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Existence and stability of periodic solution for BAM neural networks with discontinuous neuron activations

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ABSTRACT

In this paper, by using the fixed point theorem of differential inclusion theory and constructing suitable Lyapunov functions, the existence, uniqueness and global exponential stability of the periodic solution for BAM neural networks with discontinuous neuron activations are investigated. Moreover, the global convergence in finite time of the networks is discussed. The conditions that ensure the existence and stability of periodic solution are given. The obtained results extend previous work on global stability of BAM neural networks with Lipschitz continuous neuron activations but without delays, and show that Forti's conjecture is true for BAM neural networks without delays.

1. Introduction

During the last two decades, various neural network models such as Hopfield neural networks, Cellular neural networks and Bi-directional associative memory (BAM) neural networks were extensively investigated and successfully applied to signal processing, pattern recognition, associative memory and optimization problems [1–4]. In the application of neural networks either as associative memories or as optimization solvers, the stability of networks is a prerequisite. Particularly, when neural networks are employed as associative memories, the equilibrium points represent the stored patterns, and, the stability of each equilibrium point means that each stored pattern can be retrieved even in the presence of noise. When employed as an optimization solver, the equilibrium points of neural networks correspond to possible optimal solutions, and the stability of networks then ensures the convergence to optimal solutions. Also, stability of neural networks is fundamental to network designs. Due to these, stability analysis of neural networks has received extensive attention from a lot of scholars so far [5–14]. It is well known that studies on neural networks not only involve discussions of stability property of equilibrium point, but also involve investigations of other dynamics behaviors such as periodic oscillation, bifurcation and chaos. In many applications, knowing the property of periodic oscillatory solutions is very interesting and valuable. For example, the human brain is often in periodic oscillatory or chaos state, hence it is of prime importance to study periodic oscillatory and chaos phenomenon of neural networks for understanding the function of the human brain. In the existing literature, almost all results on the stability of periodic solutions of neural networks with or without time delays are conducted under some special assumptions on neuron activation functions. These assumptions frequently include those such as Lipschitz conditions, bounded and/or monotonic increasing property (see, for instance, [6–8] and the references therein). Recently, in Refs. [10–14], the authors discussed global stability for Hopfield neural networks with discontinuous neuron activations. Particularly, in [14], Forti conjectures that all solutions of Hopfield neural networks with discontinuous

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neuron activations converge to an asymptotically stable limit cycle (periodic solution) whenever the neuron inputs are periodic functions. As far as we know, there are few papers dealing with this conjecture for BAM neural networks. The purpose of this paper is to study the existence, uniqueness and global exponential stability of periodic solution of BAM neural networks with discontinuous neuron activations by using the fixed point theorem of differential inclusion theory and some new analysis techniques, and by constructing suitable Lyapunov functions. The conclusions obtained in this paper can be thought of as a generalization of the previous results established for BAM neural networks possessing smooth neuron activations [6–8]. We have proved that Forti's conjecture in [14] is true for BAM neural networks without delays.

For later discussion, we introduce the following notations.

Let $x = (x_1, \ldots, x_n)'$, $y = (y_1, \ldots, y_n)'$, $x, y \in R^n$, where the prime means the transpose. By $x > 0$ (respectively, $x \geq 0$) we mean that $x_i > 0$ (respectively, $x_i \geq 0$) for all $i = 1, \ldots, n$. $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ denotes the Euclidean norm of x. $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, $\langle \cdot, \cdot \rangle$ denotes the inner product. By the Cauchy inequality, it easily follows

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|.$$

Given a set $Q \subset R^n$, by $K[Q]$ we denote the closure of the convex hull of Q. If $\hat{x} \in R^n$ and $r > 0$, $B(\hat{x}, r) = \{x \in R^n : \|x - \hat{x}\| < r\}$ denotes the ball with radius r and center $\hat{x}$. $\mu(Q)$ denotes the Lebesgue measure in R^n of Q. Let X be a Banach space. $\|x\|_X$ denotes the norm of x, $\forall x \in X$. By $L^1([0, \omega], R^n)$, $\omega \leq +\infty$, we denote the Banach spaces of the Lebesgue integrable functions $x(\cdot)$: $[0, \omega] \to R^n$ equipped with the norm $\int_0^{\omega} \|x(t)\| \mathrm{d}t$. Let $V : R^n \to R$ be a locally Lipschitz continuous function. Clarke's generalized gradient [15] of V at x is defined by

$$\partial V(x) = K[\lim \nabla V(x_i) : x_i \to x,\ x_i \in R^n \setminus \Omega_V \cup \mathcal{N}],$$

where $\Omega_V \subset R^n$ is the set of Lebesgue measure zero where ∇V does not exist, and $\mathcal{N} \subset R^n$ is an arbitrary set with measure zero. For example, if $V : R \to R$ is given by $V(x) = |x|$, then we have

$$\partial V(x) = K[\mathrm{sign}(x)] = \begin{cases} 1, & x > 0, \\ [-1, 1], & x = 0, \\ -1, & x < 0. \end{cases}$$

The rest of this paper is organized as follows. In Section 2, a new BAM neural network model considered in this paper is developed, and some preliminaries also are given. In Section 3, the proof on the existence of periodic solution for BAM neural network is presented. Section 4 discusses global exponential stability and convergence in finite time for the neural networks. The conditions that ensure the stability of periodic solution are given. Illustrative examples are provided to show the effectiveness of the obtained results in Section 5. Some conclusions and hints are drawn in Section 6.

2. Preliminaries

The model we consider in the present paper is the BAM neural networks modelled by the differential equation

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{m} p_{ji} f_j(y_j(t)) + c_i(t), & i = 1, 2, \ldots, n, \\ \dot{y}_j(t) = -b_j y_j(t) + \sum_{i=1}^{n} q_{ij} g_i(x_i(t)) + d_j(t), & j = 1, 2, \ldots, m, \end{cases} \tag{1}$$

where $a_i > 0$, $b_j > 0$, they denote the neural self-inhibitions; $x_i(t), y_j(t)$ are the states of the ith neurons and the jth neurons, respectively; p_{ji}, q_{ij} are the connection weights; f_j and g_i represent the neuron input–output activations; and $c_i(t), d_j(t)$ are continuous ω-periodic functions which denote the external inputs at time t.

