B_i ∪ B'_i and K_i = δ_i ∪ δ'_i (i = 1, 2), then K_i is a knot in the 3-sphere S^3_i (i = 1, 2). In the following, we show that K_i has a 2-bridge decomposition and K_j has tunnel number at most two.

Since there is a nonseparating disk, say A, in X, such that A ∩ A is an arc and there are two nonseparating disks, say A_i and A_2, in Y such that A ∩ F is an arc (i = 1, 2) (cf. Fig. 7), by the same argument as that in Case II, we see that K_i (= δ_i ∪ δ'_i) has a 2-bridge decomposition.

Next we denote the images of D^*_i, D^*_2, A_1, F_1 and F_2 in ∂X_1, ∂X_2 and ∂Y_1 by the same notations. Let F_3 be the annulus in ∂Y_2 bounded by a component of ∂F_1 and a component of ∂F_2. Put F = F_1 ∪ F_3 ∪ F_2, then F is an annulus in ∂Y_2. Let ε be a trivial arc properly embedded in Y_2 which is obtained by pushing an essential arc properly embedded in F into Y_2, and let N(ε) be a regular neighbourhood of ε in Y_2. Then cl(Y_2 - N(ε)) is a genus three handlebody. By the definition of ε, there is a nonseparating disk, say D, properly embedded in cl(Y_2 - N(ε)) such that D ∩ F = ∂D ∩ F is an essential arc properly embedded in F. Then since X_2 ∩ V has a 2-handle for cl(Y_2 - N(ε)). Hence X_2 ∩ cl(Y_2 - N(ε)) is a genus two handlebody, δ_2 is contained in X_2 ∩ cl(Y_2 - N(ε)) as a trivial arc and D^*_2 ∪ F_2 is a disk in X_2 ∩ cl(Y_2 - N(ε)) (i = 1, 2).

Regard B'_i as D^2 × I and δ'_i as {x_2} × I, where D^2 is a 2-disk, x_2 is a point in int(D^2) and I = [0, 1]. Choose the glueing map f of ∂B'_1 to ∂B_2 so that f(0 × I) = D^*_1 ∪ F_1, f(0 × I) = A_1 and f(1 × I) = D^*_2 ∪ F_2. Put W_1 = (X_2 ∩ cl(Y_2 - N(ε))) ∪ f(D^2 × {0}) ∪ (D^2 × {1}) and W_2 = X_3 ∩ N(ε). Then (W_1, W_2) is a genus three Heegaard splitting of the 3-sphere S^3_1 (= B_2 ∪ B'_1). Moreover by the above construction, K_2 (= δ_2 ∪ δ'_2) is a core of a handle of W_1. Hence K_2 has tunnel number at most two.

In the rest of this section, we show that l = 1, that D_1 and V_2 ∩ S^{(1)} satisfy the above conditions and that the knot K_2 is prime. Then by the uniqueness of prime decomposition of knots, we have the conclusion (2) of Theorem. In the following proof, put D^* = D^*_1 ∪ D^*_2 and D = {D_i}_{l=1}.

Lemma 4.1. We may assume that if D has separating disks, then those all are mutually parallel.

Proof. Suppose D has more than one parallel classes consisting of separating disks. Then, since V_1 is a genus three handlebody, D has exactly two such parallel classes, say D_1 and D_2. Let D_i be a disk in D (i = 1, 2). We may assume that D_1 cuts off a solid torus not containing K, say X_1. Let X_2 be the solid torus bounded by D_1 and D_2. If D has a nonseparating disk in X_2, then by Lemma 2.3, b_1 is contained in X_1. Then by the argument in the proof of Lemma 2.4, we have a contradiction. Thus D has no nonseparating disk in X_2, and we can put D = D_1 ∪ D_2 ∪ D_3, where D_3 is an empty set or consists of nonseparating disks in X_1. Thus by the argument in the proof of Lemma 3.3, we can put V_1 ∩ S^{(1)} = D^* ∪ {A_i}_{l=1} ∪
Fig. 17.

\[ \mathcal{D}_j \cup \mathcal{D}_3 \ (j = 1 \text{ or } 2) \] for some \( r \). By Lemma 2.1, we see that there is a nonseparating disk of \( V_1 \) in \( X_2 \) which intersects \( A_i \) (\( 1 \leq i \leq r \)) in a single arc. Then, by the argument in the proof of Lemma 3.3, and since in Fig. 17(1) and (2) the disk \( D_0 \) is isotopic to \( D_0' \), we have the conclusion of the lemma. \( \square \)

