Pergamon

Appl. Math. Lett. Vol. 11, No. 2, pp. 85–89, 1998

0893-9659/98 $19.00 + 0.00

PII: S0893-9659(98)00016-0

Coincidence Theorems Involving Composites of Acyclic Mappings in Contractible Spaces

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(Received March 1997; accepted April 1997)

Abstract—Some coincidence theorems involving a new class of set-valued mappings containing compact composites of acyclic mappings defined on a contractible space is proved.

Keywords—Coincidence theorem, Contractible space, Acyclic mapping, Local intersection property.

1. INTRODUCTION AND PRELIMINARIES

Let X and Y be two nonempty sets and let $T : X \to 2^Y$ and $S : Y \to 2^X$ be two set-valued mappings, where 2^X denotes the family of all subsets of X. Following Browder [1], a point $(x_0, y_0) \in X \times Y$ is said to be a coincidence point if $y_0 \in T(x_0)$ and $x_0 \in S(y_0)$.

Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \dots, e_n$. If J is a nonempty subset of $\{0, 1, \dots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. A topological space X is said to be contractible if the identity mapping I_X of X is homotopic to a constant function. A topological space X is said to be an acyclic space if all of its reduced Cech homology groups over the rationals vanish. In particular, any contractible space is acyclic, and hence any convex or star-shaped set in a topological vector space is acyclic. For a topological space X, we shall denote by $ka(X)$ the family of all compact acyclic subsets of X.

Let X and Y be two topological spaces. For a given class $\mathbf{L}$ of set-valued mappings, define

$$\mathbf{L}(X,Y) = \left\{T : X \to 2^Y \mid T \in \mathbf{L}\right\}, \qquad \mathbf{L}_c = \left\{T = T_m T_{m-1} \cdots T_1 \mid T_i \in \mathbf{L}\right\}.$$

Using the above notation, we have the following definitions.

(1) T is an acyclic mapping, written $F \in \mathbf{V}(X, Y)$, if $F : X \to ka(Y)$ is upper-semicontinuous.
(2) $T \in \mathbf{V}^+(X, Y)$ if for any σ-compact subset K of X there exists a $T^* \in \mathbf{V}(K, Y)$ such that $T^*(x) \subset T(x)$ for all $x \in K$.
(3) $T \in \mathbf{V}_c^+(X, Y)$ if for any σ-compact subset K of X there exists a $T^* \in \mathbf{V}_c(K, Y)$ such that $T^*(x) \subset T(x)$ for all $x \in K$.

In 1986, by using Browder's fixed point theorem [1] for the set-valued mapping with open inverse values, Komiya [2] proved the following coincidence theorem.

THEOREM A. *(See [2].) Let X be a nonempty convex subset of a Hausdorff topological vector space E, and let Y be a nonempty compact convex subset of a Hausdorff topological vector space W. Suppose $A : X \to 2^Y$ is upper-semicontinuous with closed and convex values and*

This project was supported by the Natural Science Foundation of Sichuan Educational Committee, P.R. China.

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$B : Y \to 2^X$ has convex values such that $B^{-1}(x)$ is open in Y for each $x \in X$. Then there exists $(x_0, y_0) \in X \times Y$ such that $x_0 \in B(y_0)$ and $y_0 \in A(x_0)$.

Ding and Tarafdar [3] generalized Komiya's theorem to H-space (see [4]) and to the mapping with compact acyclic values. Tarafdar and Yuan [5] and Yuan [6] generalized Komiya's theorem to contractible space and to the mapping with compact contractible values. By using the generalization of the classic Knaster-Kuratowski-Mazurkiewicz theorem, Horvath [7] gave a number of coincidence theorems in which both mappings involve the property of open inverse values (or open-image values), for example, see Theorem 3 and Corollaries 3–6 in [7]. In Theorem A and Theorem 1 of [5], one of two mappings still involve the property of open inverse values. Recently Ding [8] generalized the results in [2,5] and obtained some new coincidence theorems for set-valued mappings, both without convex values and open inverse values in contractible spaces.