For the neuron activations f_j and g_i, we assume that

H_1: (1) f_j and g_i, $i = 1, \ldots, n$, $j = 1, \ldots, m$ are piecewise continuous, i.e., f_j and g_i are continuous in R except a countable set of jump discontinuous points, and in every compact set of R, have only a finite number of jump discontinuous points.

(2) f_j and g_i are nondecreasing and bounded.

Set $z(t) = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))'$, $I(t) = (c_1(t), \ldots, c_n(t), d_1(t), \ldots, d_m(t))'$, $h(z(t)) = (g_1(x_1(t)), \ldots, g_n(x_n(t)), f_1(y_1(t)), \ldots, f_m(y_m(t)))'$,

$P = (p_{ij})_{m\times n}$, $Q = (q_{ji})_{n\times m}$, $B = \begin{pmatrix} & P' \\ Q' & \end{pmatrix}$, and $D = \mathrm{diag}(a_1, \ldots, a_n, b_1, \ldots, b_m)$.

Eq. (1) can be equivalently represented by

$$\dot{z}(t) = -Dz(t) + Bh(z(t)) + I(t). \tag{2}$$

Under the assumption H_1, $h(z)$ is undefined at the points where $h(z)$ is discontinuous. It is obvious that

$$K[h(z)] = (K[g_1(x_1)], \ldots, K[g_n(x_n)], K[f_1(y_1)], \ldots, K[f_m(y_m)])',$$

where $K[g_i(x_i)] = [g_i(x_i^-), g_i(x_i^+)]$, $K[f_j(y_j)] = [f_j(y_j^-), f_j(y_j^+)]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Eq. (2) is a differential equation with a discontinuous right-hand side. For differential Eq. (2), we adopt the following definition of the solution in the sense of Filippov [16] in this paper.

Consider the differential equation with discontinuous right-hand side

$$\frac{dx}{dt} = f(x, t) \tag{3}$$

where $x \in R^n, f : R^n \times R^+ \to R^n$ is a discontinuous function on x.

Definition 2.1. Define the set-valued map ϕ by

$$\phi(x, t) = \bigcap_{r>0} \bigcap_{\mu(\mathcal{N})=0} K[f(B(x, r) \setminus \mathcal{N}, t)],$$

where $\mathcal{N}$ is an arbitrary set with measure zero. A solution $x(t)$ of Eq. (3) on an interval $[0, T), T \in (0, +\infty]$ with the initial conditions $x(0) = x_0$ is an absolutely continuous function defined on $[0, T)$, such that $x(0) = x_0$, and which satisfies the differential inclusion

$$\begin{cases} \dot{x}(t) \in \phi(x, t), & \text{for a.e. } t \in [0, T), \\ x(0) = x_0. \end{cases}$$

By Definition 2.1, we can get the definition of the Filippov solution of Eq. (2) as follows:

Definition 2.2. Under the assumption H_1, a solution of Eq. (2) on an interval $[0, T)$ with the initial condition $z(0) = z_0$ is an absolutely continuous function satisfying

$$\begin{cases} \dot{z}(t) \in -Dz(t) + BK[h(z(t))] + I(t), & \text{for a.e. } t \in [0, T) \\ z(0) = z_0. \end{cases}$$

It is easy to see that $\phi(z, t)$: $(z, t) \hookrightarrow -Dz + BK[h(z)] + I(t)$ is an upper semicontinuous set-valued map with nonempty compact convex values, hence it is measurable [17]. By the measurable selection theorem [18], we can get that if $z(t)$ is a solution of Eq. (2), then there exists a bounded measurable function $\eta(t) \in K[h(z(t))]$ such that

$$\dot{z}(t) = -Dz(t) + B\eta(t) + I(t), \quad \text{for a.e. } t \in [0, T). \tag{4}$$

Consider the following differential inclusion

$$\begin{cases} \dot{z}(t) \in -Dz(t) + BK[h(z(t))] + I(t), & \text{for a.e. } t \in [0, \omega], \\ z(0) = z(\omega). \end{cases} \tag{5}$$

It easily follows that if $z(t)$ is a solution of (5), then $z^*(t)$ defined by

$$z^*(t) = z(t - j\omega), \quad t \in [j\omega, (j + 1)\omega], \ j \in N$$

is an ω-periodic solution of Eq. (2). Hence, for the neural network (1), finding the periodic solutions is equivalent to finding solutions of (5).

Definition 2.3. The periodic solution $z^*(t) = (x_1^*(t), \ldots, x_n^*(t), y_1^*(t), \ldots, y_m^*(t))'$ of the neural network (2) is said to be globally exponentially stable, if, for any solution $z(t)$ of Eq. (2), there exist constants $\alpha > 0$ and $\lambda > 0$ such that

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| + \sum_{j=1}^{m} |y_j(t) - y_j^*(t)| \le \lambda e^{-\alpha t}.$$

Definition 2.4. An $n \times n$ matrix A is said to be an M-matrix, if: (1) $a_{ii} > 0, \ i = 1, \ldots, n$; (2) $a_{ij} \le 0, \ i \ne j, i, j = 1, \ldots, n$; (3) all successive principal minors of A are positive.

If A is an M-matrix, then there exists a positive vector $\beta \in R^n$ such that $\beta' A > 0$ [19].

Definition 2.5. An $n \times n$ matrix $C = (c_{ij})$ is said to be an H-matrix, if its comparison matrix $\widetilde{C} = (\widetilde{c}_{ij})$ defined by, for $i, j = 1, \ldots, n$

$$\widetilde{c}_{ij} = \begin{cases} |c_{ii}|, & i = j \\ -|c_{ij}|, & i \ne j \end{cases}$$

is an M-matrix.