**Lemma 4.2.** If \( \mathcal{D} \) has a separating disk, then it cuts off a solid torus containing \( K \).

**Proof.** Suppose \( \mathcal{D} \) has a separating disk, say \( D_1 \), which cuts off a solid torus not containing \( K \). Let \( X_1, X_2 \) and \( X_3 \) be the three components of \( V_1 - (D_1^* \cup D_2^* \cup D_i) \), where \( X_1 \) is a 3-ball, \( X_2 \) is the solid torus bounded by \( D_1^* \cup D_2^* \cup D_i \) and \( X_3 \) is the solid torus cut off by \( D_1 \). If \( \mathcal{D} \) has a nonseparating disk in \( X_2 \), then by Lemma 2.3, \( b_1 \) is contained in \( X_3 \). Then by the argument in the proof of Lemma 2.4 we have a contradiction. Hence by Lemma 4.1 we can put \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \), where \( \mathcal{D}_1 \) consists of separating disks parallel to \( D_1 \) and \( \mathcal{D}_2 \) is an empty set or consists of nonseparating disks in \( X_3 \). Then we can put \( V_1 \cap S^{(r)} = \mathcal{D}^* \cup \{ A_i \}_{i=1}^r \cup \mathcal{D}_2 \) for some \( r \), where \( A_1, A_2, \ldots, A_r \) are all mutually parallel separating annuli. If \( \mathcal{D}_2 \) is an empty set, then since \( \alpha_{r+1} \) is of type 1, \( b_{r+1} \) connects two annuli or an annulus and one of \( D_1^* \) and \( D_2^* \). We note here that \( b_{r+1} \) does not meet a single annulus. If the former occurs, then we can change the order of \( \{ \alpha_i \}_{i=1}^r \) so that \( \alpha_{r+1} \) is a \( d \)-arc as in the proof of Lemma 3.3. This contradicts Fact 1.1. If the latter occurs, then since \( b_{r+1} \) does not run over the bands \( b_1, \ldots, b_r \), we can push back \( b_1, \ldots, b_r \) into \( V_2 \) leaving \( b_{r+1} \) in \( V_1 \). Then since \( b_{r+1} \) connects two disks of \( S_1 \), the number \( \#(S_1) \) is reduced, a contradiction. Hence \( \mathcal{D}_2 \) is not an empty set, and we can put
\[ V_1 \cap S^{(t)} = \emptyset \cup \{ A_i \}_{i=1}^{r} \cup \{ A_i \}_{i=r+1}, \text{ where } \{ A_i \}_{i=r+1} \text{ consists of nonseparating annuli and has at most two parallel classes.} \]

Suppose \( \{ A_i \}_{i=r} \text{ consists of two parallel classes as illustrated in Fig. } 18. \)

Let \( A_s \) and \( A_t \) be the two nonseparating annuli such that \( \partial (A_s \cup A_t) \) bounds a 2-sphere with four holes in \( \partial V_1 \) disjoint from \( \partial ((V_1 \cap S^{(t)}) - (A_s \cup A_t)) \). Since \( \alpha_{t+1} \text{ is of type } 1 \), and by the same argument as the above, \( b_{i+1} \) connects \( A_s \) and \( A_t \) and runs over the bands \( b_s \) and \( b_t \). Since \( V_2 \cap S^{(t)} \) consists of \( l+1 \) annuli, \( V_2 \cap S^{(t+1)} \) consists of \( l+1 \) annuli and a disk, say \( G \). Then \( \partial G \) is identified with the loop produced by a fusion of two components of \( \partial (A_s \cup A_t) \) via \( b_{i+1} \).

Suppose \( G \) is a nonseparating disk of \( V_2 \). Let \( M_i \) be a nonseparating disk of \( V_1 \) which is contained in \( X_i \) (\( i = 1, 2, 3 \)). Then \( \{ M_1, M_2, M_3 \} \) is a complete meridian disk system of \( V_1 \). Clearly \( \partial G \) does not intersect \( M_i \). And by Fig. 19, \( b_{i+1} \) does not contribute to calculation of the algebraic intersection number of \( \partial G \) and \( \{ M_1, M_2, M_3 \} \). Hence the algebraic intersection number of \( \partial G \) and \( \{ M_1, M_2, M_3 \} \) is equal to that of \( a_s \cup a_t \) (or \( a_s \cup (-a_t) \)) and \( M_2 \), where \( a_s \) and \( a_t \) are cores of \( A_s \) and \( A_t \) respectively. Moreover it is equal to 0 or 2 (the algebraic intersection number of \( a_s \) and \( M_2 \)) because the algebraic intersection number of \( a_s \) and \( M_2 \) (\( a_t \) and \( M_2 \)) is equal to that of \( \pm a_t \) and \( M_2 \), where \( a_t \) is a core of \( A_t \). Then, since \( \partial G \) is a loop of a Heegaard diagram of \( (V_1, V_2) \), we have the following presentation of \( H_i(V_1 \cup V_2; Z) \):

\[
H_i(V_1 \cup V_2; Z) = \begin{pmatrix} x, y, z \end{pmatrix} \begin{pmatrix} 0 & 2n & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

for some integer \( n \).

