Discrete Mathematics 309 (2009) 4772–4783

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Existence of r-fold perfect $(v, K, 1)$-Mendelsohn designs with $K \subseteq \{4, 5, 6, 7\}$

F.E. Bennett[a,*], Nicole Zwicker[a], Yanxun Chang[b]

[a] Department of Mathematics, Mount Saint Vincent University, Halifax, Nova Scotia B3M 2J6, Canada

[b] Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, People's Republic of China

ARTICLE INFO

Article history:
Received 27 October 2006
Accepted 4 June 2008
Available online 7 July 2008

Keywords:
Perfect Mendelsohn design
r-fold perfect Mendelsohn design
Holey perfect Mendelsohn design

ABSTRACT

Let v be a positive integer and let K be a set of positive integers. A $(v, K, 1)$-Mendelsohn design, which we denote briefly by $(v, K, 1)$-MD, is a pair $(X, \mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of cyclically ordered subsets of X (called *blocks*) with sizes in the set K such that every ordered pair of points of X are consecutive in exactly one block of $\mathbf{B}$. If for all $t = 1, 2, \ldots, r$, every ordered pair of points of X are t-apart in exactly one block of $\mathbf{B}$, then the $(v, K, 1)$-MD is called an r-*fold perfect* design and denoted briefly by an r-fold perfect $(v, K, 1)$-MD. If $K = \{k\}$ and $r = k - 1$, then an r-fold perfect $(v, \{k\}, 1)$-MD is essentially the more familiar $(v, k, 1)$-*perfect Mendelsohn design*, which is briefly denoted by $(v, k, 1)$-PMD. In this paper, we investigate the existence of r-fold perfect $(v, K, 1)$-Mendelsohn designs for a specified set K which is a subset of $\{4, 5, 6, 7\}$ containing precisely two elements.

1. Introduction

A set of k elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be t-*apart* in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$ is taken modulo k.

Let v, k and λ be positive integers. A (v, k, λ)-*Mendelsohn design*, denoted briefly by (v, k, λ)-MD, is a pair $(X, \mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathbf{B}$. If for all $t = 1, 2, \ldots, k - 1$, every ordered pair of points of X are t-apart in exactly λ blocks of $\mathbf{B}$, then the (v, k, λ)-MD is called a *perfect* design and denoted briefly by (v, k, λ)-PMD.

In graph-theoretic terms, a (v, k, λ)-PMD is equivalent to a decomposition of the complete directed multigraph λDK_v on v vertices into k-circuits such that for any r, $1 \le r \le k - 1$, and for any two distinct vertices x and y there are exactly λ circuits along which the (directed) distance from x to y is r.

If we ignore the cyclic order of the points, then a $(v, k, 1)$-PMD becomes a *balanced incomplete block design* with parameters v, k and $\lambda = k - 1$, briefly denoted by $(v, k, k - 1)$-BIBD. Therefore, we can consider a perfect Mendelsohn design as a generalization of balanced incomplete block designs. Mendelsohn [34] first introduced the concept of a perfect cyclic design. This concept has been further studied by various authors, including Hsu and Keedwell [31], where the designs were called perfect Mendelsohn designs. We have since adapted this terminology.

It is easy to see that the number of blocks in a (v, k, λ)-PMD is $\lambda v(v - 1)/k$. Consequently, a basic necessary condition for the existence of a (v, k, λ)-PMD is the following:

$$\lambda v(v - 1) \equiv 0 \pmod{k}.$$

* Corresponding author.
E-mail addresses: frank.bennett@msvu.ca (F.E. Bennett), nzwicker@eastlink.ca (N. Zwicker), yxchang@center.njtu.edu.cn (Y. Chang).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.06.012

The above basic necessary condition is known to be sufficient in most cases, but certainly not in all. For convenience and future reference, we present a summary of the known existence results. For $k = 3, 4$, the problem of existence of (v, k, λ)-PMDs has been completely settled. We have the following conclusive results.

Theorem 1.1. (1) [16,33] *The necessary conditions for the existence of a* $(v, 3, \lambda)$*-PMD, namely,* $v \geq 3$ *and* $\lambda v(v-1) \equiv 0 \pmod 3$*, are also sufficient, except for the non-existing* $(6, 3, 1)$*-PMD.*

(2) [24,27,35,39] *The necessary conditions for the existence of a* $(v, 4, \lambda)$*-PMD, namely,* $v \geq 4$ *and* $\lambda v(v-1) \equiv 0 \pmod 4$*, are also sufficient, except for* λ *odd when* $v = 4$*, and* $\lambda = 1$ *when* $v = 8$*.*

For $(v, 5, \lambda)$-PMDs, investigations by various authors have resulted in the following almost conclusive result.

Theorem 1.2 (*[18,20–22,25]*). *The necessary condition for the existence of a* $(v, 5, \lambda)$*-PMD, namely,* $\lambda v(v-1) \equiv 0 \pmod 5$ *and* $v \geq 5$*, is also sufficient, except for* $\lambda = 1$, $v \in \{6, 10\}$*, and possibly for* $\lambda = 1$ *and* $v \in \{15, 20\}$*.*

The results for $k = 6$ are not so conclusive. In particular, for $k = 6$ and $\lambda = 1$, where the necessary condition for the existence of a $(v, 6, 1)$-PMD is $v \equiv 0, 1, 3$ or $4 \pmod 6$, only the case of $v \equiv 1 \pmod 6$ has been resolved completely. It is known that a $(6, 6, 1)$-PMD does not exist and the non-existence of a $(10, 6, 1)$-PMD was recently established in [1]. The existence of $(v, 6, 1)$-PMDs was investigated by Miao, Zhu [36] and Abel, Bennett, Zhang [9]. The results for $(v, 6, 1)$-PMDs can now be summarized in the following theorem.

Theorem 1.3 (*[1,9,36]*). *The necessary conditions for the existence of a* $(v, 6, 1)$*-PMD, namely* $v \equiv 0, 1, 3$ *or* $4 \pmod 6$ *and* $v \geq 6$*, are sufficient except for the cases* $v = 6, 10$*, and possibly for the following cases:*

1. $v \equiv 0 \pmod 6$ *and* $v \in \{12, 18, 24, 30, 48, 54, 60, 72, 84, 90, 96, 102, 108, 114, 132, 138, 150, 162, 168, 180, 192, 198\}$.
2. $v \equiv 3 \pmod 6$ *and either* $v \in \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}$ *or* v *is in one of the following intervals:* $[9, 135]$, $[153, 183]$.
3. $v \equiv 4 \pmod 6$ *and either* $v \in \{16, 22, 34\}$ *or* v *is in the interval* $[52, 148]$.

In an attempt to fill the apparent gap in the above existence results for $(v, 6, \lambda)$-PMDs, the following result was recently established:

Theorem 1.4 (*[1]*). *Necessary conditions for the existence of a* $(v, 6, \lambda)$*-PMD are* (1) $v \geq 6$*, and* (2) $v \equiv 0$ *or* $1 \pmod 3$ *if* $\lambda \not\equiv 0 \pmod 3$*. For* $\lambda > 1$*, these are sufficient except for the known impossible case of* $v = 6$ *and either* $\lambda = 2$ *or* λ *odd.*

In contrast to the case of $k = 6$, the problem of existence of $(v, 7, \lambda)$-PMDs has been reduced to relatively few possible exceptions. We now have the following theorem.

Theorem 1.5 (*[1,3,8,26]*). *Necessary conditions for the existence of a* $(v, 7, \lambda)$*-PMD are* $v \geq 7$ *if* $\lambda \equiv 0 \pmod 7$ *or* $v \equiv 0$ *or* $1 \pmod 7$, $v \geq 7$ *if* $\lambda \not\equiv 0 \pmod 7$*. These conditions are sufficient except possibly for* $\lambda = 1$ *and* $v \in \{14, 15, 21, 22, 28, 35, 36, 42, 70, 84, 98, 99, 126, 140, 141, 147, 148, 154, 182, 183, 196, 238, 245, 273, 294\}$.

Let v and λ be positive integers and let K be a set of positive integers. A (v, K, λ)-Mendelsohn design, which we denote briefly by (v, K, λ)-MD, is a pair $(X, \mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of cyclically ordered subsets of X (called *blocks*) with sizes in the set K such that every ordered pair of points of X are consecutive in exactly λ blocks of $\mathbf{B}$. If for all $t = 1, 2, \ldots, r$, every ordered pair of points of X are t-apart in exactly λ blocks of $\mathbf{B}$, then the (v, K, λ)-MD is called an *r-fold perfect* design and denoted briefly by an r-fold perfect (v, K, λ)-MD. If $K = \{k\}$ and $r = k - 1$, then an r-fold perfect $(v, \{k\}, \lambda)$-MD is essentially a (v, k, λ)-PMD. The notion of r-fold perfect $(v, K, 1)$-MDs was discussed in [17]. However, apart from $r = 2$ and subsets K of the set $\{3, 4, 5, 7\}$, which include 3, no concerted effort was made to initially specify the set K and then address the question for what values of v there exists an r-fold perfect $(v, K, 1)$-MD. In this paper, we shall investigate the existence of r-fold perfect $(v, K, 1)$-MDs for a specified set K which is a subset of $\{4, 5, 6, 7\}$ containing precisely two elements.

Before proceeding, we state the existence results for 2-fold perfect $(v, K, 1)$-MDs, which are contained in the proof of Theorem 3.1 of [17].

Theorem 1.6 (*[17]*).