Lemma 2.1 ([20]). *Let X, Y be two Banach spaces, $T : X \to Y$ be a bounded linear operator. If T is one to one and surjective, then the inverse operator of T, $T^{-1} : Y \to X$ is a bounded linear operator, i.e., there exists a constant $M > 0$, such that*

$$\|T^{-1}y\|_X \le M\|y\|_Y, \quad \forall\, y \in Y.$$

Lemma 2.2 ([21]). *Let X be a Banach space, and $P_{kc} = \{C \subset X,\ \text{nonempty, compact and convex}\}$. If $G : X \to P_{kc}$ is an upper semicontinuous set-valued map which maps bounded sets into relatively compact sets, then one of the following statements is true:*

(a) *the set $\Gamma = \{x \in X : x \in \lambda G(x),\ \lambda \in (0, 1)\}$ is unbounded.*
(b) *the $G(\cdot)$ has a fixed point, i.e., there exists $x \in X$, such that $x \in G(x)$.*

Define

$$W^{1,1}([0,\omega], R^{m+n}) = \{z(t) : z(t) \text{ is absolute continuous},\ t \in [0,\omega]\},$$
$$W_p^{1,1}([0,\omega], R^{m+n}) = \left\{z(t) \in W^{1,1}([0,\omega], R^{m+n}) \mid z(0) = z(\omega)\right\},$$
$$\|z\|_{W^{1,1}} = \int_0^\omega \|z(t)\|\mathrm{d}t + \int_0^\omega \|\dot{z}(t)\|\mathrm{d}t, \quad \forall z(t) \in W^{1,1}([0,\omega], R^{m+n}),$$

then $\|\cdot\|_{W^{1,1}}$ is a class of norms of $W^{1,1}([0,\omega], R^{m+n})$, $W^{1,1}([0,\omega], R^{m+n})$ and $W_p^{1,1}([0,\omega], R^{m+n}) \subset W^{1,1}([0,\omega], R^{m+n})$ are Banach spaces under the norm $\|\cdot\|_{W^{1,1}}$.

Lemma 2.3 ([22]). *If $S \subset W^{1,1}([0,\omega], R^{m+n})$ is a bounded set, then S is a relatively compact subclass of $L^1([0,\omega], R^{m+n})$.*

3. Existence of periodic solution

In this section, by using the fixed point theorem (Lemma 2.2) of differential inclusion theory, we shall give the proof of the existence of periodic solution for the neural network (1).

Proposition 3.1. *Under the assumption H_1, for any $z_0 \in R^{m+n}$, the neural network (1) has at least one solution satisfying the initial condition $z(0) = z_0$.*

Proof. See Appendix A.

Proposition 3.1 shows the existence of solutions of the neural network (1). In the following, we shall prove that the neural network (1) has an ω-periodic solution. □

Theorem 3.1. *Under the assumption H_1, there exists a solution for the differential inclusion system (5). i.e. the neural network (1) has an ω-periodic solution.*

Proof. By $Lz = \dot{z} + Dz$, we define linear operator $L : W_p^{1,1}([0,\omega], R^{m+n}) \to L^1([0,\omega], R^{m+n})$, then L is a bounded linear operator. Moreover, L is also one to one and surjective (the proof of this conclusion can been seen in Appendix B).

By Lemma 2.1, we can get that $L^{-1} : L^1([0,\omega], R^{m+n}) \to W_p^{1,1}([0,\omega], R^{m+n})$ is a bounded linear operator.

For any $z \in L^1([0,\omega], R^{m+n})$, define the set-valued map $\mathcal{M}$ as

$$\mathcal{M}(z) = \left\{v(t) \in L^1([0,\omega], R^{m+n}) \mid v(t) \in \psi(t, z(t)), \text{ for a.e. } t \in [0,\omega]\right\}.$$

Since $\psi(t,z) = BK[h(z)] + I(t)$ is an upper semicontinuous set-valued map with nonempty compact convex values on z, we can get that the set-valued map $\mathcal{M} : L^1([0,\omega], R^{m+n}) \hookrightarrow L^1([0,\omega], R^{m+n})$ has nonempty compact convex values. In particular, it is also upper semicontinuous.

Consider the set-valued map

$$L^{-1} \circ \mathcal{M} : L^1([0,\omega], R^{m+n}) \hookrightarrow L^1([0,\omega], R^{m+n}).$$

Since L^{-1} is continuous and $\mathcal{M}$ is upper semicontinuous, the set-valued map $L^{-1} \circ \mathcal{M}$ is upper semicontinuous.

Let $K \subset L^1([0,\omega], R^{m+n})$ be a bounded set, then

$$\mathcal{M}(K) = \bigcup_{z \in K} \mathcal{M}(z)$$

is a bounded subset of $L^1([0,\omega], R^{m+n})$. Since L^{-1} is a bounded linear operator, $L^{-1}(\mathcal{M}(K))$ is a bounded subset of $W_p^{1,1}([0,\omega], R^{m+n})$. By Lemma 2.3, $L^{-1}(\mathcal{M}(K))$ is a relatively compact subset of $L^1([0,\omega], R^{m+n})$. This shows that $L^{-1} \circ \mathcal{M}$ is the upper semicontinuous set-valued map which maps bounded sets into relatively compact sets.

Set

$$\Gamma = \left\{z(\cdot) \in L^1([0,\omega], R^{m+n}) : z \in \lambda L^{-1} \circ \mathcal{N}(z),\ \lambda \in (0, 1)\right\},$$

then Γ is a bounded subset of $L^1([0,\omega], R^{m+n})$ (the proof of this conclusion can been seen in Appendix C).

By Lemma 2.2, we can get that the set-valued map $L^{-1} \circ \mathcal{M}$ has a fixed point, i.e., there exists $z^* \in L^1([0, \omega], R^{m+n})$ such that $z^* \in L^{-1} \circ \mathcal{M}(z^*)$. Hence, we have $Lz^* \in \mathcal{M}(z^*)$, i.e, there exists a measurable selection $\eta^*(t) \in K[h(z^*(t))]$, such that

$$\dot{z}^*(t) + Dz^*(t) = B\eta^*(t) + I(t). \tag{6}$$

According to the definition of L^{-1}, we can get $z^* \in W_p^{1,1}([0, \omega], R^{m+n})$. Moreover, by Definition 2.2 and (6), we can get that $z^*(t)$ is a solution of the differential inclusion (5), i.e., the neural network (1) has an ω-periodic solution. The proof is completed. □

4. Global exponential stability of neural network

In this section, we shall establish the conditions that ensure global exponential stability of periodic solution for the neural networks (1) under the assumption H_1. Furthermore, we shall derive a result on global convergence in finite time for the neural network (1).

