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Stability and continuation of solutions to obstacle problems

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Abstract

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In this paper we will give a summary of some of our results which we have obtained recently. We mainly consider the question whether solutions to variational inequalities with an eigenvalue parameter are stable in the sense defined in Section 1. More precisely, we ask whether a solution to the variational inequality yields a strict local minimum of an associated energy functional defined on a closed convex subset of a real Hilbert space. This nonlinearity of the space of admissible vectors implies a new and interesting stability behavior of the solutions which is not present in the case of equations.

Moreover, it is noteworthy that optimal regularity properties of the solutions to the variational inequality are needed for the stability criterion which we will describe in Section 2. Applications to the beam and plate are considered in Sections 4 and 5. In the case of a plate, numerical computations are crucial because it is impossible to find an analytical expression for a branch of solutions to the variational inequality which is not also a solution to the free problem. Closely connected to the question of stability of a given solution to a variational inequality is the question of the continuation of this solution, which we will discuss in Section 3.

In Section 6 a survey will be given on the methods used for the computation of stability bounds. This includes in particular a short introduction to continuation algorithms for both equations and variational inequalities.

Frequent references will be made to the literature of direct relevance to the material presented. A few additional related research papers or monographs have been included in the bibliography (Courant and Hilbert (1962/1968), Fichera (1972), Funk (1962), Glowinski et al. (1981), Kikuchi and Oden (1988), Landau and Lifschitz (1970), Lions (1971) and Lions and Stampacchia (1967)).

Keywords: Variational inequality, unilateral problem, contact problem, beam buckling, plate buckling, eigenvalue problem, bifurcation, stable solution, continuation method.

1. Introduction

In many applications one is interested in critical points of functions I which are defined on a real *linear* space H. It is well known that a necessary condition for $u \in H$ to be a local minimum

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of the real functional I is the equation

$$I'(u)(v)=0 \tag{1.1}$$

for all $v \in H$, where I', I'' denote Gâteaux or Fréchet derivatives of I. We assume that all derivatives which we shall need exist and are continuous. A simple example is the case $H=\mathbb{R}^n$ and $I:\mathbb{R}^n \to \mathbb{R}$. Then (1.1) implies $I_{x_j}(u)=0$, $j=1,\dots,n$, for $u=(x_1,\dots,x_n)^{\mathrm{T}}$.

Equation (1.1) is a consequence of the Taylor expansion

$$I(u+v)=I(u)+I'(u)(v)+\tfrac{1}{2}I''(u)(v,v)+\mathrm{O}(\|v\|^3) \tag{1.2}$$

for fixed u, $v \in H$ and $\|v\|<\rho$, $\rho>0$ small. Here and in the following we assume that H is a real Hilbert space. By $\|v\|$ we denote a norm on H.

Now, let u be a solution to (1.1), that is, we assume that u satisfies the necessary condition for a local minimum. We are interested in the question whether u defines a local minimum of I. One sees from (1.2) that a sufficient (second-order) condition for u to define a local minimum of I is

$$I''(u)(v,v) \geqslant c\|v\|^2$$

for all $v \in H$ with a positive constant c which does not depend on v.

Example 1.1 (*Bending problem for the linear beam*). Set for $0<l<\infty$,

$$H=H_0^1(0,l) \cap H^2(0,l).$$

This Sobolev space contains all functions which are continuously differentiable and satisfy the boundary conditions $v(0)=v(l)=0$. By $k(x)$ we denote a force per unit of length which acts perpendicularly to the x-axis, see Fig. 1.1. The associated energy functional is here, see [14],

$$I(v)=\tfrac{1}{2}\int_0^l [v''(x)]^2 \,\mathrm{d}x-\int_0^l k(x)v(x)\,\mathrm{d}x.$$

It is well known that there exists a unique solution to (1.1) and that this solution defines a strict local minimum of I.

Example 1.2 (*Buckling problem for the linear Euler beam*). Here the beam is compressed by a force P acting in the direction of the negative x-axis, see Fig. 1.2. The energy functional is, see [14],

$$I_\lambda(v)=\tfrac{1}{2}\int_0^l [v''(x)]^2 \,\mathrm{d}x-\tfrac{1}{2}\lambda\int_0^l [v'(x)]^2 \,\mathrm{d}x.$$

In contrast to the bending problem, the functional I depends on a positive real parameter $\lambda=P/EJ$, EJ being the bending stiffness. It is well known that for all λ satisfying $-\infty<\lambda<$

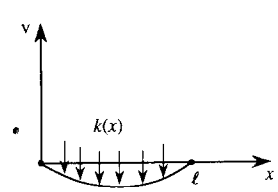

Fig. 1.1. Beam bending under lateral load.

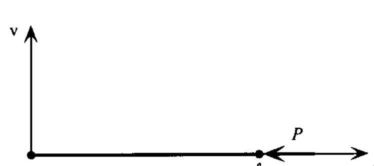

Fig. 1.2. Beam under axial load.