1. *For* $K = \{3, 4\}$ *or* $\{3, 5\}$*, a 2-fold perfect* $(v, K, 1)$*-MD exists for all* $v \geq 3$*, except possibly for* $v \in \{6, 8\}$*.*
2. *For* $K = \{3, 4, 7\}$ *or* $\{3, 5, 7\}$*, a 2-fold perfect* $(v, K, 1)$*-MD exists for all* $v \geq 3$*, except possibly for* $v = 6$*.*

2. Auxiliary designs

In order to establish our main result, we shall employ both direct and recursive constructions. In this section, we shall define some terminology and describe some of the auxiliary designs to be used in our constructions. For more detailed information on some of these related combinatorial structures, the reader is referred to [28,37].

Let $\mathrm{DK}_{n_1,n_2,\ldots,n_h}$ be the complete multipartite directed graph with vertex set $X = \cup_{1\le i\le h} X_i$, where X_i ($1 \le i \le h$) are disjoint sets with $|X_i| = n_i$, $v = \Sigma_{1\le i\le h}\, n_i$, and where two vertices x and y from different sets X_i and X_j are joined by exactly one arc from x to y and one arc from y to x.

Let K be a set of positive integers and $r < \min\{k : k \in K\}$. An *r-fold perfect holey Mendelsohn design* (or r-fold perfect HMD) with block size in K is an ordered pair $(X, \mathbf{A})$ where $\mathbf{A}$ is a set of cyclically ordered subsets of X, called *blocks*, which form an arc-disjoint decomposition of $\mathrm{DK}_{n_1,n_2,\ldots,n_h}$ with the property that for any integer t ($1 \le t \le r$) and any two vertices x and y from different sets X_i and X_j, there is exactly one circuit $c \in \mathbf{A}$ such that the directed distance along c from x to y is r. Each X_i ($1 \le i \le h$) is called a *hole* (or *group*) of the design and the multiset $\{n_1, n_2, \ldots, n_h\}$ is called the *type* of the design. We denote the design by r-fold perfect $(v, K, 1)$-HMD (or r-fold perfect K-HMD) and use an "exponential" notation to describe its type in general: a type $1^i 2^r 3^k \ldots$ denotes i occurrences of 1, r occurrences of 2, etc.

If $K = \{k\}$ and $r = k - 1$, then an r-fold perfect $(v, \{k\}, 1)$-HMD of type T is usually referred to as a $(v, k, 1)$-*holey perfect Mendelsohn design* of type T and briefly denoted by $(v, k, 1)$-HPMD (or k-HPMD).

If $\mathbf{H} = \{X_1, X_2, \ldots, X_h, H\}$, where $\{X_1, X_2, \ldots, X_h\}$ is a partition of X, then an HPMD with hole set $\mathbf{H}$ is called an *incomplete* HPMD, denoted by $(v, |H|, h, 1)$-IHPMD, and its *type* is defined to be the multiset $\{(|X_i|, |X_i \cap H|) : 1 \le i \le h\}$. We also use an "exponential" notation to describe types of IHPMDs.

A $(v, k, 1)$-HPMD of type 1^v is essentially a $(v, k, 1)$-PMD. A $(v, k, 1)$-HPMD of type $1^{v-n}n^1$ is called an *incomplete perfect Mendelsohn design*, denoted by $(v, n, k, 1)$-IPMD or more briefly by k-IPMD(v, n).

We shall make use of the following existence results for 4-IPMDs, 5-HPMDs, 5-IPMDs, and 6-IPMDs. Note that the existence of a 5-HPMD of type 5^6 is essentially established in Lemma 3.2 of [6] in the form of what is described as a 5-FPMD, and this was inadvertently omitted from the update given in [5].

Theorem 2.1 (*[23,24,40]*). *The necessary conditions for the existence of a* 4-IPMD(v, n)*, namely,* $v \ge 3n + 1$ *and* $(v - n)(v - 3n - 1) \equiv 0 \pmod 4$*, are also sufficient, except for* $(v, n) = (4, 0), (4, 1), (7, 2), (8, 0), (8, 1)$ *and possibly for* $(v, n) \in \{(19, 2), (27, 2)\}$.

Theorem 2.2 (*[5,6,18]*). *The necessary conditions for the existence of a 5-HPMD of type h^n, namely, $n \ge 5$ and $n(n-1)h^2 \equiv 0 \pmod 5$, are also sufficient, except possibly for the following cases:*

(1) $h \equiv 1, 3, 7$ *or* $9 \pmod{10}$, $h \ne 3$ *and* $n \in \{6, 10, 20\}$;

(2) $h \in \{1, 13, 17, 19\}$ *and* $n = 15$;

(3) $h = 3$ *and* $n \in \{6, 30, 56\}$;

(4) *the pairs* $(h, n) \in \{(15, 6), (15, 18), (15, 28)\}$.

Theorem 2.3. 1. *The necessary conditions for the existence of a* 5-IPMD$(v, 2)$*, namely,* $v \ge 9$ *and* $v \equiv 2, 4 \pmod 5$*, are also sufficient, except possibly for the cases* $v \in \{12, 17\}$.
2. *The necessary conditions for the existence of a* 5-IPMD$(v, 3)$*, namely,* $v \ge 13$ *and* $v \equiv 3 \pmod 5$*, are also sufficient.*
3. *There exists a* 5-IPMD(v, n) *for* $(v, n) \in \{(24, 4), (26, 5), (58, 13), (68, 13), (84, 19), (92, 22), (107, 24), (127, 24)\}$.
4. *There exists a* 5-IPMD(v, n) *for* $(v, n) \in \{(17, 4), (33, 8), (53, 8)\}$.
5. *There exists a* 5-IPMD$(25, 6)$.

Proof. See [18,25] for the first three assertions. For the fourth assertion, the constructions are essentially contained in Table 2 of [29]. Finally, a 5-IPMD$(25, 6)$ on $Z_{19} \cup \{\infty_1, \infty_2, \ldots, \infty_6\}$ with a hole on $\{\infty_1, \ldots, \infty_6\}$ is constructed by listing its base blocks as follows:

$$(0, 1, -1, 5, \infty_1), \quad (0, -1, 1, -5, \infty_2), \quad (0, 7, -7, -3, \infty_3),$$
$$(0, -7, 7, 3, \infty_4), \quad (0, 8, -8, 2, \infty_5), \quad (0, -8, 8, -2, \infty_6). \quad \square$$

Lemma 2.4 (*[9]*). *There exist* 6- IPMD(v, n)*'s for the following values of* v *and* n*:*

1. $v = 21 + n$, *and* $3 \le n \le 5$.
2. $v = 27 + n$, *and* $2 \le n \le 6$.
3. $v = 39 + n$, *and* $1 \le n \le 7$, $n \ne 2$.
4. $(v, n) \in \{(16, 3), (22, 3), (34, 3), (37, 6), (37, 7), (38, 5), (40, 3), (42, 7), (43, 6), (46, 3), (49, 6), (52, 9), (61, 12), (64, 3)\}$.

Lemma 2.5 (*[3]*). *There exist* 7- IPMD(v, n)*'s for* $(v, n) = (19, 3), (40, 3)$.

A *pairwise balanced design* (PBD) is a pair $(X, \mathbf{A})$ such that X is a set of elements (called *points*), and $\mathbf{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of points is contained in a unique block in $\mathbf{A}$. If v is a positive integer and K is a set of positive integers, each of which is not less than 2, then we say that $(X, \mathbf{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathbf{A}$. The integer v is called the *order* of the PBD. Using this notation, we can define a $(v, k, 1)$-BIBD to be a $(v, \{k\})$-PBD. We shall denote by $B(K)$ the set of all integers v for which there exists a (v, K)-PBD. For convenience, we define $B(k_1, k_2, \ldots, k_r)$ to be the set of all integers v such that there is a $(v, \{k_1, k_2, \ldots, k_r\})$-PBD.

A *group divisible design* (GDD) is a triple $(X, \mathbf{G}, \mathbf{B})$ which satisfies the following properties:

1. $\mathbf{G}$ is a partition of a set X (of *points*) into subsets called *groups*,
2. $\mathbf{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathbf{G}\}$. As with HPMDs we use an "exponential" notation to describe group-type. A GDD $(X, \mathbf{G}, \mathbf{B})$ will be referred to as a K-GDD if $|B| \in K$ for every block B in $\mathbf{B}$.

A *transversal design* (TD) TD(k, n) is a GDD of group-type n^k and block size k. It is well known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. An *incomplete transversal design* (ITD) denoted by TD(k, m) − TD(k, n) is a TD(k, m) with a sub-TD(k, n) removed. We have the following existence result for TDs with $k = 6, 8$:

Lemma 2.6 ([12,13]).

1. *A* TD$(6, m)$ *exists for all integers* $m > 4$ *except for* $m = 6$ *and possibly for* $m \in \{10, 14, 18, 22\}$.
2. *A* TD$(8, m)$ *exists for all integers* $m \geq 7$ *except possibly for* $m \in \{10, 12, 14, 15, 18, 20\text{–}22, 26, 28, 30, 33\text{–}35, 38, 39, 42, 44, 46, 51, 52, 54, 58, 60, 62, 66, 68, 74\}$.

Let S be a set of size s, and let $\mathcal{H} = \{S_1, S_2, \ldots, S_n\}$ be a set of subsets of S. A holey Latin square having hole set $\mathcal{H}$ is an $s \times s$ array L, whose rows and columns are indexed by elements of S, and possessing the following further properties:

1. Each cell in L is either empty or contains an element of S.
2. Every element of S appears at most once in any row or column of L.
3. The subarray indexed by $S_i \times S_i$ are empty for $1 \leq i \leq n$ (these subarrays are referred to as *holes*).
4. Symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{1 \leq i \leq n}(S_i \times S_i)$.