Theorem 4.1. *If the assumption H_1 and $m = n$ hold, suppose further $p_{ii} < 0, q_{ii} < 0, i = 1, \dots, n$, and $P', Q' \in R^{n\times n}$ are H-matrices, then the ω-periodic solution of the neural network (1) is globally exponentially stable.*

Proof. Under the assumption H_1, without loss of generality, we can assume that f_j and g_i satisfy

$$\eta s \ge 0, \quad \forall\, \eta \in K[f_j(s)],\ \forall\, s \in R, \tag{7}$$

$$\zeta t \ge 0, \quad \forall\, \zeta \in K[g_i(t)],\ \forall\, t \in R. \tag{8}$$

Set $\widetilde{P'} = (\widetilde{p}_{ij})_{n\times n}$, $\widetilde{p}_{ij} = \begin{cases} -p_{ii}, & i = j, \\ -|p_{ij}|, & i \ne j, \end{cases}$ and $\widetilde{Q'} = (\widetilde{q}_{ij})_{n\times n}$, $\widetilde{q}_{ij} = \begin{cases} -q_{ii}, & i = j, \\ -|q_{ij}|, & i \ne j. \end{cases}$ Since P', Q' are H-matrices, there exist $\beta = (\beta_1, \dots, \beta_n)' > 0$ and $\gamma = (\gamma_1, \dots, \gamma_n)' > 0$ such that $\beta'\widetilde{P'} > 0, \gamma'\widetilde{Q'} > 0$.

By Theorem 3.1, we can get that the neural network (1) has a ω-periodic solution. Let $z^*(t)$ be the ω-periodic solution of the neural network (1), $z(t)$ be the solution of (1) with the initial condition $x(0) = x_0$. By (4), we can get

$$\dot{z}(t) - \dot{z}^*(t) = -D\{z(t) - z^*(t)\} + B\{\eta(t) - \eta^*(t)\}, \quad \eta(t) \in K[h(z(t))],\ \eta^*(t) \in K[h(z^*(t))]. \tag{9}$$

Moreover,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|z_i(t) - z_i^*(t)| &= \partial|z_i(t) - z_i^*(t)|(\dot{z}_i(t) - \dot{z}_i^*(t)) \\ &= v_i(t)(\dot{z}_i(t) - \dot{z}_i^*(t)), \end{aligned}$$

where $v_i(t) = \operatorname{sign}(z_i(t) - z_i^*(t))$, if $z_i(t) \ne z_i^*(t)$; while $v_i(t)$ can be arbitrarily chosen in $[-1, 1]$, if $z_i(t) = z_i^*(t)$. In particular, we choose $v_i(t)$ as follows:

$$v_i(t) = \begin{cases} \operatorname{sign}(z_i(t) - z_i^*(t)), & \text{if } z_i(t) \ne z_i^*(t), \\ \operatorname{sign}(\eta_i(t) - \eta_i^*(t)), & \text{if } z_i(t) = z_i^*(t) \text{ and } \eta_i(t) \ne \eta_i^*(t), \\ 0, & \text{if } z_i(t) = z_i^*(t) = \eta_i(t) = \eta_i^*(t) = 0, \end{cases}$$

then by (7) and (8), we can get

$$\begin{aligned} &v_i(t)(z_i(t) - z_i^*(t)) = |z_i(t) - z_i^*(t)|, \\ &v_i(t)(\eta_i(t) - \eta_i^*(t)) = |\eta_i(t) - \eta_i^*(t)|, \quad i = 1, \dots, 2n. \end{aligned} \tag{10}$$

Choose a constant ρ, such that $0 < \rho < \widetilde{d} = \min(a_1, \dots, a_n, b_1, \dots, b_n)$. Consider the following Lyapunov function $V(t)$ defined by

$$V(t) = \mathrm{e}^{\rho t}\sum_{i=1}^{n}\beta_i|x_i(t) - x_i^*(t)| + \mathrm{e}^{\rho t}\sum_{j=1}^{n}\gamma_j|y_j(t) - y_j^*(t)|. \tag{11}$$

Obviously, $V(t)$ is an absolutely continuous function. Calculate the derivative of $V(t)$ along the solution $z(t)$ of Eq. (2) with the initial condition $z(0) = z_0$. By (9) and (10), we can obtain:

$$\begin{aligned} \dot{V}(t) = {}& \rho\mathrm{e}^{\rho t}\sum_{i=1}^{n}\beta_i|x_i(t) - x_i^*(t)| + \mathrm{e}^{\rho t}\sum_{i=1}^{n}\beta_i v_i(t)(\dot{x}_i(t) - \dot{x}_i{}^*(t)) \\ & + \rho\mathrm{e}^{\rho t}\sum_{j=1}^{n}\gamma_j|y_j(t) - y_j^*(t)| + \mathrm{e}^{\rho t}\sum_{j=1}^{n}\gamma_j v_{n+j}(t)(\dot{y}_j(t) - \dot{y}_j{}^*(t)) \end{aligned}$$

$$
\begin{aligned}
&= -e^{\rho t}\sum_{i=1}^{n}\beta_i(a_i-\rho)|x_i(t)-x_i^*(t)| + e^{\rho t}\sum_{i=1}^{n}\left\{\beta_i p_{ii}|\eta_i(t)-\eta_i^*(t)|\right.\\
&\quad \left. + \sum_{j\neq i}^{n}\beta_i p_{ji} v_i(t)(\eta_j(t)-\eta_j^*(t))\right\} - e^{\rho t}\sum_{j=1}^{n}\gamma_j(b_j-\rho)|y_j(t)-y_j^*(t)|\\
&\quad + e^{\rho t}\sum_{j=1}^{n}\left\{\gamma_j q_{jj}|\eta_{n+j}(t)-\eta_{n+j}^*(t)| + \sum_{i\neq j}^{n}\gamma_j q_{ij} v_{n+j}(t)(\eta_{n+i}(t)-\eta_{n+i}^*(t))\right\}\\
&\le -e^{\rho t}\sum_{i=1}^{n}\beta_i(\widetilde{d}-\rho)|x_i(t)-x_i^*(t)| + e^{\rho t}\sum_{i=1}^{n}\left\{\beta_i p_{ii}|\eta_i(t)-\eta_i^*(t)|\right.\\
&\quad \left. + \sum_{j\neq i}^{n}\beta_i |p_{ji}||\eta_j(t)-\eta_j^*(t)|\right\} - e^{\rho t}\sum_{j=1}^{n}\gamma_j(\widetilde{d}-\rho)|y_j(t)-y_j^*(t)|\\
&\quad + e^{\rho t}\sum_{j=1}^{n}\left\{\gamma_j q_{jj}|\eta_{n+j}(t)-\eta_{n+j}^*(t)| + \sum_{i\neq j}^{n}\gamma_j |q_{ij}||\eta_{n+i}(t)-\eta_{n+i}^*(t)|\right\}\\
&= -e^{\rho t}\sum_{i=1}^{n}\beta_i(\widetilde{d}-\rho)|x_i(t)-x_i^*(t)| - e^{\rho t}\sum_{j=1}^{n}\gamma_j(\widetilde{d}-\rho)|y_j(t)-y_j^*(t)|\\
&\quad - e^{\rho t}\beta'\widetilde{P}'(|\eta_1(t)-\eta_1^*(t)|,\ldots,|\eta_n(t)-\eta_n^*(t)|)'\\
&\quad - e^{\rho t}\gamma'\widetilde{Q}'(|\eta_{n+1}(t)-\eta_{n+1}^*(t)|,\ldots,|\eta_{2n}(t)-\eta_{2n}^*(t)|)'\\
&\le 0, \quad \text{for, a.e. } t\ge 0.
\end{aligned} \tag{12}
$$