$(\pi/l)^2$ there exists a unique solution to (1.1), namely $u \equiv 0$ on $(0, l)$ and that this solution defines a strict local minimum of I_λ, $-\infty < \lambda < (\pi/l)^2$. The function $u \equiv 0$ is always a solution to (1.1). It is also known that (1.1) has nontrivial solutions if and only if $\lambda_n = (n\pi/l)^2$, $n = 1, 2, \ldots$. These numbers are said to be the *eigenvalues* to the above problem. If $\lambda > \lambda_1$, then the solutions to (1.1) do not define a local minimum of the functional I_λ. In the case $\lambda = \lambda_1$, all solutions, that is, the zero solution $u_0 \equiv 0$ and the eigenfunctions $u_1 \equiv c \sin(\pi/l)x$, $c =$ const., yield a local minimum but not a strict local minimum.

There is an important characterization of the first eigenvalue:

$$\lambda_1 = \min_{v \in H\setminus\{0\}} \frac{\int_0^l [v''(x)]^2 \, dx}{\int_0^l [v'(x)]^2 \, dx}. \tag{1.3}$$

The higher eigenvalues may be characterized by minimum-maximum principles, see, for example, [6].

If the admissible deflections are restricted by unilateral side conditions, then one has an interesting new stability behavior of the beam, see Section 4, and for the corresponding problem for the plate, see Section 5.

Now, we assume, in contrast to the above, that the functional $I(v)$ is defined on a closed convex subset V of H. Then a necessary condition for $u \in V$ to be a local minimum of I is the variational inequality

$$I'(u)(v-u) \geqslant 0 \tag{1.4}$$

for all $v \in V$.

This follows easily from the expansion (1.2) and from the definition of a local minimum. Let $v \in V$ be fixed and $0 < \epsilon < 1$; then $u_\epsilon \equiv u + \epsilon(v-u) \in V$. For $\epsilon < \epsilon_0$, ϵ_0 sufficiently small, u_ϵ belongs to a small neighborhood of u. Thus,

$$I(u_\epsilon) - I(u) \geqslant 0,$$

since u defines a local minimum by assumption. The expansion (1.2) yields

$$I(u_\epsilon) = I(u) + \epsilon I'(u)(v-u) + \mathrm{O}(\epsilon^2).$$

Combining this with the above inequality we arrive at

$$\epsilon I'(u)(v-u) + \mathrm{O}(\epsilon^2) \geqslant 0,$$

which implies the variational inequality (1.4).

In this paper, we are interested in the question whether a solution u to the variational inequality (1.4) defines a local minimum of the given functional I which depends on an eigenvalue parameter.

A rough criterion follows easily from the expansion (1.2). According to this expansion we have

$$I(v) = I(u) + I'(u)(v-u) + \tfrac{1}{2}I''(u)(v-u, v-u) + \mathrm{O}(\|v-u\|^3)$$

for $v \in V$ with $\|v-u\| < \rho$.

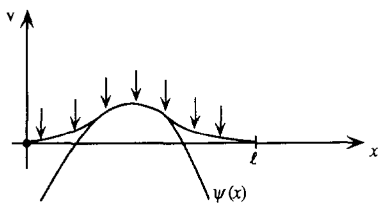

Fig. 1.3. Bending in the presence of obstacles.

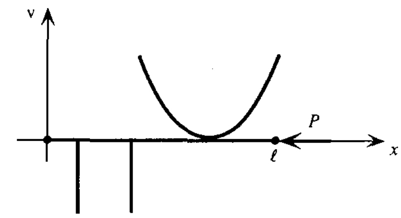

Fig. 1.4. Buckling in the presence of obstacles.

Then u defines a local minimum of I if

$$I''(u)(v-u,\ v-u) \geqslant c\,\|v-u\|^2$$

is satisfied for all $v \in V$, $\|v-u\| < \rho$. But this criterion does not take into account that u solves a variational inequality instead of an equation. In the case of a variational inequality, $I'(u)(v-u)$ is possibly positive for some $v \in V$. Thus the above criterion is too rough. In Section 2 we will give a criterion which suits our case of variational inequalities and especially our applications to the beam and plate.

An analogous problem to Example 1.1 for variational inequalities is given, if we take

$$V = \left\{ v \in H_0^1(0,\ l) \cap H^2(0,\ l)\colon\ v(x) \geqslant \psi(x) \text{ on } (0,\ l) \right\},$$

where ψ is a given function on $(0, l)$, see Fig. 1.3.

One obtains a problem corresponding to Example 1.2 for variational inequalities by setting

$$V = \left\{ v \in H_0^1(0,\ l) \cap H^2(0,\ l);\ \psi_1(x) \leqslant v(x) \leqslant \psi_2(x) \text{ on } (0,\ l) \right\},$$

where ψ_1, ψ_2 are given functions satisfying $\psi_1(x) \leqslant 0 \leqslant \psi_2(x)$ on $(0, l)$, see Fig. 1.4.

