Available online at www.sciencedirect.com
ScienceDirect
International Journal of Solids and Structures 45 (2008) 1332–1351
INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES
www.elsevier.com/locate/ijsolstr

Exact solution of the multi-cracked Euler–Bernoulli column

S. Caddemi *, I. Caliò'

Dipartimento di Ingegneria Civile ed Ambientale, Università di Catania, Catania, Italy

Received 8 March 2007; received in revised form 18 July 2007
Available online 1 October 2007

Abstract

The use of distributions (generalized functions) is a powerful tool to treat singularities in structural mechanics and, besides providing a mathematical modelling, their capability of leading to closed form exact solutions is shown in this paper. In particular, the problem of stability of the uniform Euler–Bernoulli column in presence of multiple concentrated cracks, subjected to an axial compression load, under general boundary conditions is tackled. Concentrated cracks are modelled by means of Dirac's delta distributions. An integration procedure of the fourth order differential governing equation, which is not allowed by the classical distribution theory, is proposed. The exact buckling mode solution of the column, as functions of four integration constants, and the corresponding exact buckling load equation for any number, position and intensity of the cracks are presented. As an example a parametric study of the multi-cracked simply supported and clamped–clamped Euler–Bernoulli columns is presented.

Keywords: Closed form solution; Euler–Bernoulli column; Stability; Concentrated cracks; Singularities; Distribution theory

1. Introduction

In many engineering problems, involving beam-like structures, continuity of the physical and geometrical properties can be interrupted by singularities due to the presence of concentrated cracks. The effect of concentrated cracks has been widely studied in the literature and models aiming at describing the variation of the flexural stiffness of the beam in the vicinity of the crack have been proposed. Without the claim of being exhaustive, some examples are recalled in what follows. In particular, a stiffness reduction due to the presence of a crack with an exponential variation law, that is not restricted to a local influence, has been proposed by Christides and Barr (1984). On the contrary, a stiffness reduction with a local effect governed by a triangular variation has been proposed by Sinha et al. (2002). Furthermore, Cerri and Vestroni (2003) proposed a constant stiffness reduction, due to a concentrated crack, limited to an effective length around the crack. Bilello (2001) modelled locally the effect of a concentrated crack by means of an ineffective area delimited by a linear reduction of its height starting from the cracked

* Tel.: +39 0957382266; fax: +39 0957382297.
E-mail address: scaddemi@dica.unict.it (S. Caddemi).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2007.09.022

section. Chondros et al. (1998) modelled the crack as a continuous flexibility by using the displacement field in the vicinity of the crack, found with fracture mechanics methods. The consistent continuous model proposed by Chondros et al. (1998) is a generalization of the Christides and Barr (1984) cracked beam theory that is based on experimental determination of the exponent of the stress field. Other authors presented models of beams with transverse cracks showing that cracked structural members can be represented by a consistent static flexibility matrix (Gudmundson, 1984) and evaluating alternative expressions by including coupling terms of the flexibility influence coefficients (Okamura et al., 1969; Rice and Levy, 1972; Dimarogonas and Massouros, 1981). Papadoupolos and Dimarogonas (1987) introduced a full matrix for an arbitrary loading of a cracked beam computed by means of fracture mechanics method.

The effect of concentrated damages on the flexural stiffness in the vicinity of the crack was also treated in the literature in a macroscopic way by means of the idea of an equivalent rotational spring connecting two adjacent segments of the beam (Irwin, 1957a,b; Freund and Hermann, 1976; Gounaris and Dimarogonas, 1988; Rizos et al., 1990; Ostachowicz and Krawczuk, 1991; Paipetis and Dimarogonas, 1986). This model, based on fracture mechanics concepts, is able to capture the slope discontinuity at the cross-sections where the cracks occur. A review can be found in (Dimarogonas, 1996) with particular reference to dynamics, where a distinction between the behaviour of a notch and a crack is evidenced. In particular, a thin cut, although used to model cracks, leads to a local flexibility less than that associated to a crack.

According to the approach based on an equivalent rotational spring, the cracked beam models with continuous flexibility in the vicinity of the crack can be approximated as models with lumped flexibility by imposing that the rotation discontinuity due to the concentrated flexibility reproduces the relative rotation of the cross-sections affected by the crack.

Whatever model is adopted to describe the influence of concentrated cracks, the effects of the reduced stiffness on the deflection, on the dynamic characteristics, on the load carrying capacity, etc. should be evaluated by means of procedures able to provide accurate results with low computational effort. In particular contributions orientated towards explicit solutions are desirable for the engineering practice.

The classical method for solving problems in presence of singularities, such as concentrated cracks, relies on integration of the governing equations between singularities and on enforcement of the continuity conditions at those sections where singularities occur. Studies aiming at providing integration procedures able to treat singularities more efficiently than classical methods have been proposed in the literature to solve both static (Yavari et al., 2000, 2001a,b; Falsone, 2002) and dynamic (Dimarogonas, 1996; Gounaris and Dimarogonas, 1988; Quian et al., 1990; Morassi, 1993; Shifrin and Ruotolo, 1999) governing equations.

In particular, the study conducted in Yavari et al. (2000, 2001a,b) is very appealing since use of the distribution theory is made in the integration procedure; it provides a formulation of the governing equations over a unique integration domain however requiring, to be solved, the enforcement of a single continuity condition at each singularity.

The effects of concentrated cracks on the stability characteristics of beam structures have been also investigated in the literature by making use of the local flexibility formulation (Liebowitz et al., 1967; Liebowitz and Claus, 1968; Okamura et al., 1969; Anifantis and Dimarogonas, 1983; Takahashi, 1999; Yavari and Sarkani, 2001; Li, 2002; Fan and Zheng, 2003). Single and multiple cracks have been considered on uniform and non-uniform beams, both for Euler–Bernoulli and Timoshenko beams. Among them, procedures providing exact solutions either require enforcement of continuity conditions at each singularity or, if continuity conditions are avoided, are not able to treat all the boundary conditions. Otherwise approximate solutions able to approach functions with internal discontinuities, have been proposed.

The adoption of the distribution theory seems to be an interesting approach for problems with singularities. More precisely, besides the use of the distribution rules in the integration procedure, modelling cracks directly through distributions, such as the Dirac's delta, could lead to a robust approach.

Recently the model of flexural stiffness with singularities, represented by Dirac's deltas, has been shown to be equivalent to an internal hinge endowed with a rotational spring (Biondi and Caddemi, 2005, 2007), hence it can be adopted to model concentrated cracks (Buda and Caddemi, 2007). According to the latter approach, modelling cracks by means of Dirac's deltas in the flexural stiffness seems to be a promising approach since exact closed form solutions can be formulated for the static governing equations. Non-trivial extension to more complex contexts such as stability and dynamics are not straightforward and should be studied.

In this paper the problem of stability for the uniform Euler–Bernoulli column in presence of multiple concentrated cracks under general boundary conditions is tackled by making use of the macroscopic approach that models cracks as equivalent rotational springs. Aim of the work is showing that the model of cracks by means of Dirac's deltas can be successfully employed to obtain exact closed form solutions in a context broader than the linear static problem. In particular, an integration procedure of the fourth order governing differential equation is proposed by making use of a new definition of the product of Dirac's deltas (Bagarello, 1995, 2002) which is not allowed by the classical distribution theory (Bremermann and Durand, 1961; Colombeau, 1984). The exact explicit expressions of the eigenfunctions (buckling modes) of the column, as functions of four integration constants, are presented, from which the exact buckling load equation for any number, position and intensity of the cracks can be easily derived. A parametric study of multi-cracked simply supported and clamped–clamped Euler–Bernoulli columns, for different numbers, positions and damage intensities, is developed and discussed. It is shown that unpredictable results are encountered particularly for the clamped–clamped boundary conditions.

2. The Euler–Bernoulli column with multiple singularities

In this section the fourth order governing differential equation of the uniform Euler–Bernoulli columns, subjected to an axial compression load, in presence of multiple concentrated cracks is presented. In particular, concentrated cracks are modelled as Dirac's delta singularities superimposed to an uniform flexural stiffness. The fourth order governing differential equation is reduced to a second order differential equation, by making use of a new definition of the product of two Dirac's deltas, under a form suitable to be integrated in closed form.

The following model of uniform flexural stiffness with Dirac's delta singularities is adopted in this study:

$$E(x)I(x) = E_0 I_0\left[1 - \sum\nolimits_{i=1}^{n} \gamma_i \delta(x - x_{0i})\right] \tag{1}$$

where n singularities, given by Dirac's deltas centred at abscissae $x_{0i},\ i = 1, \ldots, n$, represent n concentrated cracks. The parameters $\gamma_i,\ i = 1, \ldots, n$ introduced in Eq. (1) multiplying the Dirac's deltas are related to the depth of the cracks as shown later.

The fourth order governing differential equation for buckling of the Euler–Bernoulli column with multiple singularities, under an axial compression load N, may be written as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[E_0 I_0\left(1 - \sum_{i=1}^{n} \gamma_i \delta(x - x_{0i})\right)\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\right] + N\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = 0 \tag{2}$$

where $u(x)$ is the transversal displacement function and x is the axial coordinate spanning from 0 to the length L of the column.

