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Hadamard product of certain meromorphic starlike and convex functions

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ABSTRACT

In this paper, the authors establish certain results concerning the Hadamard product for two classes related to starlike and convex univalent meromorphic functions of order α and type β with positive coefficients.

1. Introduction

Throughout this paper, let the functions of the form :

$$\varphi(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \quad (c_1 > 0, c_n \geq 0), \tag{1.1}$$

and

$$\psi(z) = d_1 z - \sum_{n=2}^{\infty} d_n z^n \quad (d_1 > 0, d_n \geq 0) \tag{1.2}$$

be regular and univalent in the unit disc $U = \{z : |z| < 1\}$; and let

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_0 > 0, a_n \geq 0), \tag{1.3}$$

$$f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z \quad (a_{0,i} > 0, a_{n,i} \geq 0), \tag{1.4}$$

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \quad (b_0 > 0, b_n \geq 0), \tag{1.5}$$

and

$$g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^n \quad (b_{0,j} > 0, b_{n,j} \geq 0), \tag{1.6}$$

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be regular and univalent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$.

For a function $f(z)$ defined by (1.3) (with $a_0 = 1$) we define

$$I^0 f(z) = f(z),$$
$$I^1 f(z) = zf'(z) + \frac{2}{z},$$
$$I^2 f(z) = z(I^1 f(z))' + \frac{2}{z}$$

and for $k = 1, 2, 3, \ldots\ldots$

$$I^k f(z) = z\,(I^{k-1} f(z))' + \frac{2}{z}$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} n^k a_n z^n.$$

The operator I^k was introduced by Frasin and Darus [1].

With the help of the differential operator I^k, we define the classes $\sum S_0^*(k, \alpha, \beta)$ and $\sum C_0(k, \alpha, \beta)$ as follows :

Denote by $\sum S_0^*(k, \alpha, \beta)$, the class of functions $f(z)$ which satisfy the condition

$$\left| \frac{\frac{z(I^k f(z))'}{I^k f(z)} + 1}{\frac{z(I^k f(z))'}{I^k f(z)} + 2\alpha - 1} \right| < \beta \tag{1.7}$$

$(z \in U^*, 0 \le \alpha < 1, 0 < \beta \le 1, k \in N_0)$.

Let $\sum C_0^*(k, \alpha, \beta)$ be the class of functions $f(z)$ for which $-zf'(z) \in \sum S_0^*(k, \alpha, \beta)$.

We note that :

(i) $\sum S_0^*(0, \alpha, \beta) = \sum S_0^*(\alpha, \beta)$, is the class of of meromorphic starlike functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$with $a_0 = 1$; studied by Mogra et al. [2];

(ii) $\sum C_0^*(0, \alpha, \beta) = \sum C_0^*(\alpha, \beta)$, is the class of meromorphic convex functions of order $\alpha (0 \le \alpha < 1)$ and typeβ $(0 < \beta \le 1)$ with positive cofficients;

(iii) $\sum S_0^*(k, \alpha, 1) = \sum^*(k, \alpha)$ (Frasin and Darus [1]).

Using similar arguments as given in [1], we can easily prove the following results for functions in the classes $\sum S_0^*(k, \alpha, \beta)$ and $\sum C_0^*(k, \alpha, \beta)$.

A function$f(z) \in \sum S_0^*(k, \alpha, \beta)$ if, and only if,

$$\sum_{n=1}^{\infty} n^k [(1+\beta)n + (2\alpha - 1)\beta + 1] a_n \le 2\beta(1-\alpha)a_0; \tag{1.8}$$

and $f(z) \in \sum C_0^*(k, \alpha, \beta)$ if, and only if,

$$\sum_{n=1}^{\infty} n^{k+1} [(1+\beta)n + (2\alpha - 1)\beta + 1] a_n \le 2\beta(1-\alpha)a_0. \tag{1.9}$$

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa [3–5], Kumar [6–8], Mogra [9,10], Aouf and Darwish [11,12], Hossen [13] and Sekine [14]. Accordingly, the quasi-Hadamard product of two functions $\varphi(z)$ and $\psi(z)$ given by (1.1) and (1.2) is defined by

$$\varphi * \psi(z) = c_1 d_1 z - \sum_{k=2}^{\infty} c_n d_n z^n.$$

Let us define the Hadamard product of two meromorphic univalent functions $f(z)$ and $g(z)$ by

$$f * g(z) = \frac{a_0 b_0}{z} + \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.10}$$

The Hadamard product of more than two meromorphic functions can similarly be defined.

In [10], Mogra obtained certain results concerning the quasi-Hadamard product of two or more functions in $\sum S_0^*(0, \alpha, \beta) = \sum S_0^*(\alpha, \beta)$ and $\sum C_0^*(0, \alpha, \beta) = \sum C_0^*(\alpha, \beta)$.

In this paper, we introduce the following class of meromorphic univalent functions in U^*.