By (11) and (12), we can get

$$
\sum_{i=1}^{n}|x_i(t)-x_i^*(t)| + \sum_{j=1}^{n}|y_j(t)-y_j^*(t)| \le \frac{V(t)}{\delta}e^{-\rho t} \le \frac{V(0)}{\delta}e^{-\rho t}, \quad t>0,
$$

where $\delta = \min(\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n)$. This implies global exponential stability of the periodic solution of the neural network (1). The proof is completed. □

Arguing like in the proof Theorem 4.1, we can obtain

Proposition 4.1. *If the assumption of Theorem 4.1 holds, then for any $z_0 \in R^{m+n}$, the neural network (1) has a unique solution which satisfies the initial condition $z(0) = z_0$. In particular, the periodic solution of the neural network (1) is unique.*

In the following, we give the analysis of convergence in finite time for the neural network (1). To do so, we further give the hypothesis

H_2: There exists a finite amount of time $t_1,\ldots,t_l$ in $[0,\omega)$, such that $z^*(t_i)$, $i=1,\ldots,l$ is a discontinuous point of $h(z)$, and

$$
h_i(z_i^*(t_k)^-) - \eta_i^*(t_k) < 0 < h_i(z_i^*(t_k)^+) - \eta_i^*(t_k), \quad i=1,\ldots,2n,\ k=1,\ldots,l.
$$

Set

$$
\begin{aligned}
&\delta_{ik}^+ = h_i(z_i^*(t_k)^+) - \eta_i^*(t_k), \qquad \delta_{ik}^- = \eta_i^*(t_k) - h_i(z_i^*(t_k)^-),\\
&\Delta = \min_{i=1,\ldots,2n.k=1,\ldots,l}\left\{\min\left\{\delta_{ik}^+,\delta_{ik}^-\right\}\right\}.
\end{aligned}
$$

Theorem 4.2. *If the assumption of Theorem 4.1 holds, suppose further H_2 is satisfied, then the solution of the neural network (1) with initial condition $z(0) = z_0$ converges to the unique periodic solution in finite time, i.e., there exists a constant $t_h \ge 0$, such that $z(t) = z^*(t)$ for $t \ge t_h$,*

$$
t_h = \frac{1}{\rho}\ln\left(1+\frac{\rho V(0)}{(l_1+l_2)n\Delta}\right).
$$

$V(0) = \sum_{i=1}^{n}\beta_i|x_i(0)-x^*(0)| + \sum_{j=1}^{n}\gamma_j|y_j(0)-y_j^*(0)|$. $l_1 = \min_{i=1,\ldots,n}\sum_{j=1}^{n}\beta_j\widetilde{p}_{ij}$ is the smallest entry of $\beta'\widetilde{P}'$, and $l_2 = \min_{i=1,\ldots,n}\sum_{j=1}^{n}\gamma_j\widetilde{q}_{ij}$ is the smallest entry of $\gamma'\widetilde{Q}'$.

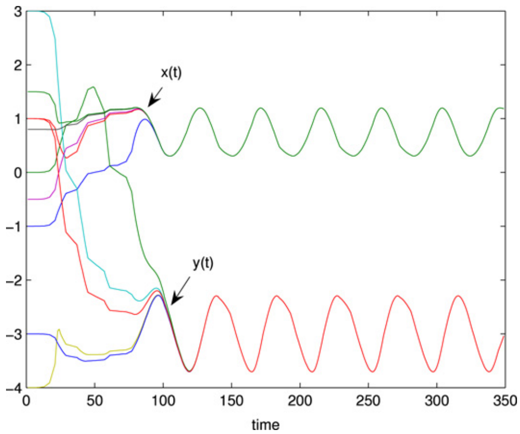

Fig. 1. Time-domain behavior of the state variables x and y.

Proof. By the assumption H_2, we can get $\Delta > 0$, and when $z(t) \neq z^*(t)$, $|\eta_i(t) - \eta_i^*(t)| \geq \Delta,\ i = 1, \ldots, 2n$. From the proof of Theorem 4.1, we have

$$\begin{aligned} \dot{V}(t) &\leq -e^{\rho t}\beta'\widetilde{P}'(|\eta_1(t) - \eta_1^*(t)|, \ldots, |\eta_n(t) - \eta_n^*(t)|)' - e^{\rho t}\gamma'\widetilde{Q}'(|\eta_{n+1}(t) - \eta_{n+1}^*(t)|, \ldots, |\eta_{2n}(t) - \eta_{2n}^*(t)|)' \\ &\leq -e^{\rho t}(l_1 + l_2)n\Delta, \quad \forall t \in \{t : z(t) \neq z^*(t), t \geq 0\}. \end{aligned} \tag{13}$$

An integration between 0 and t for (13) leads to

$$V(t) \leq V(0) - \frac{l_1 + l_2}{\rho} n\Delta \left(e^{\rho t} - 1\right).$$

By (11), we can get

$$\sum_{i=1}^{n} |x_i(t) - x^*(t)| + \sum_{j=1}^{n} |y_j(t) - y_j^*(t)| \leq \frac{e^{-\rho t}}{\delta} \left\{V(0) - \frac{l_1 + l_2}{\rho} n\Delta \left(e^{\rho t} - 1\right)\right\}, \quad t \geq 0.$$

Hence, if $t \geq t_h$, then

$$\sum_{i=1}^{n} |x_i(t) - x^*(t)| + \sum_{j=1}^{n} |y_j(t) - y_j^*(t)| \leq 0,$$

i.e., $z(t) = z^*(t)$. This completes the proof. □

5. Illustrative examples

In this section, we give two examples to illustrate the effectiveness of the results obtained in this paper.