For simplicity, by considering the dimensionless coordinate $\xi = x/L$, and indicating with the apex the differentiation with respect to ξ, the differential equation (2) takes the following form:

$$\left[\left(1 - \sum_{i=1}^{n} \frac{\gamma_i}{L}\delta(\xi - \xi_{0i})\right)u''(\xi)\right]'' + \frac{NL^2}{E_0 I_0}u''(\xi) = 0. \tag{3}$$

In Eq. (3) the property $\delta(x - x_{0i}) = \delta[L(\xi - \xi_{0i})] = (1/L)\delta(\xi - \xi_{0i})$ of the Dirac's delta distribution has been exploited (Guelfand and Chilov, 1972; Hoskins, 1979; Lighthill, 1958; Zemanian, 1965). Furthermore, by introducing the dimensionless parameters $\hat{\gamma}_i = \gamma_i/L$ and the axial load parameter $\sigma^2 = NL^2/E_0 I_0$, Eq. (3) may be written as follows:

$$\left[\left(1-\sum_{i=1}^{n}\hat{\gamma}_i\delta(\xi-\xi_{0i})\right)u''(\xi)\right]''+\sigma^2u''(\xi)=0. \tag{4}$$

Double integration of Eq. (4) leads to:

$$\left[1-\sum_{i=1}^{n}\hat{\gamma}_i\delta(\xi-\xi_{0i})\right]u''(\xi)+\sigma^2u(\xi)=b_1+b_2\xi \tag{5}$$

where b_1 and b_2 are integration constants. Eq. (5) can also be written as follows:

$$u''(\xi)=b_1+b_2\xi-\sigma^2u(\xi)+\sum_{i=1}^{n}\hat{\gamma}_iu''(\xi)\delta(\xi-\xi_{0i}). \tag{6}$$

Eq. (6) cannot be solved explicitly, under this form, with respect to $u''(\xi)$. However, by multiplying both sides of Eq. (6) by $\delta(\xi-\xi_{0j})$, the following expression is obtained:

$$u''(\xi)\delta(\xi-\xi_{0j})=\left[b_1+b_2\xi-\sigma^2u(\xi)\right]\delta(\xi-\xi_{0j})+\sum_{i=1}^{n}\hat{\gamma}_i\delta(\xi-\xi_{0i})\delta(\xi-\xi_{0j})u''(\xi). \tag{7}$$

The product of two Dirac's delta distributions appearing in the last term of Eq. (7) can be treated by exploiting the following definition proposed by Bagarello (1995, 2002):

$$\delta(\xi-\xi_{0i})\delta(\xi-\xi_{0j})=\begin{cases}A\delta(\xi-\xi_{0i}) & i=j\\ 0 & i\neq j\end{cases} \tag{8}$$

according to which the product of two Dirac's deltas, both centred at ξ_{0i}, can be reduced to a single Dirac's delta multiplied by a constant A. It is worth noticing that the value of the constant A is irrelevant in this study since the solution, here proposed, will be shown in the sequel to be independent of the constant A.

Eq. (7), in view of Eq. (8) and accounting for the standard properties of the Dirac's delta, after some algebraic manipulations, leads to the following expression:

$$u''(\xi)\delta(\xi-\xi_{0j})=\frac{1}{1-\hat{\gamma}_jA}(b_1+b_2\xi-\sigma^2u(\xi))\delta(\xi-\xi_{0j})=\frac{1}{1-\hat{\gamma}_jA}(b_1+b_2\xi_{0j}-\sigma^2u(\xi_{0j}))\delta(\xi-\xi_{0j}). \tag{9}$$

Substitution of Eq. (9) into Eq. (6) leads to:

$$u''(\xi)+\sigma^2u(\xi)=b_1+b_2\xi+\sum_{i=1}^{n}B_i(u(\xi_{0i}))\delta(\xi-\xi_{0i}) \tag{10}$$

where, for simplicity, the following positions have been introduced:

$$B(u(\xi_{0i}))=\lambda_i[b_1+b_2\xi_{0i}-\sigma^2u(\xi_{0i})] \tag{11a}$$

$$\lambda_i=\frac{\hat{\gamma}_i}{1-\hat{\gamma}_iA}. \tag{11b}$$

In particular, the dimensionless parameters $\lambda_i,\ i=1,\dots n$, defined by Eq. (11b), will be considered in the sequel as "*damage parameters*" and adopted in the applications in order to represent the concentrated damages. The choice of the parameters $\lambda_i,\ i=1,\dots n$, as representative of concentrated damages, rather than the parameters $\hat{\gamma}_i=\gamma_i/L$ multiplying the Dirac's deltas in the governing Eq. (4), avoids the adoption of a specific value for the constant A, in fact the closed form solution reported in the following section is independent of the constant A.

Furthermore, the choice of the damage parameters λ_i is justified in Appendix where physical evidence is provided and its relationship with the crack depth is obtained by making use of the models available in the literature.

The fourth order governing differential equation of the Euler–Bernoulli column given by Eq. (2) containing singularities in the flexural stiffness has been reduced to the second order differential equation (10) under a form which is suitable for closed form integration as shown in the next section.

3. Integration procedure

The general solution of Eq. (10) is given by the solution of the corresponding homogeneous equation, $u_{\rm h}(\xi)$ and a particular integral $u_{\rm p}(\xi)$ as follows:

$$u(\xi) = u_{\rm h}(\xi) + u_{\rm p}(\xi) = \widehat{C}_3 \sin\sigma\xi + \widehat{C}_4 \cos\sigma\xi + u_p(\xi) \tag{12}$$

where $\widehat{C}_3,\ \widehat{C}_4$ are integration constants. We seek a particular integral $u_{\rm p}(\xi)$ under the following form:

$$u_{\rm p}(\xi) = d_1(\xi)\sin\sigma\xi + d_2(\xi)\cos\sigma\xi + C_1 + C_2\xi \tag{13}$$

$C_1,\ C_2$ being integration constants, and $d_1(\xi), d_2(\xi)$ unknown functions, of the normalized variable ξ, to be determined such that Eq. (10) is verified. The first derivative of $u_{\rm p}(\xi)$ given by Eq. (13) is:

$$u'_{\rm p}(\xi) = d_1(\xi)\sigma\cos\sigma\xi - d_2(\xi)\sigma\sin\sigma\xi + d'_1(\xi)\sigma\sin\sigma\xi + d'_2(\xi)\sigma\cos\sigma\xi + C_2. \tag{14}$$

The search of the particular solution, besides enforcement of the second order governing equation (10), will be performed under the following additional condition:

$$d'_1(\xi)\sigma\sin\sigma\xi + d'_2(\xi)\sigma\cos\sigma\xi = 0 \tag{15}$$

involving the first derivatives of the functions $d_1(\xi),\ d_2(\xi)$. Accounting for Eq. (15) leads to the following constrained form for the first derivative $u'_{\rm p}(\xi)$ of the particular integral:

$$u'_{\rm p}(\xi) = d_1(\xi)\sigma\cos\sigma\xi - d_2(\xi)\sigma\sin\sigma\xi + C_2 \quad \text{s.t.} \quad d'_1(\xi)\sigma\sin\sigma\xi + d'_2(\xi)\sigma\cos\sigma\xi = 0 \tag{16}$$

in which the derivatives of the unknown functions $d_1(\xi)$ and $d_2(\xi)$ are not involved. In view of Eq. (16) the second derivative $u''_{\rm p}(\xi)$ of the particular integral may be written as:

$$u''_{\rm p}(\xi) = -d_1(\xi)\sigma^2\sin\sigma\xi - d_2(\xi)\sigma^2\cos\sigma\xi + d'_1(\xi)\sigma\cos\sigma\xi - d'_2(\xi)\sigma\sin\sigma\xi. \tag{17}$$

Eq. (17), in view of Eq. (13), may be written as follows:

$$u''_{\rm p}(\xi) = -\sigma^2[u_{\rm p}(\xi) - C_1 - C_2\xi] + d'_1(\xi)\sigma\cos\sigma\xi - d'_2(\xi)\sigma\sin\sigma\xi. \tag{18}$$

By means of substitution of Eq. (18) into the equilibrium equation (10), the following expression is obtained

$$\begin{aligned} &-\sigma^2 u_{\rm p}(\xi) + \sigma^2 C_1 + \sigma^2 C_2\xi + d'_1(\xi)\sigma\cos\sigma\xi - d'_2(\xi)\sigma\sin\sigma\xi + \sigma^2 u_{\rm p}(\xi) \\ &\quad = b_1 + b_2\xi + \sum_{i=1}^{n} B_i(u(\xi_{0i}))\delta(\xi - \xi_{0i}). \end{aligned} \tag{19}$$

Since $u_{\rm p}(\xi)$ must satisfy Eq. (19) the following further conditions on constants b_1 and b_2 must be imposed

$$b_1 = \sigma^2 C_1; \quad b_2 = \sigma^2 C_2. \tag{20}$$

Eqs. (15) and (19) represent a linear first order differential system of equations with unknowns functions $d_1(\xi),\ d_2(\xi)$ written as follows:

$$\begin{cases} d'_1(\xi)\sigma\sin\sigma\xi + d'_2(\xi)\sigma\cos\sigma\xi = 0 \\ d'_1(\xi)\sigma\cos\sigma\xi - d'_2(\xi)\sigma\sin\sigma\xi = \sum\limits_{i=1}^{n} B_i(u(\xi_{0i}))\delta(\xi - \xi_{0i}) \end{cases} \tag{21}$$

where Eq. (20) has been accounted for. Solution of the differential system given by Eq. (21) leads to the following expressions for $d_1(\xi),\ d_2(\xi)$:

$$\begin{cases} d_1(\xi) = \dfrac{1}{\sigma}\displaystyle\sum_{i=1}^{n} B_i(u(\xi_{0i}))\cos\sigma\xi_{0i}U(\xi-\xi_{0i}) + c_1 \\ d_2(\xi) = -\dfrac{1}{\sigma}\displaystyle\sum_{i=1}^{n} B_i(u(\xi_{0i}))\sin\sigma\xi_{0i}U(\xi-\xi_{0i}) + c_2 \end{cases} \tag{22}$$