A function $f(z) \in \sum_h^*(\alpha, \beta)$ if, and only if,

$$\sum_{n=1}^{\infty} \left\{ n^h [(1+\beta)n + (2\alpha - 1)\beta + 1] a_n \right\} \le 2\beta(1-\alpha)a_0 \tag{1.11}$$

where $0 \le \alpha < 1$, $0 < \beta \le 1$ and h is any fixed nonnegative real number.

Evidently, $\sum_k^*(\alpha, \beta) \equiv \sum S_0^*(k, \alpha, \beta)$ and $\sum_{k+1}^*(\alpha, \beta) = \sum C_0^*(k, \alpha, \beta)$.

Further, $\sum_h^*(\alpha, \beta) \subset \sum_\varphi^*(\alpha, \beta)$ if $h > \varphi \ge 0$, the containment being proper. Moreover, for any positive integer $h > k+1$, we have the following inclusion relation

$$\sum_h^*(\alpha, \beta) \subset \sum_{h-1}^*(\alpha, \beta) \subset \cdots \subset \sum_{k+2}^*(\alpha, \beta) \subset \sum^* C_0(k, \alpha, \beta) \subset \sum^* S_0^*(k, \alpha, \beta).$$

We also note that, for every nonnegative real number h, the class $\sum_h^*(\alpha, \beta)$ is nonempty as the functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} n^{-h} \left\{ \frac{2\beta(1-\alpha)}{(1+\beta)n + (2\alpha - 1)\beta + 1} \right\} a_0 \lambda_n z^n \tag{1.12}$$

where $a_0 > 0$, $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \lambda_n \le 1$, satisfy the inequality (1.11).

The objective of this paper is to establish certain results concerning the Hadamard product of meromorphic univalent functions in U^*.

2. The main theorems

Theorem 1. *Let the functions $f_i(z)$ belong to the class $\sum C_0^*(k, \alpha, \beta)$ for every $i = 1, 2, \ldots, m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\sum_{m(k+2)-1}^*(\alpha, \beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty} \left\{ n^{m(k+2)-1} [(1+\beta)n + (2\alpha - 1)\beta + 1] \prod_{i=1}^{m} a_{n,i} \right\} \le 2\beta(1-\alpha) \left[\prod_{i=1}^{m} a_{0,i} \right]. \tag{2.1}$$

Since $f_i(z) \in \sum C_0^*(k, \alpha, \beta)$, we have

$$\sum_{n=1}^{\infty} n^{k+1} [(1+\beta)n + (2\alpha - 1)\beta + 1] a_{n,i} \le 2\beta(1-\alpha) a_{0,i}. \tag{2.2}$$

for every $i = 1, 2, \ldots, m$. Therefore,

$$n^{k+1} [(1+\beta)n + (2\alpha - 1)\beta + 1] a_{n,i} \le 2\beta(1-\alpha) a_{0,i}.$$

or

$$a_{n,i} \le \left\{ \frac{2\beta(1-\alpha)}{n^{k+1}[(1+\beta)n + (2\alpha - 1)\beta + 1]} \right\} a_{0,i},$$

for every $i = 1, 2, \ldots, m$. The right-hand expression of the last inequality is not greater than $n^{-(k+2)} a_{0,i}$. Hence

$$a_{n,i} \le n^{-(k+2)} a_{0,i} \tag{2.3}$$

for every $i = 1, 2, \ldots, m$.

Using (2.3) for $i = 1, 2, \ldots, m-1$, and (2.2) for $i = m$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\{ n^{m(k+2)-1} [(1+\beta)n + (2\alpha - 1)\beta + 1] \prod_{i=1}^{m} a_{n,i} \right\} \\
&\quad \le \sum_{n=1}^{\infty} \left\{ n^{m(k+2)-1} [(1+\beta)n + (2\alpha - 1)\beta + 1] \left(n^{-(k+2)(m-1)} \prod_{i=1}^{m-1} a_{0,i} \right) a_{n,m} \right\} \\
&\quad = \left[\prod_{i=1}^{m-1} a_{0,i} \right] \sum_{n=1}^{\infty} \left\{ n^{k+1} [(1+\beta)n + (2\alpha - 1)\beta + 1] a_{n,m} \right\} \\
&\quad \le 2\beta(1-\alpha) \left[\prod_{i=1}^{m} a_{0,i} \right].
\end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \sum_{m(k+2)-1}^*(\alpha, \beta)$. □

Theorem 2. *Let the functions $f_i(z)$ belong to the class $\sum S_0^*(k, \alpha, \beta)$ for every $i = 1, 2, \ldots, m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\sum_{m(k+1)-1}^*(\alpha, \beta)$.*

Proof. Since $f_i(z) \in \sum S_0^*(k, \alpha, \beta)$, we have

$$\sum_{n=1}^{\infty} \left\{ n^k [(1+\beta)n + (2\alpha - 1)\beta + 1] a_{n,i} \right\} \leq 2\beta(1-\alpha) a_{0,i} \tag{2.4}$$

for every $i = 1, 2, \ldots, m$. Therefore

$$a_{n,i} \leq \left\{ \frac{2\beta(1-\alpha)}{n^k[(1+\beta)n + (2\alpha-1)\beta + 1]} \right\} a_{0,i},$$

and hence

$$a_{n,i} \leq n^{-(k+1)} a_{0,i} \tag{2.5}$$

for every $i = 1, 2, \ldots, m$.