In this case, it is also possible to define a critical load as in (1.3) by

$$\lambda_1 = \min_{\substack{v \in C(V) \\ v \not\equiv 0}} \frac{\int_0^l [v''(x)]^2\, dx}{\int_0^l [v'(x)]^2\, dx}.$$

Here $C(V)$ denotes the tangential cone of V at zero, that means the closure of the set

$$\{ w;\ w = tv \text{ for all } v \in V \text{ and for all } t > 0 \}$$

with respect to the $H^2(0, l)$-norm.

It turns out that λ_1 is the first point of *bifurcation* of the variational inequality

$$I'_\lambda(u)(v-u) \equiv \int_0^l u''(v-u)''\, dx - \lambda \int_0^l u'(v-u)'\, dx \geqslant 0$$

for all $v \in V$, see [19].

Here λ_0 is said to be a bifurcation point if there exists a sequence of solutions (λ_n, u_n) to this variational inequality where $u_n \not\equiv 0$, $\lambda_n \to \lambda_0$ and $\|u_n\| \to 0$.

2. The stability criterion

Let V be a closed convex subset of a real Hilbert space H and let $u \in V$ be a solution to the variational inequality

$$u \in V\colon\ F'(u)(v-u) \geqslant G'(u)(v-u) \tag{2.1}$$

for all $v \in V$. Here F', G' are the first Gâteaux derivatives of continuous real functionals F and G defined on H. We assume that all Gâteaux or Fréchet derivatives which we shall need exist and are continuous; moreover, that

$$F''(u)(v, v) \text{ is equivalent to the given norm on } H, \tag{2.2}$$

$$G''(u)(v, v) \text{ is weakly continuous with respect to } v. \tag{2.3}$$

A solution $u \in V$ to (2.1) is said to be *stable* if it defines a strict local minimum of the functional

$$I(v) = F(v) - G(v),$$

that is, $I(v) \geqslant I(u)$ holds for all $v \in V$ such that $\| v-u \| < \rho$ for a sufficiently small $\rho > 0$, and equality takes place only for $v = u$.

Thus, the variational inequality (2.1) yields a necessary condition for a stable state because for u, $v \in V$ and $0 < \epsilon < 1$ one has

$$I(u + \epsilon(v-u)) = I(u) + \epsilon I'(u)(v-u) + \mathrm{O}(\epsilon^2).$$

For a given $t > 0$ we set

$$V_t(u) = \{ w \in H;\ u + tw \in V \}$$

and assume that $V \neq \{u\}$. We remark that each $v \in V$, $v \neq u$, may be written as $u + tw$ with $t > 0$, $F''(u)(w, w) = 1$ and $u + tw \in V$, where $t^2 = F''(u)(v-u, v-u)$ and $w = t^{-1}(v-u)$.

Define for a given positive constant A,

$$V_{t,A}(u) = \{ w \in V_t(u);\ F''(u)(w, w) \leqslant 1 \text{ and } I'(u)(w) \leqslant At \}.$$

Let

$$\Lambda_H^{-1} = \max_{w \in H \setminus \{0\}} \frac{G''(u)(w, w)}{F''(u)(w, w)}.$$

From (2.2) and (2.3) it follows that there is a maximizer of this problem.

We make the following hypothesis.

(H) Let $t_n \to 0$, $t_n > 0$, and let $w_n \in V_{t_n, A}(u)$ be a weakly convergent sequence $w_n \rightharpoonup w$. Then it follows that $G''(u)(w, w) < 1$.

In [26, Section 2] the following result was proved.

Theorem 2.1. *Suppose that the hypothesis* (H) *is satisfied with a constant A satisfying $2A > \Lambda_H^{-1} - 1$. Then the solution u to the variational inequality* (2.1) *defines a strict local minimum. Moreover, the inequality $I(v) - I(u) \geqslant c \| v-u \|^2$ holds for all v with $\| v-u \| \leqslant \rho$, where c and ρ are positive constants.*

In the following sections we set $F(v)=f(v)$ and $G(v)=\lambda g(v)$ where λ is a real positive parameter. Instead of $I(v)$ we shall write $I_\lambda(v)$.

3. On the continuation of solutions

We are interested in the local continuation of a given solution $(u_0, \lambda_0)\in V\times\mathbb{R}$ to the variational inequality

$$u\in V:\ f'(u)(v-u)\geqslant\lambda g'(u)(v-u) \tag{3.1}$$

for all $v\in V$.

Only in some special cases is it possible to find an analytical expression for a branch of solutions to a variational inequality of type (3.1).

In this section we will collect some theoretical results on the local continuation of parameter-dependent nonlinear variational inequalities. Associated numerical continuation methods were given by one of the authors [33–36] already some years ago.