Thus \( H_i(S^3; Z) \neq 0 \), a contradiction.

Next suppose \( G \) is a separating disk of \( V_2 \).

Let \( U \) be the solid torus in \( V_2 \) cut off by \( G \). If \( U \) contains a component of \( (V_2 \cup S^{(t+1)}) - G \), then the component is a separating annulus which cuts off a solid torus in \( V_2 \) (cf. Fig. 4). Then by the argument in the proof of Lemma 2.4, we can reduce the number \#(S_i), a contradiction. Hence \( U \cap ((V_2 \cap S^{(t+1)}) - G) = \emptyset \).
Perform an isotopy of type A from $V_i$ to $V_2$ to push back $b_{l+1}$ into $V_2$. If $b_{l+1}$ is pushed back into $U$, then we can reduce the number $\#(S_1)$ by the same argument as the above. If $b_{l+1}$ is pushed back into $V_2 - U$, then the annulus $(G \cup b_{l+1})$ splits $V_2$ into two genus two handlebodies, say $W_1$ and $W_2$, where $U \subset W_1$. Let $R$ be the component of $V_1 - (A_i \cup A_r)$ indicated in Fig. 18. Then $\partial V_2 \cap W_1$ is identified with $\partial V_1 \cap R$. This is a contradiction because $\partial V_2 \cap W_1$ is a torus with two holes and $\partial V_1 \cap R$ is a 2-sphere with four holes. This contradiction shows that $\{A_{l+r+1}\}$ has exactly one parallel class.

Suppose $A_{r+1}, A_{r+2}, \ldots, A_l$ are all mutually parallel. Since $S^{(l)}$ is a separating 2-sphere in $S^3$, $l > r + 1$. Then $A_r, A_{r+1}$ and $A_l$ are in the position as illustrated in Fig. 20.

By the minimality of $\#(S_1)$, $b_{l+1}$ connects two components of $A_r, A_{r+1}$ and $A_l$. Then by the same argument as that in the case when $\{A_{l+r+1}\}$ has two parallel classes, we see that $\#(S_1)$ is reduced or $H_f(S^3; Z) = 0$, a contradiction. This completes the proof of the lemma. \(\square\)

**Lemma 4.3.** We may assume that $\mathcal{D}$ consists of at most two parallel classes, one of which consists of separating disks and the other consists of nonseparating disks.

**Proof.** Suppose $\mathcal{D}$ has more than two parallel classes. If those all are nonseparating disks, then each component of $V_1 - S_1$ is a 3-ball, a contradiction. Hence by
Lemmas 4.1 and 4.2, \( \mathcal{D} \) has exactly three parallel classes as illustrated in Fig. 21. Then by the argument in the proof of Lemma 3.3 (cf. Fig. 10), we have the conclusion of the lemma.

Suppose \( \mathcal{D} \) consists of two parallel classes. If one of them consists of separating disks and the other consists of nonseparating disks, then we have the conclusion of the lemma. If \( \mathcal{D} \) consists of separating disks, then by Lemma 4.1 we have the conclusion of the lemma. If \( \mathcal{D} \) consists of nonseparating disks, then those are the two parallel classes in the three parallel classes illustrated in Fig. 21. Then by the argument in the proof of Lemma 3.3, \( S \) is isotopic rel. \( K \) to a 2-sphere \( S' \) such that \( S' \cap V_1 = D^*_1 \cup D^*_2 \cup \mathcal{D}' \), where \( \mathcal{D}' \) consists of one parallel class. This completes the proof of the lemma. \( \square \)

**Lemma 4.4.** \( \mathcal{D} \) consists of one parallel class.

**Proof.** Suppose \( \mathcal{D} \) has more than one parallel classes. Then by Lemmas 4.2 and 4.3, we can put \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \), where \( \mathcal{D}_1 = (D_j)_{j=1}^r \) consists of nonseparating disks and \( \mathcal{D}_2 = (D_j)_{j=r+1}^l \) consists of separating disks each of which cuts off a solid torus containing \( K \). We may assume that \( D_1, D_2, \ldots, D_l \) are ordered as illustrated in Fig. 22. Note that if we remove one parallel class consisting of nonseparating disks from Fig. 21, then the resulting figure is homeomorphic to Fig. 22. And note that \( r > 1 \) because \( S \) is a separating 2-sphere in \( S^3 \).

![Fig. 21.](image-url)
By changing the letters of $D_1, D_2, \ldots, D_r$ if necessary, we may assume that $b_1$ meets $D_1$ or $D_{r+1}$.

Claim. We may assume that $b_1$ meets $D_1$.