L is said to have order s; if $S_1, S_2, \ldots, S_n$ are disjoint, it is also said to have type $(|S_1|, |S_2|, \ldots, |S_n|)$. Alternatively, if for $1 \leq i \leq m$ there are u_i holes of size t_i, then we can write the type of S as $t_1^{u_1} t_2^{u_2} \ldots t_m^{u_m}$.

Two holey Latin squares of the same type on the same set S of size s and hole set $\mathcal{H}$ are said to be *orthogonal* if their superposition yields every ordered pair in $(S \times S) \setminus \cup_{1 \leq i \leq n}(S_i \times S_i)$. A set of k holey Latin squares is said to be orthogonal if every pair of them is orthogonal. Commonly used are the notations k MOLS(s) when there is no hole and k IMOLS(s, s_1) when there is just one hole of size s_1. If $\mathcal{H} = \{S_1, S_2, \ldots, S_n\}$ is a partition of S and $|S_i| = s_i$ for $i = 1, \ldots, n$ then k holey MOLS of type $(s_1, s_2, \ldots, s_n)$ are denoted as k HMOLS$(s_1, \ldots, s_n)$. If several holes have the same size, then the hole type $(s_1, \ldots, s_n)$ can be given exponentially as in the case of HPMDs and GDDs.

We shall make use of the following existence result for 3 HMOLS of type $4^n s^1$, which is a consequence of Lemma 2.6 and a result of R.M. Wilson (see, for example, Lemma 3.10 of [25]).

Lemma 2.7 ([25]). *Let* $E = \{6, 10, 14, 18, 22\}$ *and* $n \geq 5$ *be an integer such that* $n \notin E$. *Suppose that* $n \geq s + 1$. *Then there exist* 3 *HMOLS of type* $4^n s^1$.

We will also need the following useful result:

Lemma 2.8 ([14,20]). *Let* $F = \{6, 9, 10, 12, 14, 15, 17, 21, 24, 26, 27\}$ *and* $n \geq 6$ *be an integer such that* $n \notin F$. *Then there exist 3 HMOLS of type* $2^n 3^1$.

3. Recursive construction methods

3.1. Fill-in-hole

One importance of HPMDs and IPMDs is that their holes can frequently be filled to give a PMD. This simple but very effective approach has been frequently used in the construction of several other combinatorial structures such as GDDs and PBDs. See, for instance, [18,24,26,36,39] for other examples of this approach for PMDs. The following two constructions have been well used in the past.

Construction 3.1. *If a* k-IPMD(v, h) *and an* $(h, k, 1)$-*PMD both exist, then so does a* $(v, k, 1)$-*PMD.*

Construction 3.2. *Suppose a k-HPMD of type* $(h_1, h_2, \ldots, h_n)$ *exists. Also suppose that* $w \geq 0$ *and there exist a* k-IPMD$(h_i + w, w)$ *for* $1 \leq i \leq n - 1$, *plus an* $(h_n + w, k, 1)$-*PMD. Then a* $(v, k, 1)$-*PMD exists for* $v = \sum h_i + w$.

In most of our constructions, we shall tacitly make use of the fact that an r-fold perfect $(v, K, 1)$-MD is also a t-fold perfect $(v, K, 1)$-MD for all positive integers $t \le r$. For all practical purposes, we shall rely quite heavily on the following two constructions, which can be viewed as easy generalizations of the preceding two constructions, with straightforward proofs.

Construction 3.3. *Suppose there exist a k-IPMD(v, h) and an r-fold perfect $(h, \{k, t\}, 1)$-MD with $r \le k - 1$. Then there exists an r-fold perfect $(v, \{k, t\}, 1)$-MD.*

Construction 3.4. *Suppose an r-fold perfect K-HMD of type $(h_1, h_2, \ldots, h_n)$ exists with $r < \min\{l : l \in K\}$. Also suppose that $w \ge 0$, $k \in K$ and there exist a k-IPMD$(h_i + w, w)$ for $1 \le i \le n - 1$ plus an r-fold perfect $(h_n + w, K, 1)$-MD. Then an r-fold perfect $(v, K, 1)$-MD exists for $v = \sum h_i + w$.*

3.2. Weighting

In recursive constructions of GDDs and PBDs, the “weighting” technique and Wilson’s Fundamental GDD construction [37] are frequently used. Similar techniques are also available for constructing r-fold perfect L-HMDs. Here, we start with a master GDD and use r-fold perfect L-HMDs as ingredients for inflation. For more details on these techniques, see [22,26,36].

Construction 3.5 (*Weighting*). *Suppose $(X, \mathbf{G}, \mathbf{B})$ is a GDD and w is a function from X to $Z^+ \cup \{0\}$. Suppose there exists an r-fold perfect L-HMD of type $\{w(x) : x \in B\}$ for every $B \in \mathbf{B}$. Then there exists an r-fold perfect L-HMD of type $\{\sum_{x \in G} w(x) : G \in \mathbf{G}\}$.*

If all points in a GDD are given weight 1, we have the following construction.

Construction 3.6. *Suppose that there exist a K-GDD of group-type T. If there exists an r-fold perfect $(k, L, 1)$-MD for every $k \in K$, then there exists an r-fold perfect L-HMD of type T.*

In particular, if the given GDD is a PBD, and all points are given weight 1, we have the following well-known construction.

Construction 3.7. *If there exist a $(v, K, 1)$-PBD and a $(t, k, 1)$-PMD for all $t \in K$, then there exists a $(v, k, 1)$-PMD.*

As an easy generalization of Construction 3.7, we have the following construction from [17].

Construction 3.8 (*[17]*). *Suppose that there exists a $(v, \{k_1, k_2, \ldots, k_r\})$-PBD and for each k_i there exists an r-fold perfect $(k_i, m_i, 1)$-MD. Then there exists an r-fold perfect $(v, \{m_1, m_2, \ldots, m_r\})$-MD.*

It is also possible to start with an HPMD and inflate using TDs as in the following construction:

Construction 3.9. *Suppose a k-HPMD of type $(v_1, v_2, \ldots, v_h)$ and a TD(k, m) both exist. Then a k-HPMD of type $(mv_1, mv_2, \ldots, mv_h)$ exists.*

From [22], we have the following construction to obtain 5-HPMDs from 3 HMOLS.

Construction 3.10. *Suppose there exist 3 HMOLS of type $(h_1, h_2, \ldots, h_n)$. Then there exists a 5-HPMD of type $(5h_1, 5h_2, \ldots, 5h_n)$.*

3.3. SDP and SIP

For our recursive constructions, we shall also make use of the Singular Direct Product (SDP) and Singular Indirect Product (SIP) methods. SDP and SIP constructions for PMDs were used in [36]. These constructions are as follows:

Construction 3.11 (*SDP*). *Suppose the following designs exist: a $(g, k, 1)$-PMD, a TD$(k, m - h)$ and an $(m, h, k, 1)$-IPMD. Then a $(v, n, k, 1)$-IPMD exists for $v = g(m - h) + h$ and $n = h$ or m. If further an $(m, k, 1)$-PMD exists, then so does a $(v, k, 1)$-PMD.*

Construction 3.12 (*SIP*). *Suppose a $(g, k, 1)$-PMD, a TD$(k, m - h + a)$−TD(k, a) and an $(m, h, k, 1)$-IPMD exist where $0 \le a \le h$. Then a $(v, n, k, 1)$-IPMD exists for $v = g(m - h) + n$ and $n = h + (g - 1)a$. If further an $(n, k, 1)$-PMD exists, then so does a $(v, k, 1)$-PMD.*

To use these two constructions we need some information on known complete and incomplete TDs. In [12] a list of known TD(k, m)’s is given for $v < 100\,000$ and in [11] there is a list of known TD(k, m)-TD(k, h)’s for $v \le 1000$, $h \le 50$.

4. Working lemmas

In order to establish our main results, we shall need some working lemmas, most of which are based on the recursive constructions described above. First of all Theorem 2.2 guarantees us the following:

Lemma 4.1 ([18]). *For $h \in \{10, 20\}$, there exists a 5-HPMD of type h^n for all integers $n \geq 5$.*

Lemma 4.2. *Suppose there exists a TD(6, m). If $m \equiv 0, 1, 4 \pmod 5$, then there exists a 5-HPMD of type $10^m(2w)^1$ for $0 < w < m$. Hence a 5-IPMD$(10m + 2w + 1, 2w + 1)$ also exists.*

Proof. Adjoin a new point x to the groups of the TD(6, m) and delete another point to get a $\{6, m + 1\}$-GDD of type $5^m m^1$, where all size $m + 1$ blocks intersect the size m group at x. Give weight 0 or 2 to each point of the size m group and weight 2 to other points. If $m \equiv 4 \pmod 5$, x must receive weight 2 in order for a 5-HPMD(2^{m+1}) to exist. If $m \equiv 1 \pmod 5$, x must receive weight 0 in order for a 5-HPMD(2^m) to exist. Now a 5-HPMD of type 2^{m+1} or 2^m exists from Theorem 2.2. From the same theorem we also have 5-HPMDs of types 2^5 and 2^6. We may apply Construction 3.5 to obtain a 5-HPMD of type $10^m(2w)^1$ for $0 < w < m$. The resulting 5-IPMD comes from adjoining one infinite point to the 5-HPMD and filling in the holes with an (11, 5, 1)-PMD. □

Lemma 4.3. *Let $F = \{6, 9, 10, 12, 14, 15, 17, 21, 24, 26, 27\}$ and $n \geq 6$ be an integer such that $n \notin F$. Then there exists a 5-HPMD of type $10^n 15^1$. Hence there also exists a 5-IPMD$(10n + 18, 13)$.*

Proof. Applying Construction 3.10 with the results of Lemma 2.8, we readily obtain the desired 5-HPMD. For the resulting 5-IPMD, we adjoin 3 infinite points to this 5-HPMD. We fill in the hole of size 15 by using a 5-IPMD(18, 3), which comes from Theorem 2.3, and fill in all but one of the holes of size 10 by using a 5-IPMD(13, 3). □

Lemma 4.4. *Let $E = \{6, 10, 14, 18, 22\}$ and $n \geq 5$ be an integer such that $n \notin E$. Suppose that $n \geq s + 1$. Then there exists a 5-HPMD of type $20^n(5s)^1$. Hence a 5-IPMD$(20n + 5s + k, 5s + k)$ also exists for $k = 2, 3, 4$.*

Proof. Applying Construction 3.10 with the results of Lemma 2.7, we readily obtain the desired 5-HPMD. For the resulting 5-IPMDs, with $k = 2, 3, 4$, we adjoin k infinite points to this 5-HPMD and fill in the holes of size 20 by using a 5-IPMD$(20+k, k)$, which comes from Theorem 2.3. □

It is fairly well known that the existence of a holey Steiner pentagon system (HSPS) of type T implies the existence of a 5-HPMD of the same type T (see, for example, [18]). So the following useful lemma is an immediate consequence of Lemmas 6.8 and 6.9 of [19].