3.1. Continuation with respect to a norm

Set

$$B_\rho(u_0)=\{v\in H;\ \|u_0-v\|<\rho\},\quad \rho>0.$$

We suppose that there is a $\rho>0$ such that

(i) f is weakly lower semicontinuous on $v\cap\overline{B_\rho(u_0)}$,

(ii) g is weakly continuous on $V\cap\overline{B_\rho(u_0)}$,

(iii) $f''(v)(w, w)$ is equivalent to the given norm $\|w\|=(w, w)^{1/2}$ on H uniformly with respect to $v\in V\cap\overline{B_\rho(u_0)}$,
that is, we assume in (iii) that there exist positive constants c_1, c_2 not depending on v such that

$$c_1\|w\|^2\leqslant f''(v)(w, w)\leqslant c_2\|w\|^2$$

for all $v\in V\cap B_\rho(u_0)$,

(iv) $g''(v)(w, w)$ is weakly continuous with respect to $w\in H$ for every fixed $v\in V\cap\overline{B_\rho(u_0)}$,

(v) $f'(v)(v)\geqslant c\|v\|^2$, $c>0$, for all $v\in V\cap\overline{B_\rho(u_0)}$.

We mention that the question of continuation from trivial solutions $(0, \lambda_0)$ of (3.1) under the assumption $f'(0)=g'(0)=0$ was studied by several authors, compare [21] and its references, [2,38].

For a given $u\in V$ and $t>0$ we set

$$V_t(u)=\{w\in H;\ u+tw\in V\}$$

and assume that $V_t(u)\neq\{0\}$ for all t with $0<t\leqslant t_0$.

Let for a given constant A

$$V_{t,A}(u)=\{w\in V_t(u);\ f''(u)(w, w)\leqslant 1,\ |f'(u)(w)|\leqslant At \text{ and } |g'(u)(w)|\leqslant At\}.$$

Assume that $(u, \lambda)\in V\times\mathbb{R}_+$ is a solution to (3.1) with $f(u)=r^2$, $r>0$. We make the following hypothesis.

(A) Let $t_n \downarrow 0$ and $w_n \in V_{t_n,A}(u)$ be a weakly convergent sequence $w_n \rightharpoonup w$. Then it follows that $w \in K_u$ for a given closed convex cone with the vertex at zero.

In applications to obstacle problems for the beam or plate, see [22,23] and Sections 4 and 5, assumptions of type (A) or (H) from Section 2 imply in some cases of contact that $w = 0$ and grad $w = 0$ hold on the boundary of the contact set. Then K_u is defined through these side conditions.

We assume that

$$g''(u)(w, w) > 0$$

is satisfied for a $w \in K_u$.

Define Λ_{K_u} through

$$\Lambda_{K_u}^{-1} = \max_{w \in K_u \setminus \{0\}} \frac{g''(u)(w, w)}{f''(u)(w, w)}.$$

See [21] for this and similar problems in convex sets and for related references.

Let $(u_0, \lambda_0) \in V \times \mathbb{R}_+$ be a solution to the variational inequality (3.1) with $f(u_0) = r_0^2$, $r_0 > 0$. Set $u(r_0) = u_0$ and $\lambda(r_0) = \lambda_0$. Under the further assumption

$$g'(u(r_0))(h) > 0 \quad \text{for an } h \in H,$$

such that $u(r_0) \pm h \in V$, one has [30] the next theorem.

Theorem 3.1. *Suppose that the inequality $\lambda(r_0) \leqslant \Lambda_{K_{u_0}}$ and the hypothesis* (A) *with $u = u_0$ are satisfied. Then there exists a constant $\eta_0 > 0$ such that for every r, $|r - r_0| < \eta_0$, there exists a solution $(u(r), \lambda(r))$ to the variational inequality* (3.1) *with $f(u(r)) = r^2$. Moreover, there exists a constant c independent of r such that*

$$\| u(r) - u(r_0) \| \leqslant c |r - r_0|^{1/2} \quad \textit{and} \quad |\lambda(r) - \lambda(r_0)| \leqslant c |r - r_0|^{1/2}.$$

The method of proof of this result is based on Beckert's continuation method for eigenvalue equations [1] which is in a certain sense the variational counterpart to the method of Decker and Keller [7] in the regular case.

For the question concerning the uniqueness of the continuation see [30, Section 3].

3.2. Continuation with respect to the eigenvalue parameter

Let $(u, \lambda) \in V \times \mathbb{R}$ be a solution of (3.1). The question is whether for a given ϵ, $|\epsilon| \leqslant \epsilon_0$, $\epsilon_0 > 0$ sufficiently small, there exists a solution $u(\epsilon)$ to the eigenvalue $\lambda(\epsilon) = \lambda + \epsilon$. We have shown in [26] that there is a solution $u(\epsilon)$ of (3.1) with $\| u(\epsilon) - u \| \leqslant c\epsilon$, provided a certain eigenvalue criterion is satisfied. Moreover, one has for the local behavior $u(\epsilon) = u + \epsilon u_1 + o(\epsilon)$, where u_1 is a solution of an associated linear variational inequality over a closed convex cone with the vertex at zero, see Theorem 3.3. These generalize recent results for elliptic variational inequalities of second order by Conrad et al. [5] to more general problems.