Proof. Suppose $b_1$ meets $D_{r+1}$. Let $U$ be the solid torus in $V_1$ bounded by $D_1 \cup D_r \cup A_1$. Then by Lemma 2.1, there is a nonseparating disk of $V_1$ in $U$, say $D$, which intersects $A_1$ in a single arc. If $b_2$ is attached to a disk in $D$, then we can push back $b_1$ into $V_2$ leaving $b_2$ in $V_1$, and we can regard this situation as that $b_1$ meets $D_1$. By this observation we can put $V_1 \cap S^{(r)} = \partial^* \cup A_1 \cup \{A_i\}_{i=r+1}^{+1}$, where $A_i$ is a separating annulus produced by the isotopy of type A at $\alpha_i$ $(1 \leq i \leq l - r)$. Since $A_i$ intersects $D$ in an arc, by the deformation as demonstrated in Fig. 17(1), we see that $S$ is isotopic to $S'$ such that $V_1 \cap S' = \partial^* \cup A_1 \cup \{D_i\}_{i=r+1}^{+1}$, where $D_i$ is a separating disk in $V_1$ which cuts off a solid torus containing $\partial^*$. Then by the argument in the proof of Lemma 4.2, we have a contradiction. This completes the proof of the claim.

By Claim, we can put $V_1 \cap S^{(r)} = \partial^* \cup \{A_i\}_{i=r+1}^{+1} \cup \{A_j\}_{j=r+1}^{+1}$ as in the proof of Lemma 3.3, where $A_i$ is a nonseparating annulus. Suppose $\{A_i\}_{i=r+1}^{+1}$ consists of two parallel classes (cf. Fig. 18). Perform isotopies of type A at $\alpha_i$ $(i = r+1, r+2, \ldots)$. Let $b_k$ be the band which meets $\{A_i\}_{i=r+1}^{+1}$ such that $k = r+1$ or $k > r+1$ and $b_j$ does not meet $\{A_i\}_{i=r+1}^{+1}$ for $r+1 \leq j \leq k-1$. If $b_k$ runs over a band $b_j$ for some $j \leq r$, then $b_k$ connects two annuli in $\{A_i\}_{i=r+1}^{+1}$. Then by the argument in the proof of Lemma 4.2, we have a contradiction. If $b_k$ does not run over any band $b_j$ for $1 \leq j \leq r$, then we can push back $b_j$ $(1 \leq j \leq r)$ into $V_2$ leaving $b_k$ in $V_1$. Then $\alpha_k$ is a d-arc as in the proof of Lemma 3.3, and we have a contradiction. Hence $\{A_i\}_{i=r+1}^{+1}$ consists of one parallel class.

If $\alpha_{r+1}$ is of type I, then since $\alpha_{r+1}$ is not a d-arc, $b_{r+1}$ connects two annuli in $\{A_i\}_{i=r+1}^{+1}$. Then by the argument in the proof of Lemma 4.2, we have a contradiction. Thus $\alpha_{r+1}$ is of type III and $b_{r+1}$ meets $D_{r+1}$ as illustrated in Fig. 23. Hence by repeating these arguments, we can put $V_1 \cap S^{(r)} = \partial^* \cup \{A_i\}_{i=r+1}^{+1} \cup \{A_j\}_{j=r+1}^{+1}$. Moreover by Lemma 2.1, there is a nonseparating disk of $V_1$, say $D$, such that $A_i$
Fig. 23.

(1 \leq i \leq r) intersects \( D \) in a single arc (see Fig. 23). If \( b_j \ (r + 1 \leq j \leq l) \) does not run over \( b_r \), then we can push back \( b_j \ (1 \leq i \leq r) \) into \( V_2 \) leaving \( b_j \) in \( V_1 \). Then by Lemma 2.1 we may assume that \( b_j \ (r + 1 \leq j \leq l) \) intersects \( D \) in a single arc. Hence by the deformation illustrated in Fig. 17(1), we have the same situation as that in the proof of Lemma 4.2. Then by the argument in the proof of Lemma 4.2, we have a contradiction. Hence \( b_j \ (r + 1 \leq j \leq l) \) runs over \( b_r \) many times and intersects \( D \) many times.

Here, to complete the proof of Lemma 4.4, we prepare a technical lemma.

Lemma 4.5. Let \( V \) be a solid torus in \( S^3 \) and put \( E = \text{cl}(S^3 - V) \). Let \( C \) be an annulus in \( \partial V \) such that each component of \( \partial C \), say \( c_1 \) and \( c_2 \), is a meridian of \( V \). Put \( C' = \text{cl}(\partial V - C) \). Let \( G_1 \) and \( G_2 \) be mutually disjoint disks in \( \text{int}(C') \), and put \( P = \text{cl}(C' - (G_1 \cup G_2)) \) and \( g_i = \partial G_i \ (i = 1, 2) \). Let \( \gamma \) be an arc properly embedded in \( E \) connecting a point in \( \text{int}(G_i) \) and a point in \( \text{int}(G_2) \) (see Fig. 24).