$c_1,\ c_2$ being integration constants. In view of expressions (22) the particular solution expressed by Eq. (13) can be written as follows:

$$\begin{aligned} u_{\mathrm{p}}(\xi) &= \frac{1}{\sigma}\sum_{i=1}^{n} B_i(u(\xi_{0i}))[\sin\sigma\xi\cos\sigma\xi_{0i} - \cos\sigma\xi\sin\sigma\xi_{0i}]U(\xi-\xi_{0i}) + c_1\sin\sigma\xi + c_2\cos\sigma\xi + C_1 + C_2\xi \\ &= \frac{1}{\sigma}\sum_{i=1}^{n} B_i(u(\xi_{0i}))\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) + c_1\sin\sigma\xi + c_2\cos\sigma\xi + C_1 + C_2\xi. \end{aligned} \tag{23}$$

Therefore it easy to verify that the general solution of the equilibrium equation (10) can be written as:

$$u(\xi) = \frac{1}{\sigma}\sum_{i=1}^{n} B_i(u(\xi_{0i}))\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) + C_1 + C_2\xi + C_3\sin\sigma\xi + C_4\cos\sigma\xi \tag{24}$$

where the positions $C_3 = c_1 + \widehat{C}_3,\ C_4 = c_2 + \widehat{C}_4$ have been considered. It has to be remarked that, being $B_i(u(\xi_{0i}))$, given by expression (11a), a function of $u(\xi_{0i})$, the transversal displacement function $u(\xi)$ depends on its values $u(\xi_{0i}),\ i = 1,\ldots,n$, at abscissae $\xi_{0i},\ i = 1,\ldots,n$, which are evaluated by Eq. (24), in view of the properties of the unit step distribution, as follows:

$$u(\xi_{0i}) = \frac{1}{\sigma}\sum_{j=1}^{i-1} B(u(\xi_{0j}))\sin\sigma(\xi_{0i}-\xi_{0j}) + C_1 + C_2\xi_{0i} + C_3\sin\sigma\xi_{0i} + C_4\cos\sigma\xi_{0i}. \tag{25}$$

Therefore the expression of $B(u(\xi_{0i}))$, given by Eq. (11a), can be written, in view of Eq. (25), as follows:

$$B(u(\xi_{0i})) = \lambda_i\left[b_1 + b_2\xi_{0i} - \sigma\sum_{j=1}^{i-1} B(u(\xi_{0j}))\sin\sigma(\xi_{0i}-\xi_{0j}) + C_1 + C_2\xi_{0i} + C_3\sin\sigma\xi_{0i} + C_4\cos\sigma\xi_{0i}\right]. \tag{26}$$

It is worth noticing that the expression of $B(u(\xi_{0i}))$ involves the values $B(u(\xi_{0j})),\ j = 1,\ldots,i-1$, hence for all the abscissae ξ_{0j} with $j < i,\ i = 1,\ldots,n$. Eq. (26) can be given the following extended form:

$$\begin{aligned} B(u(\xi_{0i})) = \lambda_i\Bigg\{ & b_1 + b_2\xi_{0i} - \sigma\sum_{j=1}^{i-1}\lambda_j\Bigg[b_1 + b_2\xi_{0j} - \sigma\sum_{k=1}^{j-1} B(u(\xi_{ok}))\sin\sigma(\xi_{0j}-\xi_{ok}) + C_1 + C_2\xi_{0j} + C_3\sin\sigma\xi_{0j} \\ & + C_4\cos\sigma\xi_{0j}\Bigg]\sin\sigma(\xi_{0i}-\xi_{0j}) + C_1 + C_2\xi_{0i} + C_3\sin\sigma\xi_{0i} + C_4\cos\sigma\xi_{0i}\Bigg\}. \end{aligned} \tag{27}$$

In view of expressions (24) and (26), the general solution of the equilibrium equation (10) takes the following form:

$$\begin{aligned} u(\xi) = \frac{1}{\sigma}\sum_{i=1}^{n}\lambda_i & \left[b_1 + b_2\xi_{0i} - \sigma\sum_{j=1}^{i-1} B(u(\xi_{0j}))\sin\sigma(\xi_{0i}-\xi_{0j}) + C_1 + C_2\xi_{0i} + C_3\sin\sigma\xi_{0i} + C_4\cos\sigma\xi_{0i}\right] \\ & \sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) + C_1 + C_2\xi + C_3\sin\sigma\xi + C_4\cos\sigma\xi \end{aligned} \tag{28}$$

The explicit expression of the transversal displacement function $u(\xi)$ is obtained by Eq. (28) as follows

$$\begin{aligned} u(\xi) = C_1 + C_2\xi &+ C_3\left[\sum_{i=1}^{n}\mu_i(\xi_1,\ldots,\xi_i)\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) + \sin\sigma\xi\right] \\ &+ C_4\left[\sum_{i=1}^{n}v_i(\xi_1,\ldots,\xi_i)\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) + \cos\sigma\xi\right] \end{aligned} \tag{29}$$

where

$$\mu_i(\xi_1,\ldots,\xi_i) = -\sigma\lambda_i\left[\sum_{j=1}^{i-1}\mu_j \sin\sigma\big(\xi_{0i}-\xi_{0j}\big) + \sin\sigma\xi_{0i}\right] \tag{30}$$

$$v_i(\xi_1,\ldots,\xi_i) = -\sigma\lambda_i\left[\sum_{j=1}^{i-1}v_j \sin\sigma(\xi_{0i}-\xi_{0j}) + \cos\sigma\xi_{0i}\right]. \tag{31}$$

Eq. (29), together with the positions given by Eqs. (30) and (31), is the sought exact closed form solution of the fourth order differential buckling equation (10). The transversal displacement function $u(\xi)$, provided by Eq. (29), represents the exact buckling mode correspondent to the buckling load expressed by means of the load parameter $\sigma = (NL^2/E_0I_0)^{1/2}$. The integration constants $C_1,\ C_2,\ C_3,\ C_4$ in Eq. (29) have to satisfy the boundary conditions. The exact buckling load equation can be obtained from the fourth order determinant of boundary condition system of equations.

The derivatives of the exact closed form solution (29) can be written as

$$\begin{aligned} u'(\xi) = {} & C_2 + C_3\left[+\sigma\cos\sigma\xi + \sum_{i=1}^{n}\mu_i(\xi_1,\ldots,\xi_i)\sigma\cos\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i})\right] \\ & + C_4\left[-\sigma\sin\sigma\xi + \sum_{i=1}^{n}v_i(\xi_1,\ldots,\xi_i)\sigma\cos\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i})\right] \end{aligned} \tag{32}$$

$$\begin{aligned} u''(\xi) = {} & C_3\left\{-\sigma^2\sin\sigma\xi - \sigma\sum_{i=1}^{n}\mu_i(\xi_1,\ldots,\xi_i)[\sigma\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) - \delta(\xi-\xi_{0i})]\right\} \\ & + C_4\left\{-\sigma^2\cos\sigma\xi - \sigma\sum_{i=1}^{n}v_i(\xi_1,\ldots,\xi_i)[\sigma\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) - \delta(\xi-\xi_{0i})]\right\} \end{aligned} \tag{33}$$

$$\begin{aligned} u'''(\xi) = {} & C_3\left\{-\sigma^3\cos\sigma\xi - \sigma\sum_{i=1}^{n}\mu_i(\xi_1,\ldots,\xi_i)\big[\sigma^2\cos\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) - \delta'(\xi-\xi_{0i})\big]\right\} \\ & + C_4\left\{+\sigma^3\sin\sigma\xi - \sigma\sum_{i=1}^{n}v_i(\xi_1,\ldots,\xi_i)\big[\sigma^2\cos\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i}) - \delta'(\xi-\xi_{0i})\big]\right\}. \end{aligned} \tag{34}$$

Eqs. (32)–(34) are those to be adopted to impose the boundary conditions and derive the correspondent buckling load equation as shown in the next section.

The crack model, adopted in this study, making use of Dirac's deltas, allowed the above described integration procedure; however this model can be related to the classical crack models provided in the literature as shown in Appendix.

4. The buckling load equation of multi-cracked columns

The buckling load equation can be derived for any multi-cracked column by simply imposing the standard boundary conditions, including the general case of rotational and translational spring supports. In this section, the closed form solution presented in Eqs. (29)–(34) are adopted to treat the case of simply supported and clamped–clamped Euler–Bernoulli columns. The buckling load equations are derived and numerically solved in order to obtain the critical loads of the considered multi-cracked columns and the corresponding modes. Furthermore a parametric study for different numbers, positions and values of the damage parameters is presented.

In particular, it has to be noted that, since the damage parameters λ_i have been chosen as representative of the damage intensities, the correspondent crack depth can be easily inferred by the graph reported in Fig. A1 in Appendix, where the relationships with some of the existing damage models have been plotted according to Eq. (A17).

4.1. Simply supported column

The boundary conditions of the simply supported column are

$$u(0) = 0; \quad u''(0) = 0; \quad u(1) = 0; \quad u''(1) = 0. \tag{35}$$

In view of Eqs. (29) and (33) the following conditions for the integration constants C_1, C_2, C_3, C_4, can be written

$$C_1 = C_4 = 0 \tag{36}$$

$$C_2 + \left[\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i) \sin\sigma(1-\xi_{0i}) + \sin\sigma\right] C_3 = 0 \tag{37}$$

$$C_3 \left\{\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i) \sin\sigma(1-\xi_{0i}) + \sin\sigma\right\} = 0 \tag{38}$$

from which the buckling load equation is obtained

$$\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i) \sin\sigma(1-\xi_{0i}) + \sin\sigma = 0 \tag{39}$$

whose zeros provide the values of the critical load parameters σ_{cr_k}. Eq. (39) is the generalization to the multi-cracked column of the buckling load equation presented by Yavari and Sarkani (2001) for the column with a single internal hinge. If all the damage parameters λ_i are zero, i.e. no crack occurs, Eq. (39) reduces to the buckling load equation of the undamaged simply supported column.