We define

$$C_u(V) = \{ v \in H;\ u + v \in V \}$$

and

$$C_u^{\perp}(V) = \{v \in C_u(V);\ I'_\lambda(u)(v) = 0\},$$

and assume $C_u^{\perp}(V) \neq \{0\}$.

Let K denote the closed cone hull of $C_u^{\perp}(V)$, that is, the closure of the set

$$\{sv;\ v \in C_u^{\perp}(V) \text{ and } s \geqslant 0\}.$$

The cone K is a convex cone with the vertex at zero. We make the following hypothesis.

(A$_1$) Let $t_n \downarrow 0$ and let $w_n \in V_{t_n}(u)$ be a weakly convergent sequence $w_n \rightharpoonup w$ such that $\limsup_{n\to\infty} I'_\lambda(u)(w_n) < \infty$ holds. Then it follows that $w \in K$.

Let

$$\Lambda_K^{-1} = \max_{w \in K \setminus \{0\}} \frac{g''(u)(w, w)}{f''(u)(w, w)}.$$

Concerning the local continuation we have [26] the following theorem.

Theorem 3.2. *Assume that* (A$_1$) *and* $\lambda > \Lambda_K$ *are satisfied. Then there exists a solution* $u(\epsilon)$ *of the variational inequality by* (3.1) *for the eigenvalue* $\lambda(\epsilon) = \lambda + \epsilon$, $|\epsilon| \leqslant \epsilon_0$, $\epsilon_0 > 0$ *sufficiently small. Moreover, one has* $\| u(\epsilon) - u \| \leqslant c|\epsilon|$ *with a constant c which does not depend on* ϵ.

For the formulation of the next theorem we consider the variational inequality

$$u \in K:\ I''_\lambda(u)(h, v-h) - \sigma g'(u)(v-h) \geqslant 0 \tag{3.2}$$

for all $v \in K$. We recall that K is the closed cone hull of $C_u^{\perp}(V)$.

Let H_0 be the linear space $H_0 = K - H$. If $\lambda < \Lambda_{H_0}$ holds, where

$$\Lambda_{H_0}^{-1} = \max_{w \in H_0 \setminus \{0\}} \frac{g''(u)(w, w)}{f''(u)(w, w)},$$

then there exists a unique solution of (3.2), see [17].

Concerning a development of $u(\epsilon)$ with respect to ϵ, we have [26] the next theorem.

Theorem 3.3. *Assume that hypothesis* (A$_1$) *and* $\lambda < \Lambda_{H_0}$ *are satisfied. Then there exist eigenvectors* $u(\epsilon)$ *of* (3.1) *for eigenvalues* $\lambda(\epsilon) = \lambda + \epsilon$, *with*

$$u(\epsilon) = u + \epsilon u_1 + o(\epsilon),$$

where $\| \epsilon^{-1} o(\epsilon) \| \to 0$ *as* $\epsilon \to 0$ *and where* $u_1 \in K$ *is the unique solution of* (3.2) *with* $\sigma = 1$ *for* $\epsilon > 0$ *and* $\sigma = -1$ *if* $\epsilon < 0$.

4. Application to the beam

In this section we give a simple but illustrative application of the previous stability criterion to the unilateral beam. We consider the linear Euler beam in the case that its deflection is limited by an obstacle. Let a clamped or simply supported beam be axially compressed by a force

$P > P_0$, where P_0 denotes the critical load of Euler. The beam then contacts the obstacle. We assume that the energy of the beam, up to a multiplicative constant, is given by

$$I_\lambda(v) = f(v) - \lambda g(v).$$

4.1. The linear Euler beam

Set

$$f(v) = \tfrac{1}{2}\int_0^l v''(x)^2 \, dx, \qquad g(v) = \tfrac{1}{2}\int_0^l v'(x)^2 \, dx$$

and $\lambda = P/EJ$, EJ the bending stiffness. Here $v(x)$ denotes the deflection of the beam away from the reference line and l the length of the beam.

We consider in the following the simply supported beam, satisfying the boundary conditions $v(0) = v(l) = 0$. We assume that the admissible deflections v satisfy the inequality

$$v(x) \leqslant \psi(x) \quad \text{on } (0, l),$$

where $\psi \in C^4[0, l]$ is a given function with $\psi(x) > 0$ on $[0, l]$.

This problem was considered by Link [15] in the case $\psi(x) \equiv d =$ const. and later in [22] by using variational inequalities. [1]

Let

$$V = \{ v \in H_0^1(0, l) \cap H^2(0, l);\ v(x) \leqslant \psi(x) \}$$

be the set of admissible deflections with the usual Sobolev notation.