Suppose \( A \) is an annulus properly embedded in \( E \) disjoint from \( \gamma \), and put \( \partial A = a_1 \cup a_2 \). Then if \( a_1 \) is a core of \( C \) and \( a_2 \) is in \( P \), then \( a_2 \) is parallel in \( P \) to \( c_1 \) or \( c_2 \), and if \( a_1 \) is in \( \text{int}(G_i) \) \((i = 1 \text{ or } 2)\) and bounds a disk containing \( \gamma \cap G_i \), and \( a_2 \) is in \( P \), then \( a_2 \) is parallel in \( P \) to \( g_1 \) or \( g_2 \).

Fig. 24.
Proof. If \( a_1 \) is a core of \( C \), then \( a_2 \) is also a meridian of \( V \). If \( a_2 \) separates \( g_1 \) and \( g_2 \) in \( P \), then \( A \) intersects \( \gamma \). Hence \( a_2 \) does not separate \( g_1 \) and \( g_2 \) and is parallel in \( P \) to \( c_1 \) or \( c_2 \).

Suppose \( a_1 \) is in \( \text{int}(G_i) \) \((i = 1 \text{ or } 2)\). Without loss of generality, we may assume that \( a_1 \) is in \( \text{int}(G_1) \). Let \( F_1 \) be a disk in \( G_1 \) bounded by \( a_1 \). Since \( a_1 \cap C = \emptyset \), \( a_2 \) is also an inessential loop in \( \partial V \) and bounds a disk in \( \partial V - C \), say \( F_2 \). Since \( a_2 \) is in \( P \) and \( \partial F_1 \cap \partial F_2 = \emptyset \), we have \( F_1 \cap F_2 = \emptyset \) or \( F_1 \subset \text{int}(F_2) \). If \( F_1 \cap F_2 = \emptyset \), then \( A \cup F_1 \cup F_2 \) bounds a 3-ball in \( E \) in which \( \gamma \) is properly embedded. Then \( G_2 \subset F_2 \) and \( a_2 \) is parallel in \( P \) to \( g_2 \). Suppose \( F_1 \subset \text{int}(F_2) \) and put \( A' = \text{cl}(F_2 - F_1) \). Then, since the torus \( A \cup A' \) bounds a 3-manifold in \( E \) and \( A \cap \gamma = \emptyset \), we see that \( F_2 \cap G_2 = \emptyset \) and \( a_2 \) is parallel in \( P \) to \( g_1 \). This completes the proof of the lemma.

Recall Fig. 23. Let \( U' \) be the genus two handlebody in \( V_1 \) bounded by \( A_1 \cup A_r \cup A_{r+1} \). We denote the images of \( A_1 \), \( A_r \) and \( A_{r+1} \) in \( \partial U' \) by the same notations. Let \( N(A_{r+1}) \) be a regular neighborhood of \( A_{r+1} \) in \( U' \), and put \( U = \text{cl}(U' - N(A_{r+1})) \) and \( \tilde{A}_{r+1} = \text{cl}(\partial N(A_{r+1}) - \partial U') \)(see Fig. 25). Let \( a_1 \) and \( a_2 \) be cores of \( A_{r+1} \) and \( \tilde{A}_{r+1} \) respectively, and \( A \) the annulus in \( N(A_{r+1}) \) bounded by \( a_1 \cup a_2 \).

Since \( a_1 \) \((i = 1, r)\) is of type III, there are a component of \( \partial A_1 \), say \( g_1 \), and a component of \( \partial A_r \), say \( g_2 \), such that \( g_i \) bounds a disk \( G_i \) \((i = 1, 2)\) in \( S^{r+1} \) disjoint from \( \text{int}(A_1 \cup A_r) \) and \( G_r \) intersects \( K \) in a point. Put \( c_1 = \partial A_1 - g_1 \) and \( c_2 = \partial A_r - g_2 \), and let \( C \) be the annulus in \( S^{r+1} \) bounded by \( c_1 \cup c_2 \). Let \( B \) be the 3-ball in \( S^3 \) bounded by \( S^{r+1} \) containing \( U \). Put \( V = \text{cl}(S^3 - B) \cup U \) and \( E = \text{cl}(S^3 - V) \) \((= \text{cl}(B - U)) \). Then \( V \) is a solid torus because \( U \) is a genus two handlebody containing \( A_1 \) and \( A_r \) in the boundary and there is a complete meridian disk system of \( U \), say \( \{M_i, M_r\} \), such that \( M_i \cap A_i \) is an essential arc properly embedded in \( A_i \) \((i = 1, r)\). Put \( \gamma = E \cap K \). Then, since \( U \cap K = \emptyset \), \( \gamma \) is an arc properly embedded in \( E \), and this situation is the same as that in Lemma 4.5 (see Figs. 24, 25 and 26).