In this note, we shall establish some new coincidence theorems involving a new class of mappings containing compact composites of acyclic mappings defined in contractible spaces and the mappings do not have convex values and open inverse values. These theorems further generalize the corresponding results in [1,2,5,6,8,9].

Let X and Y be two topological spaces and $T : X \to 2^Y$ a set-valued mapping. T is said to have local intersection property (see [10]) if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$. The example in [10, p. 63] shows that a set-valued mapping with local intersection property may not have the property of open inverse values.

LEMMA 1. *Let X and Y be topological spaces and $T : X \to 2^Y$ a set-valued mapping. Then the following conditions are equivalent.*

(i) *For each $x \in X$, $T(x) \neq \emptyset$ and T has the local intersection property.*
(ii) *For each $y \in Y$, $T^{-1}(y)$ contains a open set $O_y \subset X$ (which may be empty) such that $X = \bigcup_{y \in Y} O_y$.*
(iii) *There exists a set-valued mapping $F : X \to 2^Y$ such that $F(x) \subset T(x)$ for each $x \in X$; $F^{-1}(y)$ is open in X for each $y \in Y$ and $X = \bigcup_{y \in Y} F^{-1}(y)$.*
(iv) *For each $x \in X$, there exists $y \in Y$ such that $x \in \operatorname{int}(T^{-1}(y))$.*

PROOF.
(i) $\Rightarrow$ (ii). Since for each $x \in X$, $T(x) \neq \emptyset$, by (i), there exists an open neighborhood $N(x)$ of x such that

$$M(x) = \bigcap_{z \in N(x)} T(z) \neq \emptyset.$$

It follows that there exists $y \in M(x) \subset Y$ such that $N(x) \subset T^{-1}(y)$, and hence, we have

$$X = \bigcup_{x \in X} N(x) \subset \bigcup_{y \in Y} T^{-1}(y) = X.$$

For each $y \in Y$, if $y \in M(x)$ for some $x \in X$, let $O_y = N(x)$ and if $y \notin M(x)$ for all $x \in X$, let $O_y = \emptyset$. Then the family $\{O_y\}_{y \in Y}$ of open sets satisfies Condition (ii).
(ii)$\Rightarrow$ (iii). Suppose Condition (ii) holds. Define a mapping $F : X \to 2^Y$ by

$$F(x) = \{y \in Y : x \in O_y\}.$$

For each $y \in Y$, we have

$$F^{-1}(y) = \{x \in X : y \in F(x)\} = \{x \in X : x \in O_y\} = O_y \subset T^{-1}(y).$$

It follows that for each $y \in Y$, $F^{-1}(y)$ is open in X and for each $x \in X$, $F(x) \subset T(x)$ and

$$X = \bigcup_{y \in Y} O_y = \bigcup_{y \in Y} F^{-1}(y).$$

This proves Condition (iii) holds.

(iii)$\Rightarrow$ (iv). Suppose Condition (iii) holds. Then, we have $F^{-1}(y) \subset \operatorname{int}(T^{-1}(y)) \subset T^{-1}(y)$ for each $y \in Y$ and

$$X = \bigcup_{y \in Y} F^{-1}(y) \subset \bigcup_{y \in Y} \operatorname{int}(T^{-1}(y)) \subset \bigcup_{y \in Y} T^{-1}(y) \subset X.$$

Therefore, for each $x \in X$, there exists $y \in Y$ such that $x \in \operatorname{int}(T^{-1}(y))$.

(iv)$\Rightarrow$ (i). Suppose Condition (iv) holds. For each $x \in X$, there exists a $y \in Y$ such that $x \in \operatorname{int}(T^{-1}(y))$, and hence, there exists an open neighborhood $N(x)$ of x such that $N(x) \subset \operatorname{int}(T^{-1}(y)) \subset T^{-1}(y)$. It follows that

$$y \in \bigcap_{z \in N(x)} T(z),$$

that is, T has the local intersection property.

The following lemma, the proof of which is contained in the proof of Theorem 1 of [11], will be a basic tool for our purpose (see also [12]).