Lemma 4.5.
1. *For $t = 5$ or any $t \geq 9$ and $t \neq 43, 67$, there always exists a 5-HPMD of type $10^t u^1$ or a 5-HPMD of type $20^{t/2}u^1$ for $u = 2, 4, 6, 8$.*
2. *For $t = 43, 67$, there exists a 5-HPMD of type $20^{(t-5)/2}10^5 u^1$ for $u = 2, 4, 6, 8$.*

From Theorem 2.1, we have the following lemma.

Lemma 4.6.
1. *There exists a 4-IPMD$(v, 7)$ if and only if $v \geq 22$ and $v \equiv 2$ or $3 \pmod 4$.*
2. *There exists a 4-IPMD$(v, 11)$ if and only if $v \geq 34$ and $v \equiv 2$ or $3 \pmod 4$.*

5. Existence of 3-fold perfect $(v, K, 1)$-MDs with $K \subseteq \{4, 5, 6, 7\}$

In this section, we shall investigate the existence of 3-fold perfect $(v, K, 1)$-MDs with $K \subseteq \{4, 5, 6, 7\}$, which contains precisely two elements, including 4.

Theorem 5.1. *There exists a 3-fold perfect $(v, \{4, 5\}, 1)$-MD for all integers $v \geq 5$, except possibly for $v \in \{6, 7, 8, 10, 14, 15, 18, 19, 22, 23, 27\}$.*

Proof. By Theorem 1.1, we have the results for the stated values of $v \equiv 0$ or $1 \pmod 4$. Next, for $v \in \{11, 26, 30, 31\}$, we have a $(v, 5, 1)$-PMD by Theorem 1.2. For other values of $v \equiv 2$ or $3 \pmod 4$, we know that there exists a 4-IPMD$(v, 11)$ for all $v \geq 34$ and $v \equiv 2$ or $3 \pmod 4$ by Lemma 4.6. Applying Construction 3.3, we can fill in the hole of size 11 with an (11, 5, 1)-PMD to obtain the desired result for all $v \geq 34$ and $v \equiv 2$ or $3 \pmod 4$. □

Theorem 5.2. *There exists a 3-fold perfect $(v, \{4, 6\}, 1)$-MD for all integers $v \geq 5$, except possibly for $v \in \{6, 8, 10, 11, 14, 15, 18\}$.*

Proof. By Theorem 1.1, we have the results for the stated values of $v \equiv 0$ or $1 \pmod 4$. Next, for $v \in \{7, 19\}$, we have a $(v, 6, 1)$-PMD by Theorem 1.3. For the other values of $v \equiv 2$ or $3 \pmod 4$, we know that there exists a 4-IPMD$(v, 7)$ for all $v \geq 22$ and $v \equiv 2$ or $3 \pmod 4$ by Lemma 4.6. By applying Construction 3.3, we can fill in the hole of size 7 with a (7, 6, 1)-PMD to obtain the desired result for all $v \geq 22$ and $v \equiv 2$ or $3 \pmod 4$. This completes the proof. □

Theorem 5.3. *There exists a* 3*-fold perfect* $(v, \{4, 7\}, 1)$*-MD for all integers* $v \geq 5$*, except possibly for* $v \in \{6, 10, 11, 14, 15, 18, 19\}$.

Proof. By Theorem 1.1, we have the results for the stated values of $v \equiv 0$ or 1 (mod 4). Next, for $v \in \{7, 8\}$, we have a $(v, 7, 1)$-PMD by Theorem 1.5. For the other values of $v \equiv 2$ or 3 (mod 4), we know that there exists a 4-IPMD$(v, 7)$ for all $v \geq 22$ and $v \equiv 2$ or 3 (mod 4) by Lemma 4.6. By applying Construction 3.3, we can fill in the hole of size 7 with a (7, 7, 1)-PMD to obtain the desired result for all $v \geq 22$ and $v \equiv 2$ or 3 (mod 4). This completes the proof. □

6. Existence of 4-fold perfect $(v, \{5, 6\}, 1)$-MDs

In this section, we shall investigate the existence of 4-fold perfect $(v, \{5, 6\}, 1)$-MDs.

We begin with the following result arising from the existence of $(v, 5, 1)$-PMDs in Theorem 1.2.

Lemma 6.1. *There exists a* 4*-fold perfect* $(v, \{5, 6\}, 1)$*-MD for all integers* $v \geq 5$ *and* $v \equiv 0$ *or* 1 (mod 5)*, except possibly for* $v \in \{6, 10, 15, 20\}$.

Proof. Theorem 1.2 guarantees the existence of a $(v, 5, 1)$-PMD, which is 4-fold perfect for all of the stated values of v. □

Lemma 6.2. *There exists a* 4*-fold perfect* $(v, \{5, 6\}, 1)$*-MD for all integers* $v \geq 13$ *and* $v \equiv 3$ (mod 10)*, except possibly for* $v \in \{23, 33\}$.

Proof. First of all, for $v \in \{13, 43\}$, we have a $(v, 6, 1)$-PMD by Theorem 1.3. Next, from Lemma 4.1, we have a 5-HPMD of type 10^n for all integers $n \geq 5$. In addition to this, we also have a 5-IPMD(13, 3) from Theorem 2.3. So, by applying Construction 3.4, we can adjoin 3 infinite points to the 5-HPMD of type 10^n and fill in the holes by using a 5-IPMD(13, 3) and a (13, 6, 1)-PMD to get the desired results for all $v \geq 53$. This completes the proof. □

Lemma 6.3. *There exists a* 4*-fold perfect* $(v, \{5, 6\}, 1)$*-MD for all integers* $v \geq 7$ *and* $v \equiv 7$ (mod 10)*, except possibly for* $v \in \{17, 27\}$.

Proof. First of all, if $v \in \{7, 37, 67\}$, we have a $(v, 6, 1)$-PMD from Theorem 1.3. Next, for $v = 47$, there exists a 6-HPMD of type $4^{10}6^1$ in [9]. To this we can adjoin one infinite point and fill in the holes with a (5, 5, 1)-PMD and a (7, 6, 1)-PMD to get the desired result. For $v = 77$, we take a TD(7, 11) and replace each block of size 7 with a (7, 6, 1)-PMD and each group of size 11 with an (11, 5, 1)-PMD. This gives the desired 4-fold perfect (77, {5, 6}, 1)-MD. For $v = 87$, start with a TD(6, 8) and truncate one group to one point, then give weight two to each point of the resulting {5, 6}-GDD of type $8^5 1^1$ to get a 5-HPMD of type $16^5 2^1$. Adjoin 5 new points and fill in the size 16 holes with a 5-IPMD(21, 5), which comes from a (21, 5, 1)-BIBD, and a (7, 6, 1)-PMD. We obtain 4-fold perfect (87, {5, 6}, 1)-MD. Finally, for any $v = 10t + 7$ with $t = 5$ or $t \geq 9$, we can apply Lemma 4.5 to obtain a 5-HPMD of either type $10^t 6^1$, or $20^{t/2}6^1$, or $20^{(t-5)/2}10^5 6^1$. Adjoin one new point and fill in holes with an (11, 5, 1)-PMD, a (21, 5, 1)-PMD, and a (7, 6, 1)-PMD. We thus obtain a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all integers $v = 57$ or $v \geq 97$ with $v \equiv 7$ (mod 10). This completes the proof. □

Lemma 6.4. *There exists a* 4*-fold perfect* $(v, \{5, 6\}, 1)$*-MD for all integers* $v \leq 209$ *and* $v \equiv 9$ (mod 10)*, except possibly for* $v \in \{9, 29\}$.