A necessary condition for u to be a stable state is the variational inequality

$$u \in V\colon\ I'_\lambda(u)(v - u) \geqslant 0 \tag{4.1}$$

for all $v \in V$.

Now, we consider some cases of contact of the beam with the obstacle. Let (u, λ) be a solution to (4.1) such that the coincidence set

$$C = \{ x \in (0, l);\ u(x) = \psi(x) \}$$

is a closed interval $[a, b]$, $0 < a < b < l$.

Integration by parts yields

$$I'_\lambda(u)(w) = -A_1 w(a) - A_2 w(b) + \int_a^b w L_\lambda \psi \, dx, \tag{4.2}$$

where $w = v - u$,

$$A_1 = u'''(a - 0) - \psi'''(a), \qquad A_2 = -u'''(b + 0) + \psi'''(b),$$

and

$$L_\lambda \psi = \psi^{IV} + \lambda \psi''.$$

[1] We would like to thank Professor R. Klötzler for telling one of the authors about this problem.

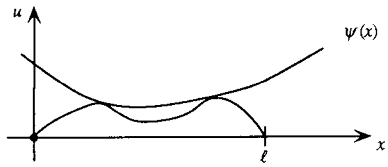

Fig. 4.1. Beam buckling with obstacle ψ, $L_\lambda \psi > 0$.

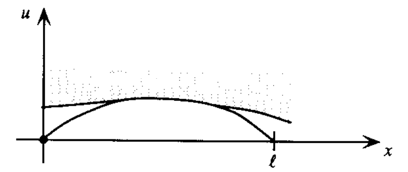

Fig. 4.2. Beam buckling with obstacle ψ, $L_\lambda \psi < 0$.

From (4.2) and (4.1) one concludes that

$$C \cap \{x \in (0, l);\ L_\lambda \psi > 0\}$$

has no interior points, see Fig. 4.1.

For the simple (standard) proof, let $x_0 \in (a, b)$ be an interior point of $[a, b]$ such that $(L_\lambda \psi)(x_0) > 0$ holds. Because of the assumption $\psi \in C^4(0, l)$, we have

$$(L_\psi)(x) \geqslant c > 0 \quad \text{on } [x_0 - h,\ x_0 + h]$$

for a constant c, h sufficiently small such that $[x_0 - h,\ x_0 + h] \subset (a, b)$.

Then we take, for example,

$$w(x;\ x_0,\ h) = \begin{cases} -[x - (x_0 - h)]^2 [x - (x_0 + h)]^2, & x_0 - h < x < x_0 + h, \\ 0, & \text{elsewhere}, \end{cases}$$

as a valid test function in (4.2) and obtain a contradiction to the variational inequality (4.1). Furthermore, from (4.2) and (4.1) it follows that $A_1 \geqslant 0$ and $A_2 \geqslant 0$. To see this, we take the above test function $w = w(x;\ a,\ h)$ and obtain

$$I'_\lambda(u)(w) = A_1 h^4 + O(h^5).$$

The assumption $A_1 < 0$ would yield a contradiction to the variational inequality (4.1) if h is small.

Thus, we assume that $L_\lambda \psi \leqslant 0$ holds on $[a, b]$. We set

$$a(u, v) = \int_0^l u'' v''\, dx, \qquad b(u, v) = \int_0^l u' v'\, dx$$

and denote by $\mu(\Omega \setminus C)$ the lowest eigenvalue of

$$u \in H:\ a(u, v) = \mu b(u, v) \tag{4.3}$$

for all $v \in H$, where

$$H = \{v \in H^2(\Omega \setminus C) \cap H_0^1(\Omega \setminus C);\ v'(a) = v'(b) = 0\}$$

with $\Omega = (0, l)$.

Remark 4.1. One has

$$\mu(\Omega \setminus C) = \min\left\{\left(\frac{a}{\tau_0}\right)^2, \left(\frac{l-b}{\tau_0}\right)^2\right\},$$

where $\tau_0 = 4.4934\ldots$ is the smallest positive root of $\tan x = x$.

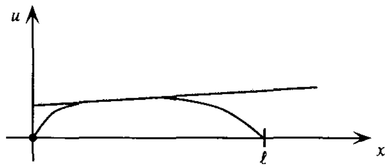

Fig. 4.3. Beam buckling with obstacle ψ, $L_\lambda \psi = 0$.

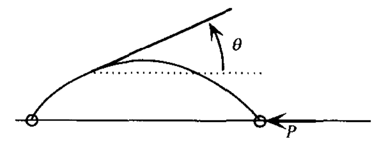

Fig. 4.4. Beam buckling under load P.

Proposition 4.2. *Suppose that $L_\lambda \psi < 0$ holds on (a, b), see Fig. 4.2. Then (u, λ) is stable if $\lambda < \mu(\Omega \setminus C)$ is satisfied.*

This result follows because the weak limit w of hypothesis (H) in Section 2 satisfies $w \equiv 0$ on $[a, b]$, see [29, Section 3].