In view of Eq. (39) solutions of the system of linear Eqs. (36)–(38) is

$$C_1 = C_2 = C_4 = 0; \quad C_3 \neq 0. \tag{40}$$

Hence the buckling modes of a simply supported multi-cracked column are given by

$$\phi_k(\xi) = C_3\left[\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i) \sin\sigma_{\mathrm{cr}_k}(\xi-\xi_{0i}) U(\xi-\xi_{0i}) + \sin\sigma_{\mathrm{cr}_k}\xi\right] \quad \text{for} \quad k = 1, 2 \ldots \infty. \tag{41}$$

In Fig. 1 the buckling behaviour of a multi-cracked simply supported column is investigated. Namely, Fig. 1a report, in sequence, the critical loads as functions of the damage parameter for different numbers of equally spaced concentrated damages with the same intensity $\lambda_i = \lambda,\ i = 1, \ldots, n$; while in Fig. 1b the corresponding buckling modes, for all the concentrated cracks, are represented.

In Fig. 2 the first critical load and the corresponding mode shapes reported in Fig. 1 are reported simultaneously for comparison.

The effect of the position of a single crack in the simply supported column is investigated in Fig. 3a where the critical load parameter is reported as function of the crack position for two values of the damage parameter $\lambda = 0.5$, 1. It can be observed that the critical load decreases when the crack moves from the ends to the middle section of the column. With reference to the same damage parameters, in Fig. 3b the influence of the position of two cracks symmetrically collocated in the column is reported. Also in this case, it can be observed as the critical load decreases as the cracks move towards the middle section of the column.

In Fig. 4 the critical load parameter of the multi-cracked simply supported column is represented as function of the number of equally spaced cracks, variable from 0 to 50, and of the relative damage parameters λ_i.

4.2. Clamped–clamped column

For a clamped–clamped column, the boundary conditions at the left and right ends are

$$u(0) = 0; \quad u'(0) = 0; \quad u(1) = 0; \quad u'(1) = 0, \tag{42}$$

by expressing the conditions (42) through the Eqs. (29) and (32), the following homogeneous linear system for the unknown integration constants can be derived

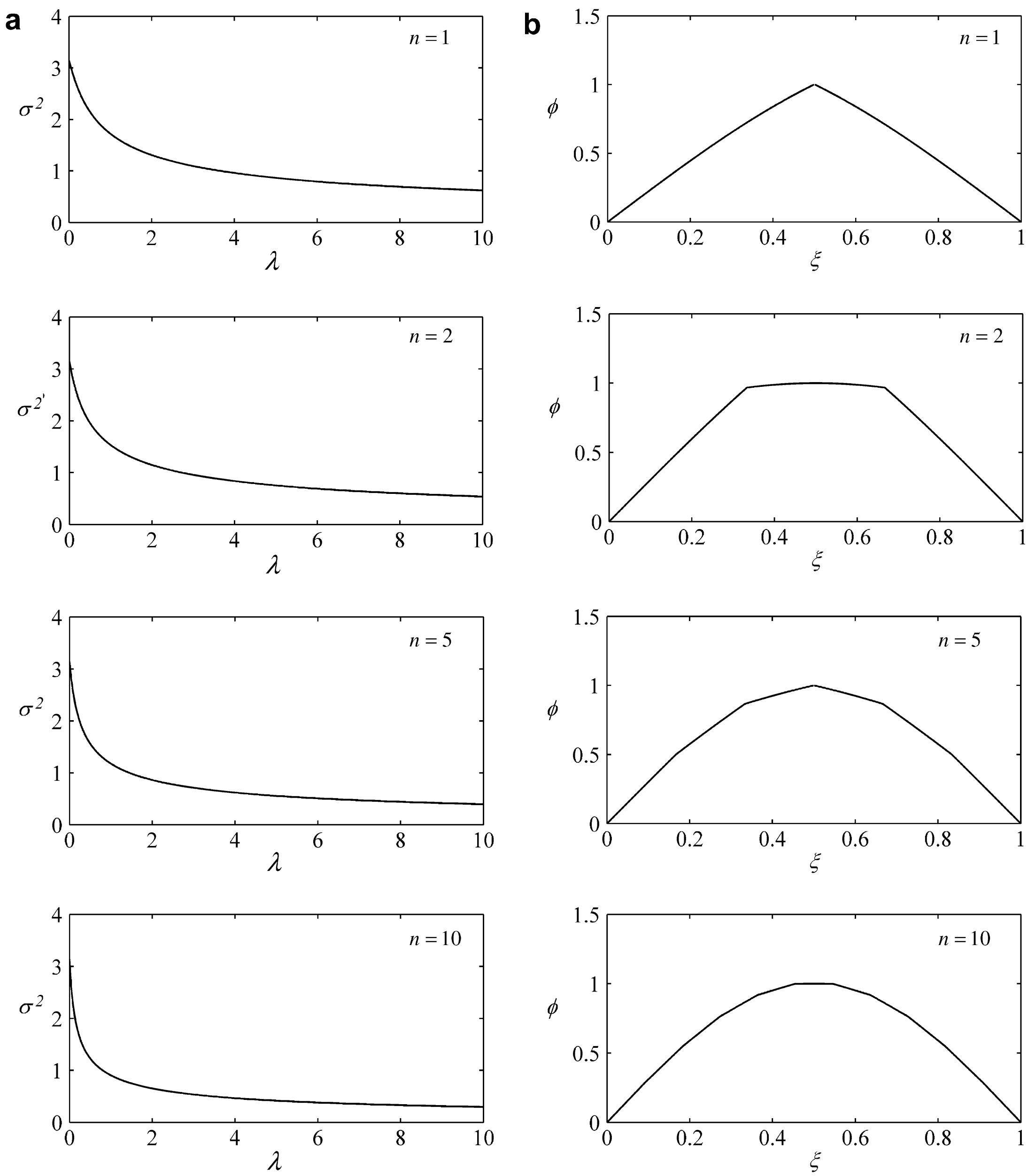

Fig. 1. Buckling behaviour of a multi-cracked simply supported column for different numbers n of equally spaced concentrated damages. (a) First critical load parameter σ^2 versus the damage parameter $\lambda_i = \lambda,\ i = 1, \ldots, n$; (b) first buckling mode ϕ for $\lambda = 1$.

$$C_1 = -C_4 \tag{43}$$

$$C_2 = -\sigma C_3 \tag{44}$$

$$C_1\left[\sum_{i=1}^{n} v_i(\xi_1, \ldots, \xi_i)\sin\sigma(1-\xi_{0i}) + \cos\sigma - 1\right] - C_3\left[\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i)\sin\sigma(1-\xi_{0i}) + \sin\sigma - \sigma\right] = 0 \tag{45}$$

$$C_1\left[\sum_{i=1}^{n} v_i(\xi_1, \ldots, \xi_i)\cos\sigma(1-\xi_{0i}) - \sin\sigma\right] - C_3\left[\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i)\cos\sigma(1-\xi_{0i}) + \cos\sigma - 1\right] = 0. \tag{46}$$

By setting equal to zero the corresponding system matrix determinant, the following expression of the exact buckling load equation is obtained

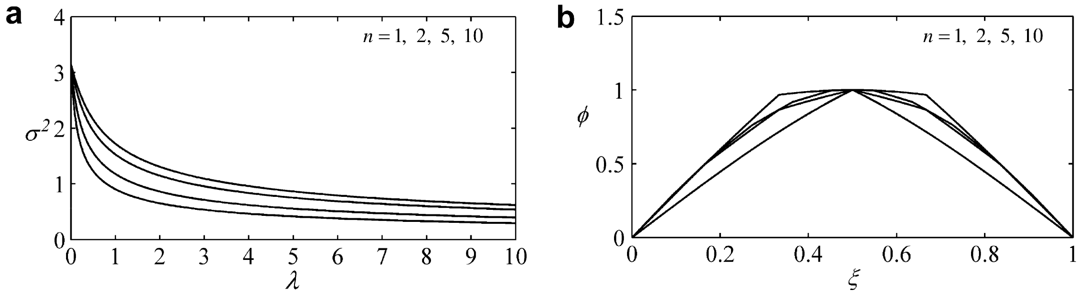

Fig. 2. Buckling behaviour of a multi-cracked simply supported column for different numbers n of equally spaced concentrated damages. (a) First critical load parameter σ^2 versus the damage parameter $\lambda_i = \lambda,\ i = 1, \ldots, n$; (b) first buckling mode ϕ for $\lambda = 1$.

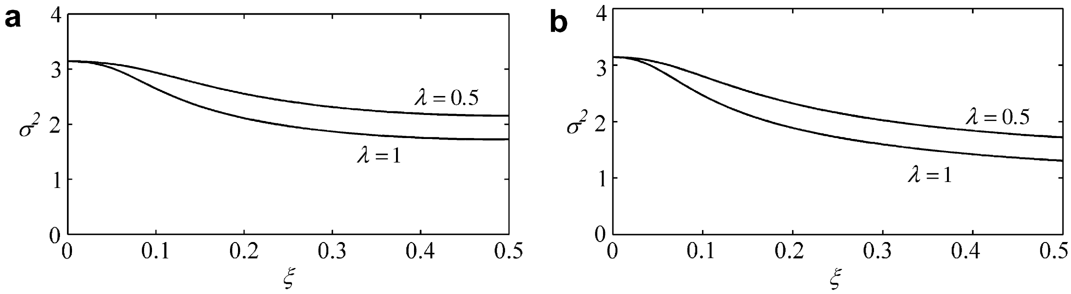

Fig. 3. Simply supported column. First critical load parameter σ^2 versus the crack positions for two values of the damage parameter $\lambda_i = 0.5,\ 1$. (a) Single-cracked column; (b) double-cracked column.