Recall Fig. 23. Let \( U' \) be the genus two handlebody in \( V_1 \) bounded by \( A_1 \cup A_r \cup A_{r+1} \). We denote the images of \( A_1 \), \( A_r \) and \( A_{r+1} \) in \( \partial U' \) by the same notations. Let \( N(A_{r+1}) \) be a regular neighborhood of \( A_{r+1} \) in \( U' \), and put \( U = \text{cl}(U' - N(A_{r+1})) \) and \( \tilde{A}_{r+1} = \text{cl}(\partial N(A_{r+1}) - \partial U') \)(see Fig. 25). Let \( a_1 \) and \( a_2 \) be cores of \( A_{r+1} \) and \( \tilde{A}_{r+1} \) respectively, and \( A \) the annulus in \( N(A_{r+1}) \) bounded by \( a_1 \cup a_2 \).

Since \( a_1 \) \((i = 1, r)\) is of type III, there are a component of \( \partial A_1 \), say \( g_1 \), and a component of \( \partial A_r \), say \( g_2 \), such that \( g_i \) bounds a disk \( G_i \) \((i = 1, 2)\) in \( S^{r+1} \) disjoint from \( \text{int}(A_1 \cup A_r) \) and \( G_r \) intersects \( K \) in a point. Put \( c_1 = \partial A_1 - g_1 \) and \( c_2 = \partial A_r - g_2 \), and let \( C \) be the annulus in \( S^{r+1} \) bounded by \( c_1 \cup c_2 \). Let \( B \) be the 3-ball in \( S^3 \) bounded by \( S^{r+1} \) containing \( U \). Put \( V = \text{cl}(S^3 - B) \cup U \) and \( E = \text{cl}(S^3 - V) \) \((= \text{cl}(B - U)) \). Then \( V \) is a solid torus because \( U \) is a genus two handlebody containing \( A_1 \) and \( A_r \) in the boundary and there is a complete meridian disk system of \( U \), say \( \{M_i, M_r\} \), such that \( M_i \cap A_i \) is an essential arc properly embedded in \( A_i \) \((i = 1, r)\). Put \( \gamma = E \cap K \). Then, since \( U \cap K = \emptyset \), \( \gamma \) is an arc properly embedded in \( E \), and this situation is the same as that in Lemma 4.5 (see Figs. 24, 25 and 26).

Fig. 25.
Since $\alpha_{r+1}$ is of type III and $A_{r+1}$ is an annulus in $\partial B$, $a_1$ is a core of $C$ or is in $\text{int}(G_i)$ ($i = 1$ or 2) and bounds a disk in $G_i$ containing $\gamma \cap G_i$. Since $A$ is an annulus properly embedded in $E$ and $a_2$ is in $\text{cl}(\partial U - (A_1 \cup A_r)) = \text{cl}(\partial V - (C \cup G_1 \cup G_2))$ ($= P$ in Lemma 4.5), by Lemma 4.5, $a_2$ is parallel in $\text{cl}(\partial U - (A_1 \cup A_r))$ to one of $c_1$, $c_2$, $g_1$, and $g_2$. Moreover, since $b_{r+1}$ runs over $b_r$, $a_2$ cannot be parallel to any component of $\partial A_1 = c_1 \cup g_1$. Hence $a_2$ is parallel to a component of $\partial A = c_2 \cup g_2$, and this means that $b_{r+1}$ runs over $b_r$ exactly once and intersects $D$ in a single arc. Hence $A_j$ ($r + 1 < j < l$) intersects $D$ in a single arc.

Now $\alpha_{r+1}$ is of type I. If $b_{r+1}$ connects two of the annuli $A_1$, $A_2$, ..., $A_l$, then by the argument in the proof of Lemma 4.2, we have a contradiction. Hence by Fact 1.3, $b_{r+1}$ connects $A_j$ and one of $D_1^+$ and $D_2^+$. Then by using the disk $D$, we can push back the bands $\{b_i\}_{i=1}^l$ into $V_2$, leaving $b_{r+1}$ in $V_1$ (cf. Fig. 11). Then $b_{r+1}$ is a band in $V_1$ connecting a disk and one of $D_1^+$ and $D_2^+$. Thus the number $\#(S_j)$ is reduced. This contradiction completes the proof of Lemma 4.4.

**Lemma 4.6.** $\mathcal{D}$ consists of separating disks.

**Proof.** Suppose $\mathcal{D}$ consists of nonseparating disks. Then by the argument in the proof of Lemma 3.3, we can put $V_1 \cap S^{(i)} = \mathcal{D}^* \cup \{A_i\}_{i=1}^l$. Then by the argument in the proof of Lemma 4.2, we see that the number $\#(S_1)$ is reduced or $H_i(S^3; \mathbb{Z}) \neq 0$. This contradiction completes the proof of the lemma.