Proof. First of all, if $v \in \{19, 49, 79, 109, 139, 199\}$, we have a $(v, 6, 1)$-PMD from Theorem 1.3. For $v \in \{39, 59\}$, we have HSPSs and hence also 5-HPMDs of types $4^8 6^1$ and $4^{13}6^1$ from [19]. To each of these HPMDs we can adjoin one infinite point and fill in the holes to get the desired results. Note that the first construction yields a 5-IPMD(39, 7) and the second produces a 5-IPMD(59, 7), and the hole of size 7 can be filled with a (7, 6, 1)-PMD for the desired result. For $v = 69$, we have an HSPS and hence also a 5-HPMD of type $1^{62}7^1$ from [2]. Equivalently, this produces a 5-IPMD(69, 7), and the hole of size 7 can be filled with a (7, 6, 1)-PMD to get the desired 4-fold perfect (69, {5, 6}, 1)-MD. Next, for $v = 89$, there exists a 5-GDD of type $12^7 4^1$ in [38]. Giving each point of this GDD a weight one, we readily obtain a 5-HPMD of type $12^7 4^1$. Now add an infinite point to this 5-HPMD and fill in the holes with a (13, 6, 1)-PMD and a (5, 5, 1)-PMD. This produces the desired 4-fold perfect (89, {5, 6}, 1)-MD. For $v \in \{129, 149, 169, 189, 209\}$, we have a $(v, \{5, 7\}, 1)$-PBD in [4]. Since we have both a (5, 5, 1)-PMD and a (7, 6, 1)-PMD, which are both 4-fold perfect, then Construction 3.8 guarantees the existence of a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for $v \in \{129, 149, 169, 189, 209\}$. For $v = 99$, we start with a TD(9, 9) and delete 8 points from four groups in such a way that the four remaining points from these groups all lie in a unique block of size 9. The resulting GDD can be viewed as a {5, 6}-GDD of type $5^8 9^1$. In this GDD we give all the points a weight of two so as to form a 5-HPMD of type $10^8 18^1$. Add an infinite point to this 5-HPMD and fill in the holes with a (11, 5, 1)-PMD and a (19, 6, 1)-PMD. This produces the desired 4-fold perfect (99, {5, 6}, 1)-MD. For $v \in \{119, 159, 179\}$, we apply Lemma 4.4 with $n = 5, 7, 8$ and $s = 3$. We first obtain 5-HPMDs of types $20^n 15^1$ where $n = 5, 7, 8$. To each of these 5-HPMDs we adjoin 4 new points and fill in holes with a 5-IPMD(24, 4)-PMD, which comes from Theorem 2.3, and a (19, 6, 1)-PMD. We thus obtain a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for $v = 119, 159, 179$, and this completes the proof. □

Lemma 6.5. *There exists a* 4*-fold perfect* $(v, \{5, 6\}, 1)$*-MD for all integers* $v \geq 9$ *and* $v \equiv 9$ (mod 10)*, except possibly for* $v \in \{9, 29\}$.

Proof. In view of Lemma 6.4, we need only consider $v \geq 219$. For this, we first apply Lemma 4.2 with $m \geq 20$, $m \equiv 0, 1, 4 \pmod 5$, and $w = 9, 19$ to get, respectively, a 5-IPMD($10m + 19$, 19) and a 5-IPMD($10m + 39$, 39) or instead a 5-IPMD($10m + 39$, 7), by filling in the hole of size 39 with a 5-IPMD(39, 7) from the construction given in Lemma 6.4. Finally, by filling in the holes of each of these 5-IPMDs with a (7, 6, 1)-PMD and a (19, 6, 1)-PMD, we thus obtain a 4-fold perfect (v, {5, 6}, 1)-MD for all integers $v \geq 219$ with $v \equiv 9 \pmod{10}$. This completes the proof. □

Lemma 6.6. *Let n and s be nonnegative integers with* $0 \leq s \leq n$. *If there exists a TD*$(8, n)$, *then there exists a* 4*-fold perfect* {5, 6}*-HMD of type* $n^7(2s)^1$.

Proof. We start with a TD$(8, n)$. In the last group, we give s points a weight of 2 and the remaining $n - s$ points a weight of zero. Give all the other points of the TD a weight of one. Note that we have a 4-fold perfect {6}-HMD of type 1^7 and a 4-fold perfect {5}-HMD of type $1^7 2^1$ from Theorem 2.3. By Construction 3.5 there exists a 4-fold perfect {5, 6}-HMD of type $n^7(2s)^1$. □

Lemma 6.7. *There exists a* 4*-fold perfect* (v, {5, 6}, 1)*-MD for* $v \in$ {102, 104, 138, 142, 152, 158, 164, 182, 202, 232, 242, 314, 322, 402, 404, 482}.

Proof. Take (v, n, s, ω) = (102, 13, 4, 3), (104, 13, 5, 3), (138, 19, 1, 3)(142, 16, 15, 0), (152, 19, 8, 3), (158, 19, 11, 3), (164, 19, 14, 3), (182, 25, 3, 1), (202, 27, 6, 1), (232, 27, 20, 3), (242, 31, 12, 1), (314, 37, 26, 3), (322, 41, 17, 1), (402, 45, 42, 3), (404, 45, 43, 3), (482, 61, 26, 3). Then, v can be written as $v = 7n + 2s + \omega$ with $0 \leq s \leq n$. By Lemma 6.6 there exists a 4-fold perfect {5, 6}-HMD of type $n^7(2s)^1$. The existence of *TD*$(8, n)$s are guaranteed by Lemma 2.6. To this, we adjoin ω infinite points by filling in the holes using a 5-IPMD($n + \omega, \omega$) or 6-IPMD($n + \omega, \omega$) from Theorem 2.3 and Lemma 2.4, and a 4-fold perfect ($2s + \omega$, {5, 6}, 1)-MD from Lemmas 6.1, 6.3 and 6.4. We get a 4-fold perfect (v, {5, 6}, 1)-MD for each v as listed in the lemma. □

Lemma 6.8. *There exists a* 4*-fold perfect* (v, {5, 6}, 1)*-MD for all integers* $v \geq 24$ *and* $v \equiv 4 \pmod{20}$, *except possibly for* $v \in$ {24, 64}.

Proof. First of all, if $v \in$ {144, 184, 204, 244, 324, 484}, we have a (v, 6, 1)-PMD from Theorem 1.3. For $v = 44$, we have a 6-IPMD(44, 5) from Lemma 2.4, and we can fill in the hole of size 5 with a (5, 5, 1)-PMD to get the desired 4-fold perfect (44, {5, 6}, 1)-MD. For $v = 84$, we have a 5-IPMD(84, 19) from Theorem 2.3, and we can fill in the hole of size 19 with a (19, 6, 1)-PMD to get the desired 4-fold perfect (84, {5, 6}, 1)-MD. For $v = 124$, a 5-HPMD of type 3^5 is given in Theorem 2.2 and a 5-HPMD of type $3^5 4^1$ can essentially be found in [30] in the context of a holey quasi-difference matrix. So we start with a TD(6, 7). In the last group, we give 4 points a weight of 4 and give the remaining 3 points a weight of zero. Give all the other points of the TD a weight of three. This construction produces a 5-HPMD of type $21^5 16^1$ and hence also a 4-fold perfect {5, 6}-HMD of type $21^5 16^1$. To this we adjoin 3 infinite points, and since we have a 6-IPMD(24, 3) from Lemma 2.4, we can fill in the hole of size 16 with a (19, 6, 1)-PMD to get the desired 4-fold perfect (124, {5, 6}, 1)-MD. Next, let E = {6, 10, 14, 18, 22} and $n \geq 9$ be an integer such that $n \notin E$. From Lemma 2.7 there exist 3 HMOLS($4^n 8^1$), and consequently we have a 5-HPMD of type $20^n 40^1$ by Construction 3.10. We can adjoin 4 infinite points to the 5-HPMD of type $20^n 40^1$ and fill in the holes by using a 5-IPMD(24, 4) and a 4-fold perfect (44, {5, 6}, 1)-MD to get the result for all of the other stated values of v, except for $v \in$ {104, 164, 404}. For v = 104, 164, 404, a 4-fold perfect (v, {5, 6}, 1)-MD exists by Lemma 6.7. □

Lemma 6.9. *There exists a* 4*-fold perfect* (v, {5, 6}, 1)*-MD for all integers* $v \geq 12$ *and* $v \equiv 14 \pmod{20}$, *except possibly for* $v \in$ {14, 54, 74}.

Proof. First of all, for $v = 34$, we have a 5-HPMD of type $1^{27}7^1$ from Lemma 2.14 of [18]. This is equivalent to a 5-IPMD(34, 7) and we can fill in the hole of size 7 with a (7, 6, 1)-PMD to get the desired 4-fold perfect (34, {5, 6}, 1)-MD. For $v = 94$, we take a TD(7, 13) and replace each block of size 7 with a (7, 6, 1)-PMD so as to form a 6-HPMD of type 13^7. To this, we adjoin 3 infinite points by filling in the holes of size 13 with a 6-IPMD(16, 3), from Lemma 2.4, and (16, 5, 1)-PMD for the desired 4-fold perfect (94, {5, 6}, 1)-MD. Next, let E = {6, 10, 14, 18, 22} and $n \geq 7$ be an integer such that $n \notin E$. Then Lemma 4.4 guarantees the existence of a 5-IPMD($20n + 34$, 34). By filling in the hole of size 34 with a 4-fold perfect (34, {5, 6}, 1)-MD, we obtain a 4-fold perfect (v, {5, 6}, 1)-MD for all of the other stated values of v, except for $v \in$ {114, 134, 154, 234, 314, 394, 474}. For $v \in$ {154, 234, 394, 474}, we have a (v, 6, 1)-PMD by Theorem 1.3. For $v = 114$, by Lemma 6.6 with (v, n, s) = (114, 13, 10) there exists a 4-fold perfect {5, 6}-HMD of type $13^7 20^1$. To this, we adjoin 3 infinite points by filling in the holes using a 6-IPMD(16, 3), and a 5-IPMD(23, 3) and a (16, 5, 1)-PMD for the desired result. For $v = 134$, we start with a TD(6, 11) and adjoin a point x to the groups. Now delete some other point and use its blocks to redefine groups. This gives a {6, 12}-GDD of type $5^{11} 11^1$. Note that the point x is the intersection of the blocks of size 12 and the group of size 11. For input into this GDD, we have 5-HPMDs of types 2^5 and 2^6 from Theorem 2.2 and a 5-HPMD of type $2^{11} 5^1$ can be found in [30] in the context of a holey quasi-difference matrix. Now in the group of size 11 of the GDD, we give weight 5 to the point x, a weight of 2 to eight points and and weight zero to two points. For all the other points of the GDD, we assign a weight of two. This construction produces a 5-HPMD of type $10^{11} 21^1$. To this HPMD, we adjoin an extra three points. We fill in the hole of size 21 with a 6-IPMD(24, 3) from Lemma 2.4. We then fill in all but one of the holes of size 10 with a 5-IPMD(13, 3)-PMD, which comes from Theorem 2.3, and the last with a (13, 6, 1)-PMD. The resulting design is the desired 4-fold perfect (134, {5, 6}, 1)-MD. Finally, for $v = 314$, a 4-fold perfect $(v, \{5, 6\}, 1)$-MD exists by Lemma 6.7. □