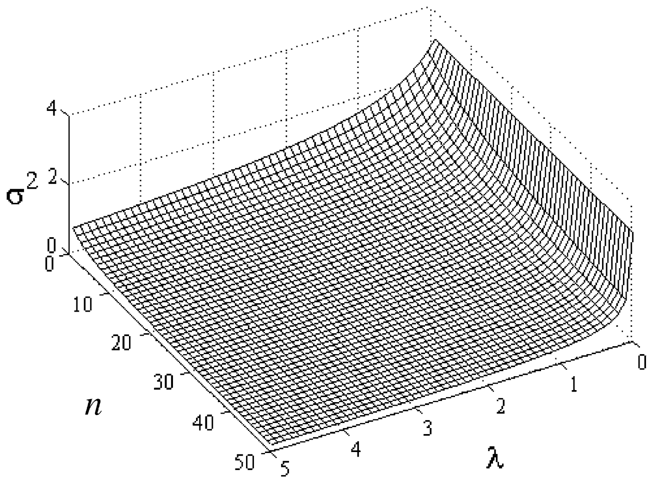

Fig. 4. Multi-cracked simply supported column. First critical load parameter σ^2 versus the number n of equally spaced cracks and the damage parameter $\lambda_i = \lambda,\ i = 1, \ldots, n$.

$$
\begin{aligned}
&(1-\cos\sigma)\left[\sum_{i=1}^{n} \upsilon_i \sin\sigma(1-\xi_{0i}) + \sum_{i=1}^{n} \mu_i \cos\sigma(1-\xi_{0i}) - 2\right] + (\sin\sigma - \sigma)\sum_{i=1}^{n} \upsilon_i \cos\sigma(1-\xi_{0i}) \\
&+ \sin\sigma\left[\sigma - \sum_{i=1}^{n} \mu_i \sin\sigma(1-\xi_{0i})\right] + \sum_{i=1}^{n} \upsilon_i \cos\sigma(1-\xi_{0i}) \sum_{j=1}^{n} \mu_j \sin\sigma(1-\xi_{0j}) \\
&- \sum_{i=1}^{n} \upsilon_i \sin\sigma(1-\xi_{0i}) \sum_{j=1}^{n} \mu_j \cos\sigma(1-\xi_{0j}) = 0 \qquad (47)
\end{aligned}
$$

whose zeros provide the values of the critical load parameters σ_{cr_k} of the multi-cracked clamped–clamped column.

By substituting the critical load parameter in the boundary condition system of Eqs. (43)–(46), the value of the integration constants that provide the buckling mode of the clamped–clamped multi-cracked column can be obtained as follows

$$C_1 = \frac{1 - \sum_{j=1}^{n} \mu_j \cos \sigma_{\mathrm{cr}_K} \left(1 - \xi_{0j}\right) - \cos \sigma_{\mathrm{cr}_K}}{\sin \sigma_{\mathrm{cr}_K}} C_3; \quad C_2 = -\sigma_{\mathrm{cr}_K} C_3; \quad C_4 = C_1. \tag{48}$$

By replacing into Eq. (29) the values of the integration constants given by Eqs. (48), the closed form expressions of the buckling modes of a clamped–clamped multi-cracked column can be written as follows

$$\begin{aligned}\phi_k(\xi) = {} & C_3 \frac{1 - \sum_{j=1}^{n} \mu_j \cos \sigma_{\mathrm{cr}_K} \left(1 - \xi_{0j}\right) - \cos \sigma_{\mathrm{cr}_K}}{\sin \sigma_{\mathrm{cr}_K}} \left[1 + \sum_{i=1}^{n} v_i(\xi_1, \ldots, \xi_i) \sin \sigma_{\mathrm{cr}_K} (\xi - \xi_{0i}) U(\xi - \xi_{0i}) + \cos \sigma_{\mathrm{cr}_K} \xi \right] \\ & + C_3 \left[\sum_{i=1}^{n} \mu_i(\xi_1, \ldots, \xi_i) \sin \sigma_{\mathrm{cr}_K} (\xi - \xi_{0i}) U(\xi - \xi_{0i}) + \sin \sigma_{\mathrm{cr}_K} \xi - \sigma_{\mathrm{cr}_K} \xi \right].\end{aligned} \tag{49}$$

Eqs. (47) and (49) allow a study of the clamped–clamped multi-cracked column which is conducted in what follows for varying number, position and intensity of the cracks.

In Figs. 5 and 6 the case of a single crack located at the middle cross-section is treated. In particular, Fig. 5a reports the first critical load as function of the damage parameter, while in Fig. 5b the correspondent buckling mode is represented. The effect of the position of a single crack is investigated in Fig. 6 where the critical load parameter is reported as function of the crack position for two values of the damage parameter $\lambda = 0.5,\ 1$. It can be observed that the critical load increases when the crack moves from the clamped ends towards the cross-section where the inflection point of the undamaged column buckling mode is encountered, reaching the buckling load of the undamaged column. Then, the buckling load decreases when the crack moves towards the middle cross-section.

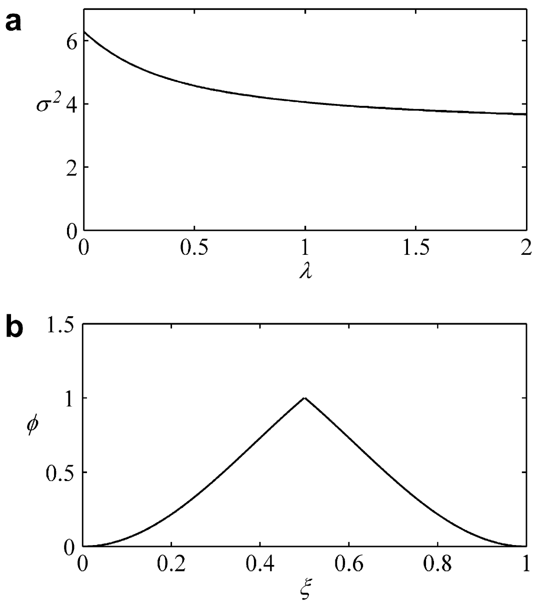

Fig. 5. Single-cracked clamped–clamped column. (a) First critical load parameter σ^2 versus the damage intensity parameter λ; (b) first buckling mode.

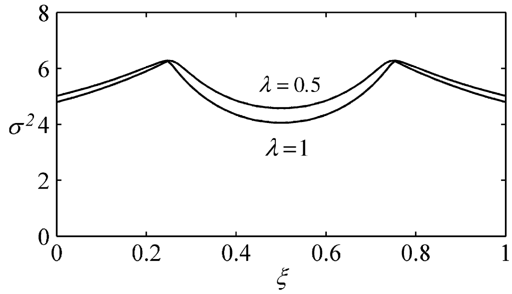

Fig. 6. Single-cracked clamped–clamped column. First critical load parameter σ^2 versus the crack positions for two values of the damage parameter $\lambda_i = 0.5,\ 1$.

In Figs. 7 and 8 the case of two cracks, equally spaced along the beam span and with the same damage parameters $\lambda_1 = \lambda_2 = \lambda$, is treated. In particular, Fig. 7a reports the first critical load as function of the damage parameter λ. The inspection of Fig. 7a reveals that the critical load curve, for the case of the double-cracked column, exhibits a corner point for the value of the damage parameter $\lambda = 0.38$. This interesting and apparently unusual behaviour can be justified by observing the stability mode shapes of the double-cracked column plotted in Fig. 7b and c. Fig. 7b shows the buckling mode concerning damage parameter values $\lambda < 0.38$,

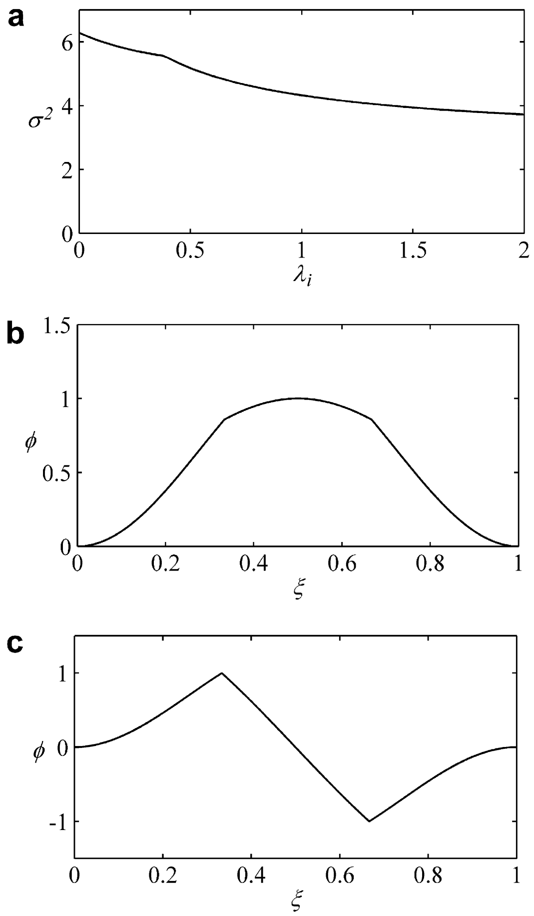

Fig. 7. Double-cracked clamped–clamped column. (a) First critical load parameter σ^2 versus the damage intensity parameter $\lambda_1 = \lambda_2 = \lambda$; (b) first symmetric buckling mode; (c) first anti-symmetric buckling mode.