**Lemma 4.7.** $l = 1$, $S_1 = D_1^+ \cup D_1^- \cup D_1^*$ and $V_2 \cap S^{(1)} = F_1 \cup F_2$, where $D_1$ is a separating disk which cuts off a solid torus in $V_1$ containing $K$, and $F_1$ and $F_2$ are nonseparating annuli in $V_2$ such that $F_1$ is a union of a nonseparating disk $G_i$ and a band $h_i$ ($i = 1$, 2, $G_1$ and $G_2$ are mutually parallel and $h_1$ and $h_2$ are not mutually parallel as illustrated in Fig. 6.

**Proof.** By Lemma 4.6, we can put $V_1 \cap S^{(i)} = \mathcal{D}^* \cup \{A_i\}_{i=1}^l$, where $A_i$ ($1 < i < l$) is a separating annulus as illustrated in Fig. 16. Then we can put $V_2 \cap S^{(i)} = \{F_j\}_{j=1}^{l+1}$, where $F_j$ is an annulus in $V_2$. 
Claim. $F_j$ is a nonseparating annulus for any $j \ (1 \leq j \leq l+1)$.

Proof. Suppose $F_k$ is a separating annulus for some $k$. Put $F_k = G_k \cup h_k$, where $G_k$ is a separating disk in $V_2$ and $h_k$ is a band. Then $G_k$ splits $V_2$ into a solid torus and a genus two handlebody. If the solid torus contains a component of $(F_j)_{j=1}^{l+1} - F_k$, then by the argument in the proof of Lemma 2.4, we have a contradiction. And by the same reason, we see that $h_k$ is contained in the genus two handlebody. Hence $F_k$ splits $V_2$ into two genus two handlebodies (cf. Fig. 3), say $Y_1$ and $Y_2$, where $Y_1 \cap ((F_j)_{j=1}^{l+1} - F_k) = \emptyset$. Then $\partial V_2 \cap Y_1$ is a torus with two holes. Since there is exactly one component of $\partial V_2 - S(t)$ which is a torus with two holes (that is the component cut off by $\partial A_i$), $\partial F_k$ is identified with $\partial A_i$. Then $F_k \cup A_i$ is a torus. This contradiction completes the proof of the claim.

By Claim we can put $F_j = G_j \cup h_j \ (1 \leq j \leq l+1)$, where $G_j$ is a nonseparating disk in $V_2$ and $h_j$ is a band in $V_2$. Perform isotopies of type $A$ at $\alpha_i \ (i \leq i \leq 2l+1)$. Then $\partial G_j \ (1 \leq j \leq l+1)$ is identified with a loop in $\partial V_i$ produced by a fusion of two components of $\partial (V_1 \cap S(t))$ via the band $b_{l+j}$. Let $M_1$ be a nonseparating disk in $V_i$ parallel to $D_1^*$, and $M_2$, $M_3$ two nonseparating disks in $V_i$ such that $M_i \cap K = \emptyset \ (i = 1, 2)$ and $\{M_1, M_2, M_3\}$ is a complete meridian disk system of $V_i$. Then we see that $b_{l+j}$ does not contribute to calculation of $H_i(V_1 \cup V_2; Z)$ (cf. Fig. 19). Let $a$ be a core of $A_i$. Then $a \cap M_1 = \emptyset$. Let $p$ be the algebraic intersection number of $a$ and $M_2$, and $q$ the algebraic intersection number of $a$ and $M_3$. Then since $V_1 \cap S(t) = D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_l$ (where $A_1, A_2, \ldots, A_l$ are mutually parallel annuli), no band connects the two disks $D_1^*$ and $D_2^*$ (Fact 1.3) and since each band connects two of the $2l+2$ loops $\partial (D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_l)$, the algebraic intersection number of $\partial G_j$ and $\{M_1, M_2, M_3\}$ is one of the following (modulo sign): (1) $(0, p, q)$ if $b_{l+j}$ connects a disk and an annulus or (2) $(0, 0, 0)$ or $(0, 2p, 2q)$ if $b_{l+j}$ connects two annuli.