Example 1. Let us consider the following neural network

$$\begin{cases} \dot{x}(t) = -2x(t) - f[y(t)] + 0.5 + \mathrm{sin}t, \\ \dot{y}(t) = -y(t) - 3g[x(t)] - \mathrm{cos}t, \end{cases}$$

where $f(\theta) = g(\theta) = \mathrm{sign}(\theta) = \begin{cases} 1, & \theta > 0 \\ -1, & \theta < 0 \end{cases}$ is discontinuous, and satisfies the assumption H_1. $P' = (-1)$, $Q' = (-3)$ are H-matrices. The conditions of Theorems 3.1 and 4.1 hold. Thus, this neural network has a unique 2π-periodic solution which is globally exponentially stable.

Figs. 1 and 2 show the convergent behavior of the solutions of this neural network with the initial values $(-1, 0)'$, $(1, 3)'$, $(-0.5, -4)'$, $(0.8, -3)'$ and $(1.5, 1)'$ respectively. It can be seen that all these solutions converge to the unique 2π-periodic solution of this neural network. This is in accordance with the conclusion of Theorem 4.1.

Example 2. Let us consider the following neural network

$$\begin{cases} \dot{x_1}(t) = -0.8x_1(t) - 0.4f_1[y_1(t)] + 0.2f_2[y_2(t)] + \mathrm{sin}t, \\ \dot{x_2}(t) = -0.5x_2(t) + 0.1f_1[y_1(t)] - 0.2f_2[y_2(t)] + \mathrm{cos}t, \\ \dot{y_1}(t) = -0.6y_1(t) - 0.3g_1[x_1(t)] + 0.2g_2[x_2(t)] + \mathrm{sin}t, \\ \dot{y_2}(t) = -0.7y_2(t) - 0.1g_1[x_1(t)] - 0.5g_2[x_2(t)] + \mathrm{cos}t, \end{cases}$$

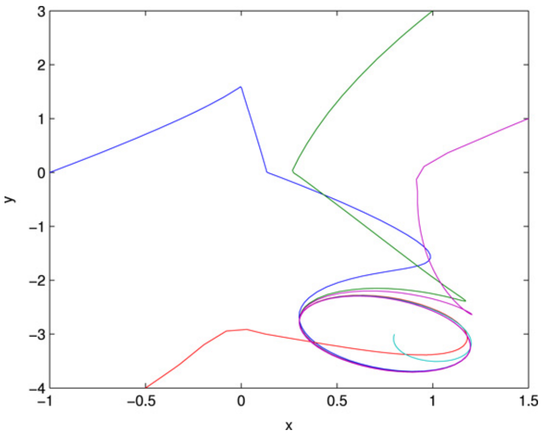

Fig. 2. Phase plane behavior of the state variables x and y.

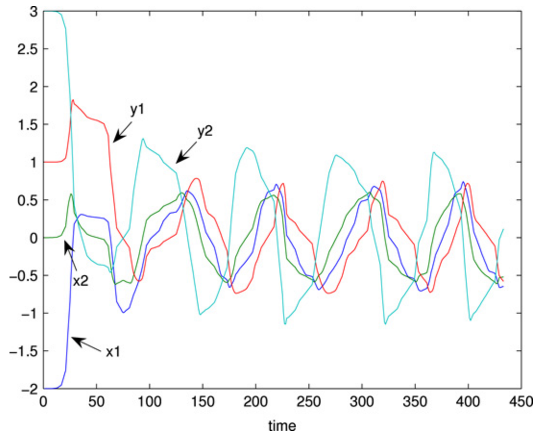

Fig. 3. Time-domain behavior of the state variables x_1, x_2, y_1 and y_2.

where

$$f_i(\theta) = \mathrm{sign}(\theta) = \begin{cases} 1, & \theta > 0 \\ -1, & \theta < 0 \end{cases}, \qquad g_i(\theta) = \begin{cases} \arctan(\theta) + 1, & \theta > 0 \\ \arctan(\theta) - 1, & \theta < 0 \end{cases}, i = 1, \ldots, n,$$

are discontinuous, and satisfy the assumption H_1. $P' = \begin{pmatrix} -0.4 & 0.2 \\ 0.1 & -0.2 \end{pmatrix}$, $Q' = \begin{pmatrix} -0.3 & 0.2 \\ -0.1 & -0.5 \end{pmatrix}$ are H-matrices. The conditions of Theorems 3.1 and 4.1 hold. Thus, this neural network has a unique 2π-periodic solution which is globally exponentially stable.

Figs. 3–8 show the convergent behavior of the solution of this neural network with the initial value $(-2, 0, 1, 3)$. It can be seen that the solution converges to the unique 2π-periodic solution of this neural network. This is in accordance with the conclusion of Theorem 4.1.

6. Conclusion

In this paper, we have presented a new BAM neural network with discontinuous neuron activations. By using the fixed point theorem of differential inclusion theory, we have proved the existence of periodic solutions for the neural network. The conditions that ensure the uniqueness and global exponential stability of periodic solution for the neural network have been established. Moreover, the conditions that guarantee global convergence in finite time of the neural network have been developed. The obtained results show that Forti's conjecture in [14] is true for BAM neural networks with discontinuous neuron activations.

How to investigate the stability of periodic solution for delayed BAM neural networks with discontinuous neuron activations, in particular, how to investigate the existence of periodic solution in the framework of differential inclusion theory will be the topic of future research.

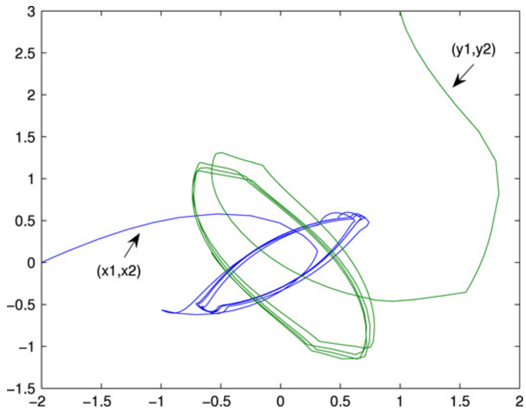

Fig. 4. Phase plane behavior of the state variables (x_1, x_2) and (y_1, y_2).