In the next proposition we assume that $L_\lambda \psi = 0$ holds on the coincidence set $C = [a, b]$. This is true, for example, for a linear object function, see Fig. 4.3. In this case we have from (4.2),

$$I'_\lambda(u)(w) = -A_1 w(a) - A_2 w(b) \tag{4.4}$$

with nonnegative constants A_1, A_2 and $w = v - u$, $v \in V$.

We denote by $\mu_{a,b}$ the lowest eigenvalue of (4.3) with

$$H = \left\{ v \in H^2(0, l) \cap H_0^1(0, l);\ v(a) = v'(a) = 0 \text{ and } v(b) = v'(b) = 0 \right\}.$$

Remark 4.3. One has

$$\mu_{a,b} = \min\left\{ \left(\frac{a}{\tau_0}\right)^2, \left(\frac{l-b}{\tau_0}\right)^2, \left(\frac{2\pi}{b-a}\right)^2 \right\},$$

where τ_0 is the same as in Remark 4.1.

Proposition 4.4. *Suppose that $L_\lambda \psi = 0$ on (a, b) and $A_1 > 0$, $A_2 > 0$ hold. Then (u, λ) is stable if $\lambda < \mu_{a,b}$ is satisfied.*

The result follows by showing $w(a) = w'(a) = 0$ and $w(b) = w'(b) = 0$ for the weak limit w in hypothesis (H), see [29, Section 3].

If the obstacle is constant, that is, $\psi(x) \equiv d = \text{const.} > 0$, then one finds by a straightforward calculation a branch of solutions to the variational inequality (3.1), see [22],

$$u = \begin{cases} \dfrac{d}{\pi}(\sqrt{\lambda}\, x + \sin \sqrt{\lambda}\, x), & 0 \leqslant x < k, \\ d, & k \leqslant x \leqslant l - k, \\ \dfrac{d}{\pi}(\sqrt{\lambda}\,(l - x) + \sin \sqrt{\lambda}\,(l - x)), & l - k < x \leqslant l, \end{cases}$$

where $0 < k \leqslant \frac{1}{2}$ and $\lambda = (\pi/k)^2$. Since $u''(k) = u''(l-k) = 0$, it follows from [29, Proposition 3.2] that (u, λ) is stable for $(2\pi/l)^2 < \lambda < (4\pi/l)^2$, in fact, for $\lambda = (2\pi/l)^2$ too, see [29, Proposition 3.3].

4.2. A nonlinear beam

We assume that the energy of the beam which is compressed by a force P is given by, see, for example, [10, p.309],

$$I_\lambda = \tfrac{1}{2}\int_0^l \theta'^2 \,\mathrm{d}x + \lambda \int_0^l \cos\theta \,\mathrm{d}x,$$

where $\lambda = P/EJ$, EJ denotes the bending stiffness, l the length of the beam and x the arclength. By $\theta(x)$ we denote the angle between the tangential direction of the beam at x and the reference line ($P = 0$), see Fig. 4.4.

If $v(x)$ denotes the deflection of the beam away from the reference line, then we have

$$v' = \sin\theta, \qquad \cos\theta = \sqrt{1 - v'^2}, \qquad \theta' = \frac{v''}{\sqrt{1 - v'^2}}$$

and, thus, for the energy

$$I_\lambda(v) = f(v) - \lambda g(v), \tag{4.5}$$

where

$$f(v) = \tfrac{1}{2}\int_0^l v''^2 \left(1 - v'^2\right)^{-1} \mathrm{d}x$$

and

$$g(v) = -\int_0^l \left(1 - v'^2\right)^{1/2} \mathrm{d}x.$$

We assume that $|v'| < 1$ is satisfied on $[0, l]$. In the case of a simply supported beam, we have the boundary conditions $v(0) = v(l) = 0$ and for the clamped beam the conditions $v(0) = v'(0) = v(l) = v'(l) = 0$ are prescribed.

For a constant $d > 0$, we define

$$V = \left\{ v \in H_0^1(0, l) \cap H^2(0, l);\ v(x) \leqslant d \text{ on } (0, l) \right\}.$$

That is, we consider the simply supported beam. A necessary condition for $u \in V$ to define a local minimum of the energy functional (4.5) is the variational inequality

$$f'(u)(v - u) \geqslant \lambda g'(u)(v - u) \tag{4.6}$$

for all $v \in V$, where

$$f'(u)(h) = \int_0^l \left\{ \left(1 - u'^2\right)^{-1} u''h'' + u''^2 u' \left(1 - u'^2\right)^{-2} h' \right\} \mathrm{d}x$$

and

$$g'(u)(h) = \int_0^l \left(1 - u'^2\right) u'h' \,\mathrm{d}x.$$

By scaling we obtain

$$\lambda(d, l) = \frac{1}{l^2} \lambda\left(\frac{d}{l}, 1\right).$$

Thus, it is possible to take $l = 1$.