Lemma 6.10. *There exists a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all integers $v \geq 22$ and $v \equiv 2 \pmod{20}$, except possibly for $v \in \{22, 62, 122\}$.*

Proof. First of all, for $v = 42$, we have a (42, 6, 1)-PMD by Theorem 1.3. Next, let $E = \{6, 10, 14, 18, 22\}$ and $n \geq 9$ be an integer such that $n \notin E$. From Lemma 2.7 there exist 3 HMOLS(4^n8^1), and hence we have a 5-HPMD of type 20^n40^1 by Construction 3.10. We can adjoin 2 infinite points to the 5-HPMD of type 20^n40^1 and fill in the holes by using a 5-IPMD(22, 2) and a (42, 6, 1)-PMD to get the result for all of the other stated values of v, except for $v \in \{82, 102, 142, 162, 182, 202, 242, 322, 402, 482\}$. For $v = 82$, we have a 5-IPMD(18, 3) by Theorem 2.3 and we first apply Construction 3.12 with $(g, k, m, h, a) = (5, 5, 18, 3, 1)$ to obtain a 5-IPMD(82, 7). We then fill in the hole with a (7, 6, 1)-PMD for the desired result. For $v \in \{102, 142, 182, 202, 242, 322, 402, 482\}$, a 4-fold perfect $(v, \{5, 6\}, 1)$-MD exists by Lemma 6.7. Finally, for $v = 162$, we start with a TD(11, 13) (or a {11}-GDD(13^{11})). In the last group of this TD, give 8 points a weight of 3, and 5 points a weight of 1. Note that we have a 5-HPMD($1^{10}1^1$) and a 5-HPMD($1^{10}3^1$) by Theorem 2.3. By Construction 3.5 there exists a 5-HPMD of type $13^{10}29^1$. To this, we adjoin 3 infinite points by filling in the holes using a 6-IPMD(16, 3) from Lemma 2.4, and a 4-fold perfect (32, {5, 6}, 1)-MD from Lemma 6.11. □

Lemma 6.11. *There exists a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all integers $v \geq 12$ and $v \equiv 12 \pmod{20}$, except possibly for $v \in \{12, 52, 92\}$.*

Proof. First of all, for $v = 32$, we have a 6-IPMD(32, 5) from Lemma 2.4, and we can fill in the hole of size 5 with a (5, 5, 1)-PMD to get the desired 4-fold perfect (32, {5, 6}, 1)-MD. For $v = 72$, we start with a TD(5, 14) with groups G_i for $i = 1, 2, \ldots, 5$ and a particular block $B = \{a_1, a_2, \ldots, a_5\}$ where $G_i \cap B = a_i$. To the TD we shall adjoin two infinite points x, y and construct the required design as follows. On each block of the TD other than B, we construct a (5, 5, 1)-PMD. On each group G_i together with the infinite points, we construct a 6-IPMD(16, 3) with a hole on the set $\{a_i, x, y\}$. Finally, we construct a (7, 6, 1)-PMD on the set $\{a_1, a_2, \ldots, a_5, x, y\}$ to obtain the desired 4-fold perfect (72, {5, 6}, 1)-MD. Next, let $E = \{6, 10, 14, 18, 22\}$ and $n \geq 7$ be an integer such that $n \notin E$. Then Lemma 4.4 guarantees the existence of a 5-IPMD($20n + 32$, 32). By filling in the hole of size 32 with a 4-fold perfect (32, {5, 6}, 1)-MD, we obtain a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all of the other stated values of v, except for $v \in \{112, 132, 152, 232, 312, 392, 472\}$. For $v \in \{112, 392\}$, we take a TD(7, 16) and a TD(7, 56) and replace each block of size 7 with a (7, 6, 1)-PMD, each group of size $m = 16, 56$ with an $(m, 5, 1)$-PMD. This gives the desired 4-fold perfect $(v, \{5, 6\}, 1)$-MD. For $v = 132$, we have a 5-IPMD(28, 3) by Theorem 2.3 and we first apply Construction 3.12 with $(g, k, m, h, a) = (5, 5, 28, 3, 1)$ to obtain a 5-IPMD(132, 7). We then fill in the hole with a (7, 6, 1)-PMD for the desired result. For $v = 152, 232$, a 4-fold perfect $(v, \{5, 6\}, 1)$-MD exists by Lemma 6.7. Finally, for $v = 312, 472$, we have a $(v, 6, 1)$-PMD by Theorem 1.3. □

Lemma 6.12. *There exists a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all integers $v \geq 8$ and $v \equiv 8 \pmod{10}$, except possibly for $v \in \{8, 18, 48, 108\}$.*

Proof. First of all, if $v \in \{28, 78, 178, 228\}$, we have a $(v, 6, 1)$-PMD from Theorem 1.3. For $v = 38$, we have a 6-IPMD(38, 5) from Lemma 2.4, and we can fill in the hole of size 5 with a (5, 5, 1)-PMD to get the desired 4-fold perfect (38, {5, 6}, 1)-MD. For $v = 58, 68$, we have a 5-IPMD(v, 13) from Theorem 2.3, and we can fill in the hole of size 13 with a (13, 6, 1)-PMD to get the desired 4-fold perfect $(v, \{5, 6\}, 1)$-MD. For $v = 118$, we have a 5-IPMD(26, 5) by Theorem 2.3 and we first apply Construction 3.12 with $(g, k, m, h, a) = (5, 5, 26, 5, 2)$ to obtain a 5-IPMD(118, 13). We then fill in the hole with a (13, 6, 1)-PMD for the desired result. Next, let $F = \{6, 9, 10, 12, 14, 15, 17, 21, 24, 26, 27\}$ and $n \geq 6$ be an integer such that $n \notin F$. Then Lemma 4.3 guarantees the existence of a 5-IPMD($10n + 18$, 13). By filling in the hole of size 13 with a (13, 6, 1)-PMD, we obtain a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all of the other stated values of v, except for $v \in \{138, 158, 168, 188, 258, 278, 288\}$. For $v = 138, 158$, a 4-fold perfect $(v, \{5, 6\}, 1)$-MD exists by Lemma 6.7. For $v = 168, 188, 288$, we apply Lemma 4.4 with $n = 7, 8, 13$, and $s = 5$ to first obtain a 5-IPMD(v, 28). We then fill in the hole of size 28 with a (28, 6, 1)-PMD for the desired result. Finally, for $v = 258, 278$, we apply Lemma 4.4 with $n = 11, 12$, and $s = 7$ to first obtain a 5-IPMD(v, 38). We then fill in the hole of size 38 with a 4-fold perfect (38, {5, 6}, 1)-MD for the desired 4-fold perfect $(v, \{5, 6\}, 1)$-MD, which completes the proof. □

Combining the results of Lemmas 6.1–6.5 and 6.8–6.12, we have just proved the following theorem:

Theorem 6.13. *There exists a 4-fold perfect $(v, \{5, 6\}, 1)$-MD for all integers $v \geq 5$, except possibly for $v \in \{6, 8, 9, 10, 12, 14, 15, 17, 18, 20, 22, 23, 24, 27, 29, 33, 48, 52, 54, 62, 64, 74, 92, 108, 122\}$.*

7. Existence of 4-fold perfect $(v, \{5, 7\}, 1)$-MDs

In this section, we investigate the existence of 4-fold perfect $(v, \{5, 7\}, 1)$-MDs. The following two lemmas reduce the investigation to a finite set of values of v, which can further be reduced by applying existence results, for example, those relating to $(v, 5, 1)$-PMDs in Theorem 1.2.

Table 1
Values ≥ 5 not known to belong to $B(\{5, 7, 8\})$

6	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
20	**22**	**23**	**24**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**
34	**37**	**38**	**39**	42	43	44	46	47	51	52	53
58	59	60	62	66	68	69	70	71	72	73	74
75	76	77	78	79	82	83	84	86	87	89	90
93	94	95	96	97	98	99	100	102	104	106	107
108	109	110	111	114	115	116	118	122	124	126	130
132	134	135	138	140	142	146	150	153	154	156	158
162	164	166	170	172	174	178	186	190	191	194	195
198	202	206	210	211	214	226	230	234	244	258	262
274	278	282	298	300	338	359	422	443	471	478	562

Lemma 7.1 (*[4]*). *There exists a* $(v, \{5, 7, 8\}, 1)$*-PBD for all integers* $v \geq 5$*, except possibly for those values of* v *listed in Table 1.*

As an immediate consequence of Lemma 7.1, we have the following result:

Lemma 7.2. *There exists a 4-fold perfect* $(v, \{5, 7\}, 1)$*-MD for all integers* $v \geq 5$*, except possibly for those values of* v *listed in Table 1.*

Proof. First of all, we have a (5, 5, 1)-PMD. Secondly, for $v \in \{7, 8\}$, we have a $(v, 7, 1)$-PMD by Theorem 1.5. So we can apply Construction 3.8 to the results of Lemma 7.1 to get the desired result. □

Lemma 7.3. *There exists a 4-fold perfect* $(v, \{5, 7\}, 1)$*-MD for* $v \in \{11, 16, 26, 29, 30, 31, 43, 46, 51, 60, 66, 70, 71, 75, 76, 77, 78, 86, 90, 95, 96, 100, 106, 110, 111, 115, 116, 126, 130, 134, 135, 140, 146, 150, 156, 162, 166, 170, 186, 190, 191, 195, 206, 210, 211, 226, 230, 274, 300, 471\}$.