Using (2.5) for $i = 1, 2, \ldots, m-1$, and (2.4) for $i = m$, we get

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ n^{m(k+1)-1} [(1+\beta)n + (2\alpha-1)\beta + 1] \prod_{i=1}^{m} a_{n,i} \right\} \\ &\quad \leq \sum_{n=1}^{\infty} \left\{ n^{m(k+1)-1} [(1+\beta)n + (2\alpha-1)\beta + 1] \left[n^{-(k+1)(m-1)} \prod_{i=1}^{m-1} a_{0,i} \right] a_{n,m} \right\} \\ &\quad = \left[\prod_{i=1}^{m-1} a_{0,i} \right] \sum_{n=1}^{\infty} \left\{ n^k [(1+\beta)n + (2\alpha-1)\beta + 1] a_{n,m} \right\} \\ &\quad \leq 2\beta(1-\alpha) \left[\prod_{i=1}^{m} a_{0,i} \right]. \end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \sum_{m(k+1)-1}^*(\alpha, \beta)$. □

Theorem 3. *Let the functions $f_i(z)$ belong to the class $\sum C_0^*(k, \alpha, \beta)$ for every $i = 1, 2, \ldots, m$; and let the functions $g_j(z)$ belong to the class $\sum S_0^*(k, \alpha, \beta)$ for every $j = 1, 2, \ldots, q$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $\sum_{m(k+2)+q(k+1)-1}^*(\alpha, \beta)$.*

Proof. We denote the Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ by the function $h(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left[\prod_{i=1}^{m} a_{0,i} \cdot \prod_{j=1}^{q} b_{0,j} \right] z^{-1} + \sum_{n=1}^{\infty} \left[\prod_{i=1}^{m} a_{n,i} \cdot \prod_{j=1}^{q} b_{n,j} \right] z^n.$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty} \left\{ n^{m(k+2)+q(k+1)-1} [(1+\beta)n + (2\alpha-1)\beta + 1] \left[\prod_{i=1}^{m} a_{n,i} \cdot \prod_{j=1}^{q} b_{n,j} \right] \right\} \leq 2\beta(1-\alpha) \left[\prod_{i=1}^{m} a_{0,i} \cdot \prod_{j=1}^{q} b_{0,j} \right].$$

Since $f_i(z) \in \sum C_0^*(k, \alpha, \beta)$, the inequalities (2.2) and (2.3) hold for every $i = 1, 2, \ldots, m$. Further, since $g_j(z) \in \sum S_0^*(k, \alpha, \beta)$, we have

$$\sum_{n=1}^{\infty} \left\{ n^k [(1+\beta)n + (2\alpha-1)\beta + 1] b_{n,j} \right\} \leq 2\beta(1-\alpha) b_{0,j}, \tag{2.6}$$

for every $j = 1, 2, \ldots, q$. Whence we obtain

$$b_{n,j} \leq n^{-(k+1)} b_{0,j} \tag{2.7}$$

for every $j = 1, 2, \ldots, q$.

Using (2.3) for $i = 1, 2, \ldots, m$; (2.7) for $j = 1, 2, \ldots q-1$; and (2.6) for $j = q$, we get

$$\sum_{n=1}^{\infty}\left\{n^{m(k+2)+q(k+1)-1}[(1+\beta)n+(2\alpha-1)\beta+1]\left[\prod_{i=1}^{m}a_{n,i}.\prod_{j=1}^{q}b_{n,j}\right]\right\}$$
$$\le \sum_{n=1}^{\infty}\left\{n^{m(k+2)+q(k+1)-1}[(1+\beta)n+(2\alpha-1)\beta+1]\left[n^{-m(k+2)}\prod_{i=1}^{m}a_{0,i}.\prod_{j=1}^{q}b_{n,j}\right]\right\}$$
$$\le \sum_{n=1}^{\infty}\left\{n^{m(k+2)+q(k+1)-1}[(1+\beta)n+(2\alpha-1)\beta+1]\right\}\left[n^{-m(k+2)}.n^{-(k+1)(q-1)}.\prod_{i=1}^{m}a_{0,i}.\prod_{j=1}^{q-1}b_{0,j}\right]b_{n,q}$$
$$= \left[\prod_{i=1}^{m}a_{0,i}.\prod_{j=1}^{q-1}b_{0,j}\right]\sum_{n=1}^{\infty}\left\{n^{k}[(1+\beta)n+(2\alpha-1)\beta+1]b_{n,q}\right\}$$
$$\le 2\beta(1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}.\prod_{j=1}^{q}b_{0,j}\right].$$

Hence $h(z) \in \sum^{*}_{m(k+2)+q(k+1)-1}(\alpha,\beta)$.

We note that the required estimate can also be obtained by using (2.3) for $i = 1, 2, \ldots, m-1$; (2.7) for $j = 1, 2, \ldots, q$ and (2.2) for $i = m$. □

Remark 1. Putting $k = 0$ in the above results we obtain the results obtained by Mogra [10].

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