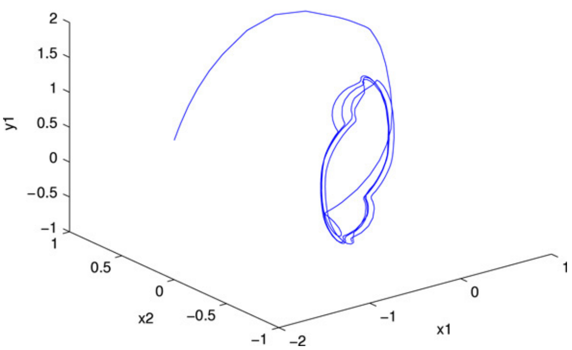

Fig. 5. Phase plane behavior of the state variables x_1, x_2 and y_1.

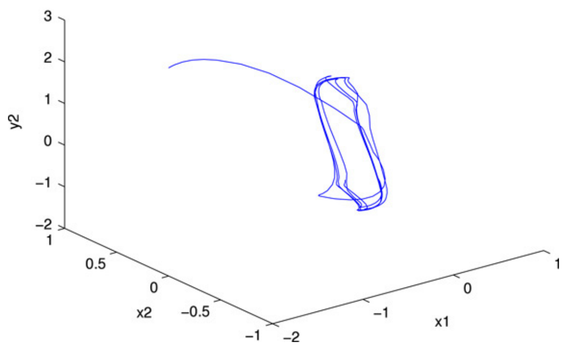

Fig. 6. Phase plane behavior of the state variables x_1, x_2 and y_2.

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Appendix A

Notice that $\phi(z, t): (z, t) \hookrightarrow -Dz + BK[h(z)] + I(t)$ is an upper semicontinuous set-valued map with nonempty compact convex values, the local existence of a solution $z(t)$ for Eq. (2) on $[0, t_0]$, $t_0 > 0$, with $z(0) = z_0$, is obvious [18].

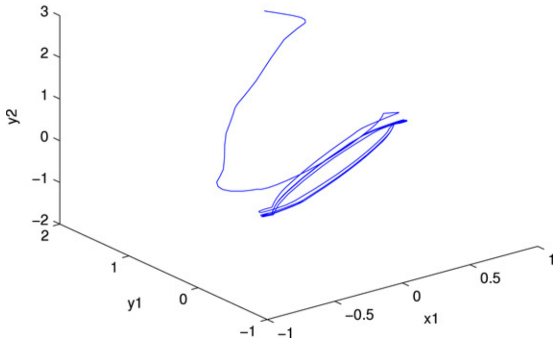

Fig. 7. Phase plane behavior of the state variables x_1, y_1 and y_2.

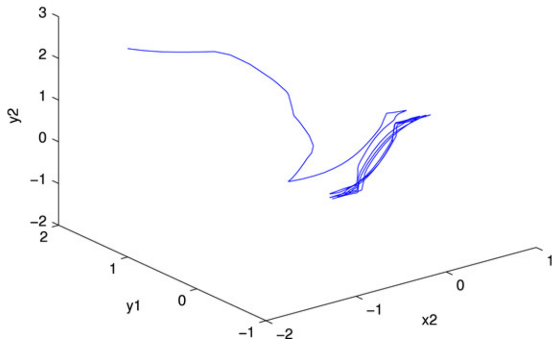

Fig. 8. Phase plane behavior of the state variables x_2, y_1 and y_2.

Set $\psi(t, z) = BK[h(z)] + I(t)$. By the assumption H_1, $h(z)$ is bounded on R^{m+n} and hence also $K[h(z)]$ is bounded on R^{m+n}. Since $I(t)$ is a continuous ω-periodic function, the set-valued map $\psi(t, z)$ is bounded, i.e., there exists a constant $M > 0$, such that

$$\sup_{z \in R^{m+n},\ t \in [0,+\infty]} \|\psi(t, z)\| \le M. \tag{14}$$

Choose $\tilde{R} > 0$, such that when $\|z(t)\| > \tilde{R}$,

$$\frac{M}{\|z(t)\|} < \frac{\widetilde{d}}{2}, \tag{15}$$

where $\widetilde{d} = \min(a_1, \dots, a_n, b_1, \dots, b_m)$. According to (4), (14) and (15), and by the Cauchy inequality, when $\|z(t)\| > \tilde{R}$, we can get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\|z(t)\|^2 &= \langle z(t), \dot{z}(t)\rangle \\
&= \langle z(t), -Dz(t) + B\eta(z) + I(t)\rangle \\
&= -\langle z(t), Dz(t)\rangle + \langle z(t), B\eta(t) + I(t)\rangle \\
&\le -\widetilde{d}\|z(t)\|^2 + M\|z(t)\| \\
&= \left(-\widetilde{d} + \frac{M}{\|z(t)\|}\right)\|z(t)\|^2 \\
&< -\frac{\widetilde{d}}{2}\|z(t)\|^2 \\
&< 0.
\end{aligned} \tag{16}$$

Let $\bar{R} = \max\{\|z_0\|, \tilde{R}\}$. By (16), we can get that $\|z(t)\| \leq \bar{R}$ on $[0, t_0]$. This means that the local solution $z(t)$ is bounded. Thus, the neural network (1) has at least a solution with initial condition $z(0) = z_0$ on $[0, +\infty)$.

Appendix B

By $Lz = \dot{z} + Dz$, we define linear operator $L: W_p^{1,1}([0, \omega], R^{m+n}) \to L^1([0, \omega], R^{m+n})$. Then

$$\begin{aligned} \|Lz\|_{L^1} &= \int_0^\omega \|\dot{z}(t) + Dz(t)\| \mathrm{d}t \\ &\leq \int_0^\omega \|\dot{z}(t)\| \mathrm{d}t + \int_0^\omega \|Dz(t)\| \mathrm{d}t \\ &\leq \int_0^\omega \|\dot{z}(t)\| \mathrm{d}t + \bar{d} \int_0^\omega \|z(t)\| \mathrm{d}t \\ &\leq \max\left\{1, \bar{d}\right\} \|z\|_{W^{1,1}}, \end{aligned} \tag{17}$$

where $\bar{d} = \max\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. By (17), we can get that L is a bounded linear operator.