Proof. This follows immediately from Theorems 1.2 and 1.5. □

Lemma 7.4. *There exists a 4-fold perfect* $(v, \{5, 7\}, 1)$*-MD for* $v \in \{33, 34, 37, 39, 42, 47, 53, 59, 68, 69, 73, 79, 83, 89, 109\}$.

Proof. First of all, for $v = 33, 53$, we have a 5-IPMD$(v, 8)$ from Theorem 2.3 and we can fill in the hole of size 8 with an (8, 7, 1)-PMD to get the desired 4-fold perfect $(v, \{5, 7\}, 1)$-MD. Similarly, for $v = 34$, we have a 5-HPMD of type $1^{27}7^1$ from Lemma 2.14 of [18]. This is equivalent to a 5-IPMD(34, 7) and we can fill in the hole of size 7 with a (7, 7, 1)-PMD to get the desired result.

For $v = 37$, we have a 5-HPMD of type 6^6 from Theorem 2.2. To this HPMD we can adjoin one infinite point and fill in the holes with a (7, 7, 1)-PMD to get the desired result.

For $v \in \{39, 47, 59, 79\}$, we have HSPSs and hence also 5-HPMDs of types $4^8 6^1$, $4^{10}6^1$, $4^{13}6^1$, and $4^{18}6^1$ from [10,19]. To each of these HPMDs we can adjoin one infinite point and fill in the holes with a (5, 5, 1)-PMD and a (7, 7, 1)-PMD to get the desired results.

For $v = 42$, we have a 5-HPMD of type 7^6 from [7]. In this HPMD, we can fill in the holes with a (7, 7, 1)-PMD to get the desired 4-fold perfect (42, {5, 7}, 1)-MD.

For $v = 68$, we have an HSPS and hence also a 5-HPMD of type $12^5 6^1$ from [5]. We can adjoin 2 infinite points to this 5-HPMD and fill in the holes by using a 5-IPMD(14, 2), which comes from Theorem 2.3, and an (8, 7, 1)-PMD to get the desired 4-fold perfect (68, {5, 7}, 1)-MD.

For $v \in \{69, 89, 109\}$, we have HSPSs and hence also 5-HPMDs of types $1^{62}7^1$, $1^{82}7^1$, and $1^{102}7^1$ from [2]. For the specified values of v, these are equivalent to a 5-IPMD$(v, 7)$, and the hole of size 7 can be filled with a (7, 7, 1)-PMD to get the desired 4-fold perfect $(v, \{5, 7\}, 1)$-MD.

For $v = 73$, we start with a 7-GDD of type 3^{15}, which can be found in [15] (see also [32]). From this 7-GDD we can actually delete one of the blocks of size 7 and an intersecting group so as to form a {5, 6}-GDD of type $3^8 2^6$. We then give weight two to each point of this GDD to get a 5-HPMD of type $6^8 4^6$. Adjoin an infinite point and fill in the size 6 and size 4 holes with a (7, 7, 1)-PMD and a (5, 5, 1)-PMD, respectively. We obtain a 4-fold perfect (73, {5, 7}, 1)-MD.

For $v = 83$, we have an HS7CS and hence also 7-HPMD of type $4^{19}6^1$ from [8]. To this 7-HPMD we can adjoin one infinite point and fill in the holes with a (5, 5, 1)-PMD and a (7, 7, 1)-PMD to get the desired result. □

Lemma 7.5. *There exists a 4-fold perfect* $(v, \{5, 7\}, 1)$*-MD for* $v \in \{87, 97, 107, 108, 118, 132, 158, 178, 174, 194, 202\}$.

Proof. For $v = 87$, start with a *TD*(6, 8) and truncate one group to one point, then give weight two to each point of the resulting {5, 6}-GDD of type $8^5 1^1$ to get a 5-HPMD of type $16^5 2^1$. Adjoin 5 new points and fill in the size 16 holes with a 5-IPMD(21, 5), which comes from a (21, 5, 1)-BIBD, and a (7, 7, 1)-PMD. We obtain a 4-fold perfect (87, {5, 7}, 1)-MD. For $v = 97$, we can apply Lemma 4.5 to obtain a 5-HPMD of type $10^9 6^1$. Adjoin one new point and fill in holes with an (11, 5, 1)-PMD and a (7, 7, 1)-PMD. We thus obtain a 4-fold perfect (97, {5, 7}, 1)-MD. For $v = 107, 108$, by Lemma 4.4 with $n = 5$, $s = 1$ and $k = 2, 3$, there exist a 5-IPMD$(100 + a, a)$ where $a = 7, 8$. Filling in the hole of size a using a $(a, 7, 1)$-PMD for the desired result. For $v = 174, 194$, by Lemma 4.4 with $n = 7, 8$, $s = 6$ and $k = 4$, there exist a 5-IPMD(174, 34) and a 5-IPMD(194, 34). Filling in the hole of size 34 using a 4-fold perfect (34, {5, 7}, 1)-MD from Lemma 7.4 for the desired result. For $v = 118$, start with a $TD(6, 11)$ and truncate one group to three points, then give weight two to each point of the resulting {5, 6}-GDD of type $11^5 3^1$ to get a 5-HPMD of type $22^5 6^1$. Adjoin 2 new points and fill in the size 22 holes with a 5-IPMD(24, 2), which comes from Theorem 2.3, and a (8, 7, 1)-PMD. We obtain a 4-fold perfect (118, {5, 7}, 1)-MD. For $v = 132$, we have a 5-IPMD(28, 3) by Theorem 2.3 and we first apply Construction 3.12 with $(g, k, m, h, a) = (5, 5, 28, 3, 1)$ to obtain a 5-IPMD(132, 7). We then fill in the hole with a (7, 7, 1)-PMD for the desired result. For $v = 158, 178$, start with a $TD(6, n)$ ($n = 15, 17$) and truncate one group to 4 points, then give weight two to each point of the resulting {5, 6}-GDD of type $n^5 4^1$ to get a 5-HPMD of type $(2n)^5 8^1$. Fill in the size $2n$ holes with a $(2n, 5, 1)$-PMD, which comes from Lemmas 7.3 and 7.4, and a (8, 7, 1)-PMD. We obtain a 4-fold perfect $(v, \{5, 7\}, 1)$-MD for $v = 158, 178$. For $v = 202$, start with a $TD(6, 5)$. Delete five points in a block, then give weight 8 to each point of the resulting {5, 6}-GDD of type $4^5 5^1$ to get a 5-HPMD of type $32^5 40^1$. Adjoin 2 new points and fill in the size 40 hole with a 5-IPMD(42, 2), fill in four holes of the size 32 with a 5-IPMD(34, 2), which comes from Theorem 2.3, and a 4-fold perfect (34, {5, 7}, 1)-MD. We obtain a 4-fold perfect (202, {5, 7}, 1)-MD. □

Lemma 7.6. *Let n and s be nonnegative integers with $0 \le s \le 2n$. If there exists a $TD(8, n)$, then there exists a 4-fold perfect $\{5, 7\}$-HMD of type $n^7 s^1$.*

Proof. We start with a TD(8, n). In the last group, we give $\lfloor s/2 \rfloor$ points a weight of 2, and $s - 2\lfloor s/2 \rfloor$ other points a weight of 1 and the remaining $n - s$ points a weight of zero. Give all the other points of the TD a weight of one. Note that we have a 4-fold perfect {7}-HMDs of types 1^7 and 1^8 from a $(t, 7, 1)$-PMD ($t = 7, 8$) and a 4-fold perfect {5}-HMD of type $1^7 2^1$ from Theorem 2.3. By Construction 3.5 there exists a 4-fold perfect {5, 7}-HMD of type $n^7 s^1$. □

Lemma 7.7. *There exists a 4-fold perfect $(v, \{5, 7\}, 1)$-MD for $v \in$ {72, 82, 84, 93, 98, 138, 214, 234, 244, 258, 262, 278, 298, 338, 359, 422, 443, 478, 562}.*

Proof. Take $(v, n, s) = (72, 8, 16), (82, 11, 5), (84, 11, 7), (93, 11, 16), (98, 11, 21), (138, 16, 26), (214, 29, 11), (234, 29, 31), (244, 29, 41), (258, 31, 41), (262, 31, 45), (278, 31, 61), (298, 41, 11), (338, 41, 51), (359, 49, 16), (422, 56, 30), (443, 56, 51), (478, 56, 86), (562, 71, 65)$. Then v can be written as $v = 7n + s$ with $0 \le s \le 2n$. By Lemma 7.6 there exists a 4-fold perfect $(v, \{5, 7\}, 1)$-MD. The existence of $TD(8, n)$s are guaranteed by Lemma 2.6. The needed 4-fold perfect $(n, \{5, 7\}, 1)$-MD and a 4-fold perfect $(s, \{5, 7\}, 1)$-MD are from Lemmas 7.2 and 7.3. □

Lemma 7.8. *There exists a 4-fold perfect $(v, \{5, 7\}, 1)$-MD for $v \in$ {99, 102, 124, 142, 153, 154, 164, 172, 198, 282}.*

Proof. Take $(v, n, s, \omega) = (99, 13, 4, 4), (102, 13, 7, 4), (124, 17, 3, 2), (142, 16, 27, 3), (153, 17, 32, 2), (154, 17, 33, 2), (164, 19, 25, 6), (172, 19, 33, 6)$. Then v can be written as $v = 7n + s + \omega$ with $0 \le s \le 2n$. By Lemma 7.6 there exists a 4-fold perfect {5, 7}-HMD of type $n^7 s^1$. The existence of $TD(8, n)$s are guaranteed by Lemma 2.6. To this, we adjoin ω infinite points by filling in the holes using a 4-fold perfect $(n + \omega, \omega, \{5, 7\}, 1)$-MD, which comes from Theorem 2.3 and Lemma 2.5, and a 4-fold perfect $(s + \omega, \{5, 7\}, 1)$-MD from Lemmas 7.2–7.4. We get a 4-fold perfect $(v, \{5, 7\}, 1)$-MD for $v = 124, 142, 153, 154, 164, 172$.