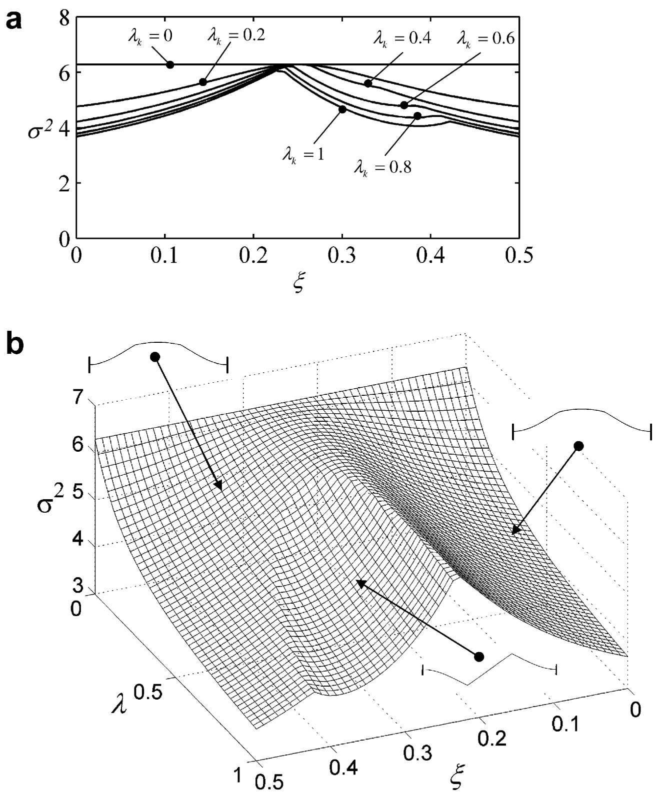

Fig. 8. Double-cracked clamped–clamped column. (a) First critical load parameter σ^2 versus the crack position for different values of the damage parameter λ_i; (b) first critical load parameter σ^2 versus the crack position ξ and the damage parameter λ.

while Fig. 7c shows the buckling mode concerning damage parameter values $\lambda > 0.38$. It can be observed that the buckling mode of the double-cracked clamped–clamped column can be symmetric or anti-symmetric and the value of $\lambda = \lambda_{\text{lim}} = 0.38$, corresponding to the corner point in Fig. 7a, represents the value which separates the two different behaviours. Specifically for $\lambda \leqslant \lambda_{\text{lim}}$ the double-cracked clamped–clamped column exhibits a symmetric mode shape while for values of $\lambda > \lambda_{\text{lim}}$ the column shows an anti-symmetric buckling mode. To the authors' knowledge, this unexpected behaviour has not been evidenced previously in the literature.

The effect of the position of two cracks equidistant ξ from the clamped ends is investigated in Fig. 8a. Precisely, in Fig. 8a the critical load parameter is reported as function of the crack positions ξ with respect to the clamped ends, when the two cracks move simultaneously towards the middle cross-section, for some values of the damage parameter $\lambda = 0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1$. It can be observed that the critical load increases according to the symmetric buckling mode reported in Fig. 7b, when the crack moves from the clamped ends towards the cross-section where the inflection point of the buckling mode is encountered, reaching the buckling load of the undamaged column. Then, the buckling load of the column decreases, switching to the anti-symmetric buckling mode depicted in Fig. 7c, when the crack moves towards the middle cross-section; finally a corner point, whose position depends on the value of the damage parameter, indicating that the symmetric buckling mode is again recovered, is encountered.

The simultaneous effect of the intensity and the position of two cracks is investigated in Fig. 8b where the surface representing the critical load values as function of the position of two cracks, equidistant ξ with respect to the clamped ends, and of the damage parameter $\lambda_1 = \lambda_2 = \lambda$ is reported. The inspection of Fig. 8b shows that a smooth region of the buckling load surface, coupled with the symmetric buckling mode, can be recognized and provides the first buckling load value for a wide range of intensity and position damage parameters. However, a local valley of the buckling surface, coupled with the anti-symmetric mode, is evident and provides an abrupt decrement of the buckling load value with respect to the expected one concerning the symmetric mode. Hence, some care must be considered for the evaluation of the first buckling load value of clamped–clamped double-cracked columns with respect to the specific values of intensity and position of the damages.

The effect of three cracks, located at cross-sections $\xi_1 = 1/3$, $\xi_2 = 1/2$, $\xi_1 = 2/3$, on a clamped–clamped column is investigated in Fig. 9. Precisely, in Fig. 9a the critical load parameter is reported as function of the damage intensity parameter λ, where two different cases have been treated: (i) the three cracks maintain the same intensity $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$; (ii) the two external cracks have been assumed with constant damage intensity $\lambda_1 = \lambda_3 = 2$, and the central crack possesses a variable intensity λ. For case (i) a regular decrement of the buckling load can be observed, coupled with the symmetric buckling mode reported in Fig. 9b. For case (ii), for the value $\lambda_2 = \lambda = 0$ (i.e. no central crack) the buckling load value $\sigma^2 = 3.725$ of the column with two equally spaced cracks (reported in Fig. 7a for $\lambda = 2$), correspondent to the anti-symmetric mode reported in Fig. 7b, is recovered. Furthermore, when $\lambda_2 = \lambda$ increases, it is worth noticing that the buckling mode maintains the anti-symmetric character, peculiar of the double-cracked column, also reported in Fig. 9c. Until the buckling mode maintains the anti-symmetric character, regardless of the presence of the central crack with $\lambda_2 = \lambda$, the buckling load keeps the double-cracked column value $\sigma^2 = 3.725$. For the value $\lambda = 0.922$ the curve shows a corner point and the buckling load starts to decrease coupled with the symmetric mode.

Further investigation concerning the multi-cracked clamped–clamped column has been conducted for an increasing number of cracks, but the results are not reported in this study for brevity. However, it has to be noted that the anti-symmetric mode of the double-cracked beam is encountered for all those cases where the external cracks possess damage intensities higher than the internal ones. While, for multi-cracks with equal

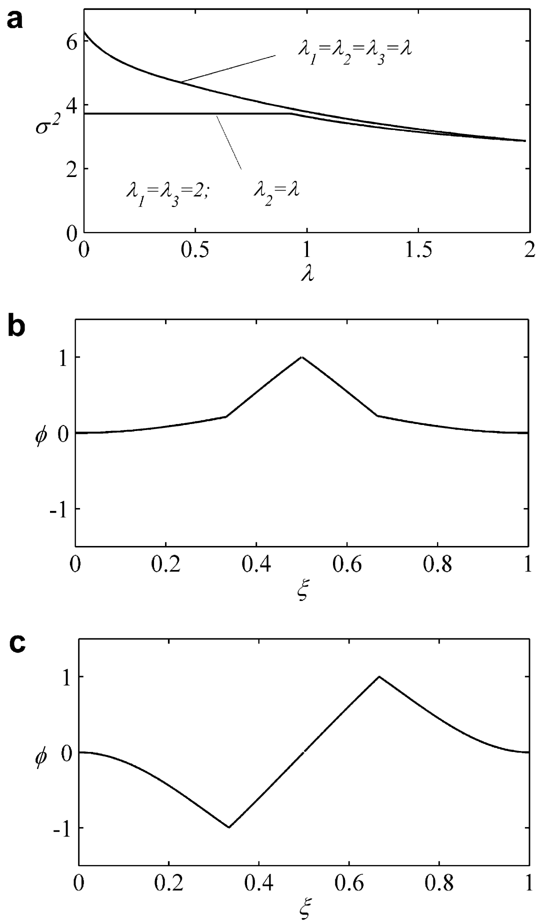

Fig. 9. Triple-cracked clamped–clamped column. (a) First critical load parameter σ^2 versus the damage intensity parameter λ; (b) first symmetric buckling mode; (c) first anti-symmetric buckling mode.

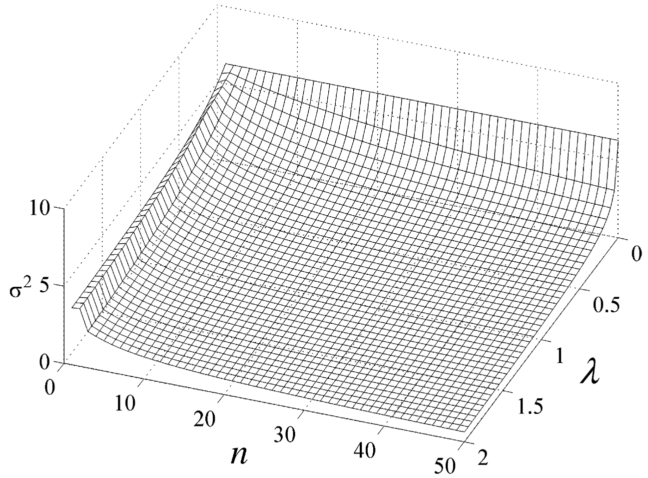

Fig. 10. Multi-cracked clamped–clamped column. First critical load parameter σ^2 versus the number n of equally spaced cracks and the damage parameter $\lambda_i = \lambda,\ i = 1, \dots, n$.

intensity, a regular behaviour (except for the case of two cracks) can be observed, as it is shown in Fig. 10 where the buckling load surface, as function of the number n of equally spaced cracks and of their intensity λ, is reported.

5. Conclusions

In this paper the exact solution of the uniform Euler–Bernoulli column in presence of multiple concentrated cracks, modelled by means of Dirac's deltas, has been derived. The exact explicit expressions of the stability mode shapes as functions of four integration constants only have been obtained by means of an original integration procedure based on a new definition of the product of Dirac's deltas, which is not allowed by the classical distribution theory.

The buckling load equation for any number, position and intensity of the cracks can be derived for any column by simply enforcing the boundary conditions at the ends of the column and without any additional continuity condition. As an example, a parametric study of the multi-cracked simply supported and clamped–clamped Euler–Bernoulli columns, for different numbers, positions and damage intensities, has been developed and discussed. While the case of multi-cracked simply supported beam shows expected regular results, on the contrary the clamped–clamped column presents some unexpected behaviour whose evidence has been possible in view of the presented exact closed form solution.

Appendix. Relationship between damage parameters and crack depth parameters

In this appendix the damage parameters λ_i, adopted in this study to represent concentrated damages and related to the singularity parameters γ_i, appearing in Eq. (1), are shown to be related to the depth of concentrated cracks by making use of the classical crack models provided in the literature.