LEMMA 2. *Let Y be a topological space. For any nonempty subset J of $\{0, 1, \dots, n\}$, let Γ_J be a nonempty contractible subset of Y. If $\emptyset \neq J \subset J' \subset \{0, 1, \dots, n\}$ implies $\Gamma_J \subset \Gamma_{J'}$, then there exists a continuous mapping $f : \Delta_n \to Y$ such that $f(\Delta_J) \subset \Gamma_J$ for each nonempty subset J of $\{0, 1, \dots, n\}$.*

The following result is quite well known in the Lefschetz fixed point theory. For details, we refer the reader to [13,14].

LEMMA 3. *Let Δ_n be an n-dimensional simplex with the Euclidean topology. If $F \in \mathbf{V}_c(\Delta_n, \Delta_n)$ then F has a fixed point.*

2. COINCIDENCE THEOREM

THEOREM 1. *Let X be a Hausdorff topological space and D a contractible subset of a topological space Y. Let $S : D \to 2^X$ and $T : X \to 2^D$ be two set-valued mappings such that*

(i) *$S \in \mathbf{V}_c^+(D, X)$ is a compact mapping,*
(ii) *for each $x \in X$, $T(x) \neq \emptyset$ and T has local intersection property,*
(iii) *for each open set $U \subset X$, the set $\bigcap_{x \in U} T(x)$ is empty or contractible.*

Then there exist $x_0 \in X$ and $y_0 \in D$ such that $x_0 \in S(y_0)$ and $y_0 \in T(x_0)$.

PROOF. By (ii) and Lemma 1, for each $y \in D$, there exists an open set $O_y \subset X$ (which may be empty) such that $O_y \subset T^{-1}(y)$ and $X = \bigcup_{y \in Y} O_y = \bigcup_{y \in Y} T^{-1}(y)$. Since S is a compact mapping, $\overline{S(D)}$ is a compact subset of X, there exists a finite set $\{y_0, y_1, \dots, y_n\} \subset D$ such that $\overline{S(D)} = \bigcup_{i=0}^{n} O_{y_i}$. Now for each nonempty subset J of $N = \{0, 1, \dots, n\}$, define

$$\Gamma_J = \begin{cases} \bigcap \{T(x) : x \in \bigcap_{j \in J} O_{y_j}\}, & \text{if } \bigcap_{j \in J} O_{y_j} \neq \emptyset, \\ D, & \text{otherwise.} \end{cases}$$

Note that $O_y \subset T^{-1}(y)$ for each $y \in D$, if $x \in \bigcap_{j \in J} O_{y_j}$, then $\{y_j : j \in J\} \subset T(x)$. By (iii), each Γ_J is nonempty contractible and it is clear that $\Gamma_J \subset \Gamma_{J'}$, whenever $\emptyset \neq J \subset J' \subset N$. By Lemma 2, there exists a continuous mapping $f : \Delta_n \to D$ such that

$$f(\Delta_J) \subset \Gamma_J, \qquad \text{for all } \emptyset \neq J \subset N.$$

Since $f(\Delta_n)$ is compact in D and $S \in \mathbf{V}_c^+(D, X)$, there is an $S^* \in \mathbf{V}_c(f(\Delta_n), X)$ such that $S^*(y) \subset S(y)$ for each $y \in f(\Delta_n)$. By Proposition 3.1.11 of [15], $S^*(f(\Delta_n))$ is a compact subset of $S(D)$. Hence we have

$$S^*(f(\Delta_n)) = \bigcup_{i=0}^{n} [O_{y_i} \cap S^*(f(\Delta_n))].$$

Let $\{\psi_i\}_{i\in N}$ be a continuous partition of unity subordinated to the open covering $\{O_{y_i} \cap S^*(f(\Delta_n))\}_{i\in N}$, i.e., for each $i \in N$, $\psi_i : S^*(f(\Delta_n)) \to [0,1]$ is continuous,

$$\{x \in S^*(f(\Delta_n)) : \psi_i(x) \neq 0\} \subset O_{y_i} \cap S^*(f(\Delta_n)) \subset O_{y_i} \subset T^{-1}(y_i)$$

such that $\sum_{i=0}^{n} \psi_i(x) = 1$ for all $x \in S^*(f(\Delta_n))$. Define $\psi : S^*(f(\Delta_n)) \to \Delta_n$ by

$$\psi(x) = (\psi_0(x), \psi_1(x), \dots, \psi_n(x)), \qquad \text{for all } x \in S^*(f(\Delta_n)).$$