For $v = 198, 282$, let $(v, n, s, \omega) = (198, 27, 7, 2), (282, 37, 20, 3)$. Similarly, applying Lemma 7.6, we have a 4-fold perfect {5, 7}-HMD of type $n^7 s^1$. The existence of a $TD(8, n)$ is guaranteed by Lemma 2.6. To this, we adjoin ω infinite points by filling in the holes using a 4-fold perfect $(n + \omega, \omega, \{5, 7\}, 1)$-MD and $(s + \omega, \omega, \{5, 7\}, 1)$-MD from Theorem 2.3 and Lemma 2.5, and a 4-fold perfect $(n + \omega, \{5, 7\}, 1)$-MD from Lemma 7.2. We get a 4-fold perfect $(v, \{5, 7\}, 1)$-MD for $v = 198, 282$. □

Theorem 7.9. *There exists a 4-fold perfect $(v, \{5, 7\}, 1)$-MD for all integers $v \ge 5$, except possibly for $v \in$ {6, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 27, 28, 32, 38, 44, 52, 58, 62, 74, 94, 104, 114, 122}.*

Proof. The conclusion follows from Lemmas 7.2–7.5, 7.7 and 7.8. □

8. Summary

In this paper, we have investigated the existence of r-fold perfect $(v, K, 1)$-MDs for $r = 3, 4$, and for a specified set K which is a subset of {4, 5, 6, 7} containing precisely two elements. We have by no means exhausted all the possibilities for both r and K. However, one of the objectives of this paper is to provide as conclusive a result as possible when the number of block sizes is restricted to two. The existence of 5-fold perfect $(v, \{6, 7\}, 1)$-MDs is currently being investigated.

Acknowledgements

The first author would like to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada under Grant OGP 0005320. The research of the second author was supported by an NSERC Undergraduate Student Research Award. A portion of this research was carried out while the first author was visiting Beijing Jiaotong University in May 2005, and he wishes to express his thanks to the Department of Mathematics for their services. The third author was supported by NSFC Grant No. 10771013.

References

[1] R.J.R. Abel, F.E. Bennett, The existence of $(v, 6, \lambda)$-perfect Mendelsohn designs with $\lambda > 1$, Designs Codes Crypt. 40 (2006) 211–224.
[2] R.J.R. Abel, F.E. Bennett, Super-simple Steiner pentagon systems, Discrete Appl. Math. 156 (2008) 780–793.
[3] R.J.R. Abel, F.E. Bennett, The existence of perfect Mendelsohn designs with block size 7, Discrete Math. 190 (1998) 1–14.
[4] R.J.R. Abel, F.E. Bennett, Quintessential PBDs and PBDs with prime power block sizes $\geq$8, J. Combin. Designs 13 (2005) 239–267.
[5] R.J.R. Abel, F.E. Bennett, G. Ge, Super-simple holey Steiner pentagon systems and related designs, J. Combin. Designs 16 (2008) 301–328.
[6] R.J.R. Abel, F.E. Bennett, G. Ge, Resolvable perfect Mendelsohn designs with block size five, Discrete Math. 247 (2002) 1–12.
[7] R.J.R. Abel, F.E. Bennett, G. Ge, Almost resolvable perfect Mendelsohn designs with block size five, Discrete Appl. Math. 116 (2002) 1–15.
[8] R.J.R. Abel, F.E. Bennett, G. Ge, L. Zhu, Existence of Steiner seven-cycle systems, Discrete Math. 252 (2002) 1–16.
[9] R.J.R. Abel, F.E. Bennett, H. Zhang, Perfect Mendelsohn designs with block size 6, J. Statist. Plann. Inf. 86 (2000) 287–319.
[10] R.J.R. Abel, F.E. Bennett, H. Zhang, L. Zhu, Steiner pentagon covering designs, Discrete Math. 231 (2001) 11–26.
[11] R.J.R. Abel, C.J. Colbourn, J.H. Dinitz, Incomplete MOLS, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, 1996, pp. 142–172.
[12] R.J.R. Abel, A.E. Brouwer, C.J. Colbourn, J.H. Dinitz, Mutually orthogonal latin squares, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, 1996, pp. 111–142.
[13] R.J.R. Abel, C.J. Colbourn, M. Wojtas, Concerning seven and eight mutually orthogonal Latin squares, J. Combin. Designs 12 (2004) 123–131.
[14] R.J.R. Abel, H. Zhang, Direct constructions for certain types of HMOLS, Discrete Math. 181 (1998) 1–17.
[15] R.D. Baker, An elliptic semiplane, J. Combin. Theory (A) 25 (1978) 193–195.
[16] F.E. Bennett, Direct constructions for perfect 3-cyclic designs, Ann. Discrete Math. 15 (1982) 63–68.
[17] F.E. Bennett, On r-fold perfect Mendelsohn designs, Ars Combin. 23 (1987) 57–68.
[18] F.E. Bennett, Y. Chang, J. Yin, H. Zhang, Existence of HPMDs with block size five, J. Combin. Designs 5 (1997) 257–273.
[19] F.E. Bennett, G. Ge, H. Zhang, L. Zhu, Holey Steiner pentagon systems and related designs, J. Statist. Plan. Infer. 106 (2002) 353–373.
[20] F.E. Bennett, C.J. Colbourn, L. Zhu, Existence of three HMOLS of types h^n and 2^n3^1, Discrete Math. 160 (1996) 49–65.
[21] F.E. Bennett, K.T. Phelps, C.A. Rodger, J. Yin, L. Zhu, Existence of perfect Mendelsohn designs with $k = 5$ and $\lambda > 1$, Discrete Math. 103 (1992) 129–137.
[22] F.E. Bennett, K.T. Phelps, C.A. Rodger, L. Zhu, Constructions of perfect Mendelsohn designs, Discrete Math. 103 (1992) 139–151.
[23] F.E. Bennett, H. Shen, J. Yin, Incomplete perfect Mendelsohn designs with block size 4 and one hole of size 7, J. Combin. Designs 1 (1993) 249–263.
[24] F.E. Bennett, H. Shen, J. Yin, Incomplete perfect Mendelsohn designs with block size 4 and holes of size 2 and 3, J. Combin. Designs 3 (1994) 171–183.
[25] F.E. Bennett, J. Yin, H. Zhang, R.J.R. Abel, Perfect Mendelsohn packing designs with block size five, Designs Codes Crypt. 14 (1998) 5–22.
[26] F.E. Bennett, J. Yin, L. Zhu, Perfect Mendelsohn designs with $k = 7$ and λ even, Discrete Math. 113 (1993) 7–25.
[27] F.E. Bennett, X. Zhang, L. Zhu, Perfect Mendelsohn designs with block size four, Ars Combin. 29 (1990) 65–72.
[28] T. Beth, D. Jungnickel, H. Lenz, Design Theory, Bibliographisches Institut, Zurich, 1985.
[29] C.J. Colbourn, Some direct constructions for incomplete transversal designs, J. Statist. Plann. Inf. 56 (1996) 93–104.
[30] C.J. Colbourn, Construction techniques for mutually orthogonal latin squares, in: C.J. Colbourn, E.S. Mahmoodian (Eds.), Combinatorics Advances, Kluwer Academic Press, 1995, pp. 27–48.
[31] D.F. Hsu, A.D. Keedwell, Generalized complete mappings, neofields, sequenceable groups and block designs, II, Pacific J. Math. 117 (1985) 291–312.
[32] E.R. Lamken, S.A. Vanstone, Elliptic semiplanes and group divisible designs with orthogonal resolutions, Aequationes Math. 30 (1986) 80–92.
[33] N.S. Mendelsohn, A natural generalization of Steiner triple systems, in: Computers in Number Theory, Academic Press, New York, 1971, pp. 323–338.
[34] N.S. Mendelsohn, Perfect cyclic designs, Discrete Math. 20 (1977) 63–68.
[35] N.S. Mendelsohn, Combinatorial designs as models of universal algebras, in: Recent Progress in Combinatorics, Academic Press, New York, 1969, pp. 123–132.
[36] Y. Miao, L. Zhu, Perfect Mendelsohn designs with block size six, Discrete Math. 143 (1995) 189–207.
[37] R.M. Wilson, Constructions and uses of pairwise balanced designs, Math. Centre Tracts 55 (1974) 18–41.
[38] J. Yin, A.C.H. Ling, C.J. Colbourn, R.J.R. Abel, The existence of uniform 5-GDDs, J. Combin. Des. 5 (1997) 275–299.
[39] X. Zhang, On the existence of $(v, 4, 1)$-PMD, Ars Combin. 29 (1990) 3–12.
[40] X. Zhang, Incomplete perfect Mendelsohn designs with block size four, Discrete Math. 254 (2002) 565–597.