Suppose there are two disks $G_s$ and $G_t$ for some $s$ and $t$ such that $G_s \cup G_t$ does not separate $V_2$. Then $\partial G_s$ and $\partial G_t$ are two loops of a Heegaard diagram of $(V_1, V_2)$. Hence we have one of the following presentations of $H_i(V_1 \cup V_2; Z)$:

$$H_i(V_1 \cup V_2; Z) = \left\{ \begin{array}{c}
\left\langle x, y, z \left| \begin{array}{ccc}
0 & 0 & 0 \\
* & * & * \\
* & * & *
\end{array} \right| \begin{array}{c}
x \\
y \\
z
\end{array} = \begin{array}{c}
0 \\
0 \\
0
\end{array} \right\rangle, \\
\left\langle x, y, z \left| \begin{array}{ccc}
0 & p & q \\
0 & p & q \\
* & * & *
\end{array} \right| \begin{array}{c}
x \\
y \\
z
\end{array} = \begin{array}{c}
0 \\
0 \\
0
\end{array} \right\rangle, \\
\left\langle x, y, z \left| \begin{array}{ccc}
0 & p & q \\
0 & 2p & 2q \\
* & * & *
\end{array} \right| \begin{array}{c}
x \\
y \\
z
\end{array} = \begin{array}{c}
0 \\
0 \\
0
\end{array} \right\rangle \text{ or} \\
\left\langle x, y, z \left| \begin{array}{ccc}
0 & 2p & 2q \\
0 & 2p & 2q \\
* & * & *
\end{array} \right| \begin{array}{c}
x \\
y \\
z
\end{array} = \begin{array}{c}
0 \\
0 \\
0
\end{array} \right\rangle. \end{array} \right\}$$
This shows that \( H_1(S^3; \mathbb{Z}) \neq 0 \), a contradiction. Hence \( G_1, G_2, \ldots, G_{l+1} \) are all mutually parallel or \( (G_j)_{j+1} \) consists of two parallel classes such that the two disks which are not mutually parallel split \( V_2 \) into two solid tori.

Suppose \( (G_j)_{j+1} \) consists of two parallel classes.

**Claim.** If \( G_s \) and \( G_t \) are mutually parallel for some \( s \) and \( t \), then \( F_s \) and \( F_t \) are mutually parallel.

**Proof.** We may assume that the annulus in \( \partial V_2 \) bounded by \( \partial (G_s \cup G_t) \) contains no other components of \( \partial (G_1 \cup G_2 \cup \cdots \cup G_{l+1}) - \partial (G_s \cup G_t) \). If \( F_s \) and \( F_t \) are not mutually parallel, then \( \partial (F_s \cup F_t) \) bounds a 2-sphere with four holes in \( \partial V_2 \). Since there is exactly one 2-sphere with four holes in the components of \( \partial V_1 - \partial (D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_l) \), which is bounded by \( \partial (D_1^* \cup D_2^* \cup A_1) \), \( \partial (F_s \cup F_t) \) is identified with \( \partial (D_1^* \cup D_2^* \cup A_1) \). Then \( D_1^* \cup D_2^* \cup A_1 \cup F_s \cup F_t \) is a 2-sphere. Hence \( (G_j)_{j+1} = \{G_s, G_t\} \) and this shows that \( (G_j)_{j+1} \) consists of one parallel class. This is a contradiction and completes the proof of the claim.

By Claim, we see that \( (F_j)_{j+1} \) consists of two parallel classes. Then by the argument in the proof of Case B of Lemma 3.4, we see that this case is reduced to the case when \( G_1, G_2, \ldots, G_{l+1} \) are all mutually parallel. Hence we may assume that \( (G_j)_{j+1} \) consists of one parallel class.

If \( h_1, h_2, \ldots, h_{l+1} \) are all mutually parallel, then \( \partial (F_1 \cup F_2 \cup \cdots \cup F_{l+1}) \) consists of two parallel classes. This contradicts that \( \partial (D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_l) \) consists of three parallel classes. Thus we can put \( (F_j)_{j+1} = (F_j)_{k+1} \cup (F_j)_{j-k+1} \), where these are the two parallel classes. Then we may assume that \( \partial (F_k \cup F_{k+1}) \) bounds a 2-sphere with four holes in \( \partial V_2 \) which contains no other components of \( \partial (F_1 \cup F_2 \cup \cdots \cup F_{l+1}) - \partial (F_k \cup F_{k+1}) \). Since there is exactly one 2-sphere with four holes in the components of \( \partial V_1 - \partial (D_1^* \cup D_2^* \cup A_1 \cup \cdots \cup A_l) \), which is bounded by \( \partial (D_1^* \cup D_2^* \cup A_1) \), \( \partial (F_k \cup F_{k+1}) \) is identified with \( \partial (D_1^* \cup D_2^* \cup A_1) \). Then \( D_1^* \cup D_2^* \cup A_1 \cup F_k \cup F_{k+1} \) is a 2-sphere. This shows that \( l = 1 \) and \( k = 1 \). Then by the above argument, we see that \( F_1 \) and \( F_2 \) satisfy the required conditions. This completes the proof of the lemma. \( \square \)

Now we have to show that \( K_2 \) is prime. However, by noting the existence of the separating disk \( D_1 \), we see that it can be proved by the same argument as that in this section. This completes the proof of Case III and Theorem.

**References**

[5] K. Morimoto, There are knots whose tunnel numbers go down under connected sum, Preprint.