The slope function $\varphi(x) = -\frac{du}{dx}$, written as function of the dimensionless coordinate ξ as $\varphi(\xi) = -u'(\xi)/L$, and directly obtained by Eq. (32), shows jump discontinuities $\Delta\varphi(\xi_{0k})$ at abscissae $\xi_{0k},\ k = 1, \dots, n$, that are explicitly evaluated as follows:

$$\Delta\varphi(\xi_{0k}) = \varphi(\xi_{0k}^{+}) - \varphi(\xi_{0k}^{+}) = -\frac{C_3}{L}\sigma\mu_k(\xi_1, \dots, \xi_k) - \frac{C_4}{L}v_k(\xi_1, \dots, \xi_k), \quad k = 1, \dots, n \tag{A1}$$

where ξ_{0k}^{+} and ξ_{0k}^{-} are the abscissae at the right and at the left of ξ_{0k}, respectively. The discontinuities $\Delta\varphi(\xi_{0k})$ provided by Eq. (A1) represent the relative rotations between the cross-sections at ξ_{0k}^{+} and ξ_{0k}^{-}, consequence of the adopted flexural stiffness model.

On the other hand, the bending moment $M(\xi) = -EI(\xi)u''(\xi)/L^2$, can be obtained by means of Eq. (33), after some algebraic manipulations and accounting for the product of two Dirac's deltas, as follows:

$$M(\xi) = \frac{E_0 I_0}{L^2}\sigma^2\Big\{C_3\Big[\sin\sigma\xi + \sum\nolimits_{i=1}^{n}\mu_i(\xi_1,\ldots,\xi_i)\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i})\Big] \\ +C_4\Big[\cos\sigma\xi + \sum\nolimits_{i=1}^{n}v_i(\xi_1,\ldots,\xi_i)\sin\sigma(\xi-\xi_{0i})U(\xi-\xi_{0i})\Big]\Big\}. \tag{A2}$$

The values of the bending moment $M(\xi_{0k})$ at the cross-sections of abscissae $\xi_{0k},\ k = 1,\ldots,n$, can be evaluated, according to Eq. (A2), as follows:

$$M(\xi_{0k}) = \frac{E_0 I_0}{L^2}\frac{\sigma}{\lambda_k}[C_3\mu_k(\xi_1,\ldots,\xi_k) + C_4 v_k(\xi_1,\ldots,\xi_k)], \quad k = 1,\ldots,n. \tag{A3}$$

Comparison of the bending moments given by Eq. (A3) and the slope discontinuities given by Eq. (A1) leads to

$$\Delta\varphi(\xi_{0k}) = \lambda_k\frac{L}{E_0 I_0}M(\xi_{0k}), \quad k = 1,\ldots,n. \tag{A4}$$

Eq. (A4) provides the relationship between the slope discontinuities $\Delta\varphi(\xi_{0k})$ and the bending moments $M(\xi_{0k})$ at $\xi_{0k},\ k = 1,\ldots,n$, and suggests the interpretation of the adopted flexural stiffness model as internal hinges at ξ_{0k} endowed with rotational spring stiffnesses K_k^{φ} given as:

$$K_k^{\varphi} = \frac{E_0 I_0}{\lambda_k L}, \quad k = 1,\ldots,n. \tag{A5}$$

Eq. (A5) represents the relationship between K_k^{φ} and the dimensionless damage parameters λ_k. It has to be noted that: for $\lambda_k = 0$, correspondent to the presence of no crack, Eq. (A5) provides $K_k^{\varphi} = \infty$; on the other hand, for $\lambda_k = \infty$, correspondent to an entirely damaged cross-section, Eq. (A5) provides $K_k^{\varphi} = 0$.

In view of Eq. (A5) the damage parameters λ_k can be written as follows:

$$\lambda_k = \frac{E_0 I_0}{L}\frac{1}{K_k^{\varphi}}, \quad k = 1,\ldots,n. \tag{A6}$$

Hence they represent the flexibilities, normalized with respect to $L/(E_0 I_0)$, of the rotational springs equivalent to the flexural stiffness singularities introduced into the model adopted in Eq. (1).

Eq. (A4), together with Eq. (A5), shows that the Dirac's delta distributions introduced in the flexural stiffness model lead to closed form solutions correspondent to the presence of elastic rotational springs. However, in order to represent cracks by means of the present Dirac's delta distribution approach, a relationship between the damage parameters λ_k and the crack depth has to be sought.

In the literature various models of concentrated open cracks leading to a continuous description of the beam flexibility in the vicinity of the crack have been proposed. In what follows some of them are recalled in order to show how they can be related to the distribution model presented in this work.

In particular, Christides and Barr (1984) considered the effect of a crack, located at ξ_0, on a rectangular cross-section beam and proposed the following stiffness reduction involving an exponential function:

$$EI(\xi) = E_0 I_0\frac{g_c}{g_c + (1-g_c)\exp\left[-2\frac{\alpha}{\hat{h}}|\xi-\xi_0|\right]} \tag{A7}$$

where $\hat{h} = h/L$, for a rectangular cross-section, is defined as the ratio between the cross-section height h and the length of the beam; $g_c = I_c/I_0$ is the ratio between the moment of inertia of the damaged I_c and the undamaged I_0 cross-section; α is a constant, estimated from experimental tests to be 0.667, accounting for the effective portion of the beam whose flexural stiffness is affected by the damage. The stiffness reduction proposed by Christides and Barr is not local, hence, the integration required to evaluate the stiffness matrix for the beam would have to be performed numerically for different crack positions. However most of the flexibility is local to the crack, although small changes if flexibility are encountered away from the crack.

Sinha et al. (2002) proposed a simplified form of the stiffness reduction due to the concentrated crack according to a triangular variation starting from an effective length L_c on either side of the crack location, given as follows:

$$EI(\xi) = E_0 I_0 \left[g_c + (1 - g_c) \frac{|\xi - \xi_0|}{\widehat{L}_c} \right] \quad \text{for} \quad \xi_0 - \widehat{L}_c \leqslant \xi \leqslant \xi_0 + \widehat{L}_c \tag{A8}$$

where $\widehat{L}_c = L_c/L$ is the effective length L_c normalized with respect to the length L of the beam. The effective length L_c of the stiffness reduction due to the crack was determined to be $L_c = h/\alpha = 1.5h$, obtained by making equal the integral of the stiffness reduction proposed by Christides and Barr (1984) in Eq. (A7) and that proposed in Eq. (A8). The effective length does not depend on the crack depth but depends on the beam height h.

Cerri and Vestroni (2003) proposed a constant stiffness reduction, due to a concentrated crack, equal to the flexural stiffness of the damaged cross-section, limited to an effective length L_c, as follows:

$$EI(\xi) = E_0 I_0 g_c \quad \text{for} \quad \xi_0 - \widehat{L}_c \leqslant \xi \leqslant \xi_0 + \widehat{L}_c \tag{A9}$$

where the effective length L_c is the zone characterized by constant reduced stiffness and it has been evaluated through an equivalence between this model and that proposed by Christides and Barr (1984).

Another model for the stiffness reduction has been proposed by Bilello (2001), for a rectangular cross-section. Bilello followed the idea of the identification of an ineffective area around the crack that has approximately a triangular shape the height of which corresponds to the crack depth d, while the width $2L_c$ is related to it by the relationship

$$d/L_c = \tan\theta = 0.9. \tag{A10}$$

The expression concerning the effective length L_c reported in Eq. (A10) was obtained by numerical simulations and confirmed by experimental tests.

The main disadvantage of the above-mentioned models is that experimental tests or numerical simulation are required in order to estimate the parameters involved in the stiffness reduction. In Chondros et al. (1998) this difficulty is overcome since they developed a continuous cracked beam theory extending the procedure proposed by Christides and Barr (1984) but using the results from fracture mechanics theory to estimate the *crack disturbance function*.

A different approach for modelling the effect of concentrated cracks on the flexural stiffness is based on the introduction of an elastic hinge, a local compliance, which quantifies in a macroscopic way the relation between the applied load and the strain concentration surrounding the crack (Irwin, 1957a,b; Freund and Hermann, 1976; Gounaris and Dimarogonas, 1988). By following the latter approach, direct expressions of the elastic rotational spring stiffness equivalent to the crack have been provided in the literature and some of them, dependent on the crack depth are recalled in the sequel for the case of a rectangular cross-section.

For example, when a lateral crack of uniform depth d is present in a rectangular cross-section of width b and height h, the following expression for the stiffness K^{eq}, in order to unify the treatment of the models proposed in the literature, is adopted:

$$K^{eq} = \frac{E_0 I_0}{h} \frac{1}{C(\beta)} \tag{A11}$$

where $\beta = d/h$ is defined as the ratio between the crack depth d and the cross-section height h, and $C(\beta)$ is the dimensionless local compliance.

According to Liebowitz et al. (1967), Liebowitz and Claus (1968), Okamura et al. (1969), Rizos et al. (1990), the local compliance $C(\beta)$, computed from the strain energy density function, takes the following form:

$$C(\beta) = 5.346(1.86\beta^2 - 3.95\beta^3 + 16.375\beta^4 - 37.226\beta^5 + 76.81\beta^6 - 126.9\beta^7 + 172\beta^8 - 143.97\beta^9 + 66.56\beta^{10}). \tag{A12}$$

Ostachowicz and Krawczuk (1991) instead proposed the following expression for the local compliance $C(\beta)$:

$$C(\beta) = 6\pi\beta^2(0.6384 - 1.035\beta + 3.7201\beta^2 - 5.1773\beta^3 + 7.553\beta^4 - 7.332\beta^5 + 2.4909\beta^6). \tag{A13}$$

Dimarogonas has noticed that the above expressions suggest that for small crack depth the local compliance is proportional to β^2.

It has to be noted that the crack models based on a continuous description of the beam stiffness reduction in the vicinity of the crack can be approximated by means of the approach with lumped flexibility by imposing that the rotation discontinuity due to the concentrated flexibility reproduces the relative rotation of the cross-sections affected by the crack.