Let $z^1, z^2 \in W_p^{1,1}([0, \omega], R^{m+n})$. If $Lz^1 = Lz^2$, then we have

$$\dot{z}^1(t) - \dot{z}^2(t) = -D(z^1(t) - z^2(t)). \tag{18}$$

By (18), we can get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(-\|z^1(t) - z^2(t)\|^2) &= -2\langle z^1(t) - z^2(t), \dot{z}^1(t) - \dot{z}^2(t)\rangle \\ &= 2\langle z^1(t) - z^2(t), D[z^1(t) - z^2(t)]\rangle \\ &= 2\left(\sum_{i=1}^n a_i \left(x_i^1(t) - x_i^2(t)\right)^2 + \sum_{j=1}^m b_i \left(y_i^1(t) - y_i^2(t)\right)^2\right). \end{aligned} \tag{19}$$

Noting $z^1(0) = z^1(\omega)$, $z^2(0) = z^2(\omega)$, we have

$$\int_0^\omega \frac{\mathrm{d}}{\mathrm{d}t}(-\|z^1(t) - z^2(t)\|^2)\mathrm{d}t = \|z^1(0) - z^2(0)\|^2 - \|z^1(\omega) - z^2(\omega)\|^2 = 0.$$

By (19), we can get

$$\begin{aligned} 0 &\leq 2\int_0^\omega \left(\sum_{i=1}^n a_i \left(x_i^1(t) - x_i^2(t)\right)^2 + \sum_{j=1}^m b_i \left(y_i^1(t) - y_i^2(t)\right)^2\right) \\ &= \int_0^\omega \frac{\mathrm{d}}{\mathrm{d}t}(-\|z^1(t) - z^2(t)\|^2)\mathrm{d}t \\ &= 0\,. \end{aligned}$$

It follows that $z^1(t) = z^2(t)$, $t \in [0, \omega]$. This shows that L is one to one.

Let $f(t) \in L^1([0, \omega], R^{m+n})$. In order to verify that L is surjective, in the following, we will prove that there exists $z(\cdot) \in W_p^{1,1}([0, \omega], R^{m+n})$ such that

$$Lz = f,$$

i.e., we shall prove that there exists a solution for the differential equation

$$\begin{cases} \dot{z}(t) = -Dz(t) + f(t), \\ z(0) = z(\omega). \end{cases} \tag{20}$$

Consider the initial value problem

$$\begin{cases} \dot{z}(t) = -Dz(t) + f(t), \\ z(0) = \xi. \end{cases} \tag{21}$$

It is easily checked that

$$z(t) = \mathrm{e}^{-Dt}\xi + \int_0^t \mathrm{e}^{-D(t-s)} f(s)\mathrm{d}s \tag{22}$$

is the solution of (21). By (22), choose $\xi = x(\omega)$, we can get

$$e^{-D\omega}\xi + \int_0^\omega e^{-D(\omega-s)} f(s)ds = \xi,$$

i.e.,

$$\left(I - e^{-D\omega}\right)\xi = \int_0^\omega e^{-D(\omega-s)} f(s)ds, \tag{23}$$

where I is an identity matrix. Since D is a positive diagonal matrix, $I - e^{-D\omega}$ is a nonsingular matrix. By (23), we take

$$\xi = (I - e^{-D\omega})^{-1} \int_0^\omega e^{-D(\omega-s)} f(s)ds,$$

in (22), then (22) is the solution of (20). This shows that L is surjective.

Appendix C

For any $r(z) \in K[h(z)]$, when $z \neq 0$, $\lambda \in (0, 1)$, by (14) and the Cauchy inequality, we can get

$$\begin{aligned}\langle z, -Dz + \lambda Br(z) + \lambda I(t)\rangle &= -\langle z, Dz\rangle + \langle z, \lambda Br(z) + \lambda I(t)\rangle \\ &\leq -\widetilde{d}\|x\|^2 + M\|z\| \\ &= \left(-\widetilde{d} + \frac{M}{\|z\|}\right)\|z\|^2.\end{aligned} \tag{24}$$

Choose $K_0 > 0$, such that when $\|z\| > K_0$, $\frac{M}{\|z\|} < \frac{\widetilde{d}}{2}$.

Therefore, when $\|z\| > K_0$, by (24), we can get

$$\langle z, -Dz + \lambda Br(z) + \lambda I(t)\rangle < -\frac{\widetilde{d}}{2}\|z\|^2, \quad \forall\, r(z) \in K[h(z)]. \tag{25}$$

Let $z \in \Gamma$, then $z \in \lambda L^{-1} \circ \mathcal{M}(z)$, i.e., $Lz \in \lambda\mathcal{M}(z)$. By the definition of $\mathcal{M}$, there exists a measurable selection $v(t) \in K[h(z(t))]$, such that

$$\dot{z}(t) + Dz(t) = \lambda Bv(t) + \lambda I(t). \tag{26}$$

If $z \in \Gamma$, then we can derive $\max_{t\in[0,\omega]} \|z(t)\| \leq K_0$. Otherwise, $\max_{t\in[0,\omega]} \|z(t)\| > K_0$. By $L^{-1} : L^1([0, \omega], R^{m+n}) \to W_p^{1,1}([0, \omega], R^{m+n})$, we have $z(0) = z(\omega)$. Since $z(t)$ is continuous, we can choose $t_0 \in (0, \omega]$, such that

$$\|z(t_0)\| = \max_{t\in[0,\omega]} \|z(t)\| > K_0,$$

and there exists a constant $\delta_{t_0} > 0$, such that when $t \in (t_0 - \delta_{t_0}, t_0]$, $\|z(t)\| > K_0$. By (25) and (26), we can get

$$\begin{aligned}0 &\leq \frac{1}{2}\|z(t_0)\|^2 - \frac{1}{2}\|z(t)\|^2 \\ &= \frac{1}{2}\int_t^{t_0} \frac{d}{ds}(\|z(s)\|^2)ds \\ &= \int_t^{t_0} \langle z(s), \dot{z}(s)\rangle ds \\ &= \int_t^{t_0} \langle z(s), -Dz(s) + \lambda Bv(s) + \lambda I(s)\rangle ds \\ &< -\frac{\widetilde{d}}{2}\int_t^{t_0} \|z(s)\|^2 ds \\ &< 0, \quad \text{for } t \in (t_0 - \delta_{t_0}, t_0].\end{aligned}$$

This is a contradiction. Thus, we get that for any $z \in \Gamma$, $\max_{t\in[0,\omega]} \|z(t)\| \leq K_0$. Furthermore, we have

$$\|z\|_{L^1} = \int_0^\omega \|z(s)\|ds \leq \omega K_0, \quad \forall\, z \in \Gamma.$$

This shows that Γ is a bounded subset of $L^1([0, \omega], R^{m+n})$.

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