Then $\psi(x) \subset \Delta_{J(x)}$ for all $x \in S^*(f(\Delta_n))$, where $J(x) = \{j \in N : \psi_j(x) \neq 0\}$. Therefore, we have

$$f(\psi(x)) \in f(\Delta_{J(x)}) \subset \Gamma_{J(x)} \subset T(x), \qquad \text{for all } x \in S^*(f(\Delta_n)). \tag{2.1}$$

It is easy to see $\psi \circ S^* \circ f \in \mathbf{V}_c(\Delta_n, \Delta_n)$, by Lemma 3, there exists $z \in \Delta_n$ such that $z \in \psi \circ S^* \circ f(z)$. Let $y_0 = f(z)$, then $y_0 \in f(\Delta_n)$ and $\psi^{-1}(z) \cap S^*(y_0) \neq \emptyset$. Take $x_0 \in \psi^{-1}(z) \cap S^*(y_0)$, then we have $z = \psi(x_0)$ and $x_0 \in S^*(y_0) \subset S(y_0)$. It follows from (2.1) that

$$y_0 = f(z) = f \circ \psi(x_0) \subset T(x_0).$$

This completes the proof.

REMARK 1. Theorem 1 generalizes Theorem 1 of [8] and Theorem 1 of [5] in the following aspects:

(1) X may not be compact space,
(2) S is a mapping in the new class of mappings containing the composites of acyclic mappings,
(3) T may not have the property of open inverse values.

If for each $y \in D$, $T^{-1}(y)$ is open in X, then for each $x \in X$ with $T(x) \neq \emptyset$, we take $y \in T(x)$ and let $N(x) = T^{-1}(y)$. Then $N(x)$ is a open neighborhood of x and $y \in \bigcap_{z\in N(x)} T(z)$. Hence, T has the local intersection property. Theorem 1, in turn, generalizes Theorem 2.3 of [9], Theorem 1 of [2], Theorem 4.5 of [6], and Theorem 1 of [1].

THEOREM 2. *Let X be a compact topological space and Y a contractible space. Let $S : Y \to 2^X$ and $T : X \to 2^Y$ be such that*

(i) $S \in \mathbf{V}_c^+(Y, X)$,
(ii) *for each $x \in X$, $T(x) \neq \emptyset$ and T has the local intersection property,*
(iii) *for each open set $U \subset X$, the set $\bigcap_{x\in U} T(x)$ is empty or contractible.*

Then there exist $x_0 \in X$ and $y_0 \in Y$ such that $x_0 \in S(y_0)$ and $y_0 \in T(x_0)$.

PROOF. From the compactness of X, it follows that S is a compact mapping. The conclusion follows from Theorem 1.

REMARK 2. Theorem 2 generalizes Theorem 1 of [8] from $S \in \mathbf{V}(Y, X)$ to $S \in V_c^+(Y, X)$ and the corresponding results in [2,4–6,9].

COROLLARY 1. *Let X be a Hausdorff topological space and Y a contractible space. Suppose $T : X \to 2^Y$ is such that*

(i) *for each $x \in X$, $T(x) \neq \emptyset$ and T has the local intersection property,*
(ii) *for each open set $U \in X$, the set $\bigcap_{x\in U} T(x)$ is empty or contractible.*

Then for any continuous single-valued mapping $g : Y \to X$, there exists $y_0 \in Y$ such that $y_0 \in T(g(y_0))$.

PROOF. Define a mapping $S : Y \to 2^X$ by

$$S(y) = \{g(y)\}, \qquad \text{for all } y \in Y.$$

It is easy to see that $S \in \mathbf{V}(Y, X) \subset \mathbf{V}_c^+(Y, X)$. By Theorem 1, there exist $x_0 \in X$ and $y_0 \in Y$ such that $x_0 = S(y_0) = \{g(y_0)\}$ and $y_0 \in T(x_0)$. Hence we must have $y_0 \in T(g(y_0))$.

REMARK 3. Corollary 1 generalizes Corollary 1 in [8].

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