In fact, for example, for the model proposed by Bilello (2001) the following expression for the local compliance $C(\beta)$ is obtained:

$$C(\beta) = \frac{\beta(2-\beta)}{0.9(\beta-1)^2}. \tag{A14}$$

Chondros et al. (1998) proposed a lumped cracked flexibility model equivalent to their continuous model by means of the following expression for $C(\beta)$:

$$\begin{aligned} C(\beta) = 6\pi(1-\nu^2)(&0.6272\beta^2 - 1.04533\beta^3 + 4.5948\beta^4 - 9.9736\beta^5 + 20.2948\beta^6 - 33.0351\beta^7 \\ &+ 47.1063\beta^8 - 40.7556\beta^9 + 19.6\beta^{10}). \end{aligned} \tag{A15}$$

The relationship between the model with singularities, adopted in this work, and the classical crack models can now be obtained by equating K_k^{φ}, given by Eq. (A5), to the rotational spring stiffnesses K_k^{eq} proposed by the lumped flexibility approach, given by Eq. (A11) and written for the k-th crack, after simple algebra, as follows:

$$\lambda_k = \frac{h}{L}C(\beta_k), \quad k = 1, \ldots, n. \tag{A16}$$

Eq. (A16) provides the relationship between the damage parameters λ_k and the dimensionless local compliance $C(\beta_k)$, given by the models briefly recalled in this section. Furthermore, according to Eq. (A16) the physical meaning for the damage parameters λ_k as "dimensionless local compliance", due to the cracks, normalized with respect to the ratio L/h of the beam can be inferred.

Finally, Eq. (A16) shows that the damage parameters λ_k are directly related to the crack depth β_k; hence, the flexural stiffness model with singularities, proposed in this study can effectively be adopted to model concentrated cracks.

In order to show how the damage parameters λ_k vary with the crack depth β_k, Eq. (A16) is plotted in Fig. A1 for some of the expressions of the local compliance $C(\beta_k)$.

It can be concluded that, since the Dirac's delta approach proposed in this study has been shown to be equivalent to the presence of concentrated elastic springs, it can be adopted to treat concentrated cracks according to the idea of the local compliance equivalent to the crack. The damage parameters λ_k, given by Eq. (11b) have been directly related to the crack depth β_k in order to make the present approach independent of the value of the constant A introduced in Eq. (8).

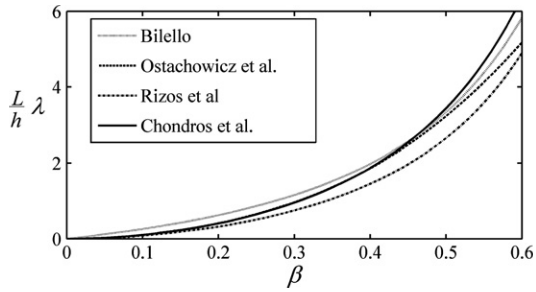

Fig. A1. Damage parameter λ versus the crack depth β, according to different models proposed in the literature.

References

Anifantis, N., Dimarogonas, A., 1983. Stability of columns with a single crack subjected to follower and vertical loads. International Journal of Solids and Structures 19 (4), 281–291.

Bagarello, F., 1995. Multiplication of distribution in one dimension: possible approaches and applications to δ-function and its derivatives. Journal of Mathematical Analysis and Applications 196, 885–901.

Bagarello, F., 2002. Multiplication of distribution in one dimension and a first application to quantum field theory. Journal of Mathematical Analysis and Applications 266, 298–320.

Bilello, C., 2001. Theoretical and experimental investigation on damaged beams under moving systems. PhD. Thesis, Università degli Studi di Palermo, Palermo, Italy.

Biondi, B., Caddemi, S., 2005. Closed form solutions of Euler–Bernoulli beams with singularities. International Journal of Solids and Structures 42, 3027–3044.

Biondi, B., Caddemi, S., 2007. Euler–Bernoulli beams with multiple singularities in the flexural stiffness. European Journal of Mechanics A/Solids. 26 (5), 789–809.

Bremermann, H.J., Durand III, L.J., 1961. On analytic continuation, multiplication, and fourier transformations of Schwartz distributions. Journal of Mathematical Physics 2, 240–257.

Buda, G., Caddemi, S., 2007. Identification of concentrated damages in Euler–Bernoulli beams under static loads. Journal of Engineering Mechanics (ASCE) 133 (8), 1–15.

Cerri, M.N., Vestroni, F., 2003. Identification of damage due to open cracks by change of measured frequencies. 16th AIMETA Congress of Theoretical and Applied Mechanics, Ferrara, Italy, 9–12 september 2003.

Chondros, T.J., Dimarogonas, A.D., Yao, J., 1998. A continuous cracked beam vibration theory. Journal of Sound and Vibration 215 (1), 17–34.

Christides, S., Barr, A.D.S., 1984. One-dimensional theory of cracked Bernoulli–Euler beams. International Journal of Mechanical Sciences 26 (11/12), 639–648.

Colombeau, J.F., 1984. New Generalized Functions and Multiplication of Distribution. North-Holland, Amsterdam.

Dimarogonas, A.D., Massouros, G., 1981. Torsional vibration of a shaft with a circumferential crack. Engineering Fracture Mechanics 15, 439–444.

Dimarogonas, A.D., 1996. Vibration of cracked structures: a state of the art review. Engineering Fracture Mechanics 55, 831–857.

Falsone, G., 2002. The use of generalised functions in the discontinuous beam bending differential equations. International Journal of Engineering Education 18 (3), 337–343.

Fan, S.C., Zheng, D.Y., 2003. Stability of a cracked Timoshenko beam column by modified Fourier series. Journal of Sound and Vibration 264, 475–484.

Freund, L.B., Hermann, G., 1976. Dynamic fracture of a beam or plate in plane bending. Journal of Applied Mechanics 76, 112–116.

Gudmundson, P., 1984. Changes in modal parameters resulting from small cracks. Proceedings 2nd International Modal Analysis Conference, Orlando, Union College, New York, Volume 2, pp. 690–697.

Gounaris, G., Dimarogonas, A.D., 1988. A finite element of a cracked prismatic beam for structural analysis. Computers and Structures 28, 309–313.

Guelfand, I.M., Chilov, G.E., 1972. Les Distributions. Dunod, Paris.

Hoskins, R.F., 1979. Generalised Functions. Ellis Horwood Limited, Chichester, England.

Irwin, G.R., 1957a. Analysis of stresses and strainsnear the end of a crack traversing a plate. Journal of Applied Mechanics 24, 361–364.

Irwin, G.R., 1957b. Relation of stresses near a crack to the crack extension force. 9th Congr. Appl. Mech., Brussels.

Li, Q.S., 2002. Buckling of an elastically restrained multi-step non-uniform beam with multiple cracks. Archive of Applied Mechanics 72, 522–535.

Liebowitz, H., Claus Jr., W.D.S., 1968. Failure of notched columns. Engineering Fracture Mechanics 1, 379–383.

Liebowitz, H., Vanderveldt, H., Harris, D.W., 1967. Carrying capacity of notched column. International Journal of Solids and Structures 3, 489–500.

Lighthill, M.J., 1958. An Introduction to Fourier Analysis and Generalised Functions. Cambridge University Press, London.

Morassi, A., 1993. Crack-induced changes in eigenfrequencies of beam structures. Journal of Engineering Mechanics 119, 1768–1803.

Okamura, H., Liu, H.W., Chorng-Shin, Chu, Liebowitz, H., 1969, A Cracked Column under Compression. Engineering Fracture Mechanics 1, 547–564.

Ostachowicz, W.M., Krawczuk, C., 1991. Analysis of the effect of cracks on the natural frequencies of a cantilever beam. Journal of Sound and Vibration 150 (2), 191–201.

Paipetis, S.A., Dimarogonas, A.D., 1986. Analytical Methods in Rotor Dynamics. Elsevier Applied Science, London.

Papadoupolos, C.A., Dimarogonas, A.D., 1987. Coupling of bending and torsional vibrations of a cracked Timoshenko shaft. Ingenieur Archiv 57, 257–266.

Rice, J.R., Levy, N., 1972. The part-through surface crack in an elastic plate. Journal of Applied Mechanics, 185.

Rizos, P.F., Aspragathos, N., Dimarogonas, A.D., 1990. Identification of crack location and magnitude in a cantilever beam from the vibration modes. Journal of Sound and Vibration 138 (3), 381–388.

Quian, G.L., Gu, S.N., Jiang, J.S., 1990. The dynamic behaviour and crack detection of a beam with a crack. Journal of Sound and Vibration 138, 233–243.

Sinha, J.K., Friswell, M.I., Edwards, S., 2002. Simplified models for the location of cracks in beam structures using measured vibration data. Journal of Sound and Vibration 251 (1), 13–38.

Shifrin, E.I., Ruotolo, R., 1999. Natural frequencies of a beam with an arbitrary number of cracks. Journal of Sound and Vibration 222, 409–423.

Takahashi, I., 1999. Vibration and stability of non-uniform cracked Timoshenko beam subjected to follower force. Computers and Structures 71, 585–591.

Yavari, A., Sarkani, S., Moyer, E.T., 2000. On applications of generalised functions to beam bending problems. International Journal of Solids and Structures 37, 5675–5705.

Yavari, A., Sarkani, S., Reddy, J.N., 2001a. On nonuniform Euler–Bernoulli and Timoshenko beams with jump discontinuities: application of distribution theory. International Journal of Solids and Structures 38, 8389–8406.

Yavari, A., Sarkani, S., Reddy, J.N., 2001b. Generalised solutions of beams with jump discontinuities on elastic foundations. Archive of Applied Mechanics 71 (9), 625–639.

Yavari, A., Sarkani, S., 2001. On applications of generalised functions to the analysis of Euler–Bernoulli beam-columns with jump discontinuities. International Journal of Mechanical Sciences 43, 1543–1562.

Zemanian, A.H., 1965. Distribution Theory and Transform Analysis. McGraw-Hill, New York.