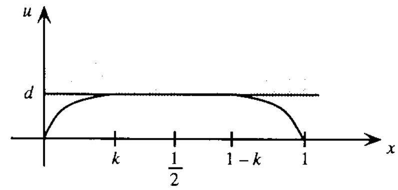

Fig. 4.5. The simply supported beam with constant obstacle.

We assume that (u, λ) is a solution of (4.6) such that

$$\begin{aligned} u &= d \quad \text{for } k \leqslant x \leqslant 1-k, \\ u &< d \quad \text{for } 0 < x < k \text{ and } 1-k < x \leqslant 1, \end{aligned} \tag{4.7}$$

where $0 < k < \frac{1}{2}$ (u, λ depend on k) and u is even with respect to $x_0 = \frac{1}{2}$, see Fig. 4.5.

Let

$$V_0 = \{ w \in H^2(0, k);\ w(0) = w(k) = w'(k) = 0 \}.$$

For the associated bilinear forms to the second Fréchet derivatives f and g we have

$$f''(u)(v, w) = \int_0^k \Big[(1-u'^2)^{-1} v''w'' + 2(1-u'^2)^{-2} u'u''\{v'w'' + v''w'\} + \{4(1-u'^2)^{-3} + (1-u'^2)^{-2}\} u''^2 v'w' \Big]\, dx$$

and

$$g''(u)(v, w) = \int_0^k \Big[(1-u'^2)^{-1/2} + (1-u'^2)^{-3/2} u'^2 \Big] v'w'\, dx.$$

Let μ_0 be the smallest eigenvalue of

$$v \in V_0\colon\ f''(u)(v, w) = \mu g''(u)(v, w) \tag{4.8}$$

for all $w \in V_0$.

We assume that $A \equiv u'''(k) = -u'''(1-k)$ satisfies $A > 0$.

Then we have, cf. [22], the next proposition.

Proposition 4.5. *Assume that*

$$\lambda < \min\left\{ \mu_0, \left(\frac{2\pi}{1-2k} \right)^2 \right\}.$$

Then u defines a strict local minimum of the functional (4.5).

We say that the critical load λ_{crit} is attained if equality holds in this inequality.

By using analytical bifurcation theory, one finds solutions of the type (4.7) to the nonlinear variational inequality (4.6) and the critical values, see [24],

$$\lambda_{crit} = (4\pi)^2(1-d^2) + O(d^4), \tag{4.9a}$$

$$k_{crit} = \tfrac{1}{4} - \tfrac{1}{8}d^2 + O(d^4). \tag{4.9b}$$

Remark 4.6. In the case of the clamped beam one easily finds a branch of solutions to the variational inequality as the one above for the simply supported beam, see [24, p.521]. However, one has $\lambda \equiv \mu_0$ along the constructed branch, see [24, Lemma 3.1]. Thus, our second-order stability criterion gives no answer about the stability of the constructed branch.

5. Application to the plate

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$. Set

$$f(v) = \tfrac{1}{2}\int_\Omega \left[(\Delta v)^2 + 2(1-\nu)\left(v_{x_1x_2}^2 - v_{x_1x_1}v_{x_2x_2} \right) \right] \mathrm{d}x,$$

$$g(v) = \tfrac{1}{2}\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x) v_{x_i} v_{x_j} \,\mathrm{d}x.$$

The functions $a_{ij} = a_{ji}$ are determined by the boundary force $P \cdot \boldsymbol{K}(s)$ which is acting in the reference plane, and ν is the Poisson ratio, $0 < \nu < \frac{1}{2}$. If the plate is simply supported on the boundary $\partial\Omega$, that is, $v = 0$ has to be described on $\partial\Omega$, then, cf., for example, [14],

$$f(v) = \tfrac{1}{2}\int_\Omega (\Delta v)^2 \,\mathrm{d}x - \tfrac{1}{2}(1-\nu)\int_{\partial\Omega} \kappa(s)\left(\frac{\partial v}{\partial n}\right)^2 \mathrm{d}s.$$

Here κ denotes the curvature with respect to the inner normal at the boundary and n is the outer unit normal on $\partial\Omega$. We define an associated bilinear form as follows:

$$a(u, v) = \int_\Omega \Delta u \,\Delta v \,\mathrm{d}x - (1-\nu)\int_{\partial\Omega} \kappa(s) \frac{\partial u}{\partial n}\frac{\partial v}{\partial n} \,\mathrm{d}s. \tag{5.1}$$

To simplify the exposition, we consider the case that $\boldsymbol{K} = -n$ holds. This choice implies that

$$g(u) = \tfrac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

Set

$$b(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x. \tag{5.2}$$

For the simply supported plate which we consider in this section we set

$$V = \left\{ v \in H_0^1(\Omega) \cap H^2(\Omega);\ v(x) \leqslant \psi(x) \text{ on } \Omega \right\}$$

with a $C^4(\overline{\Omega})$ function $\psi(x)$ which satisfies $0 < \psi(x)$ on $\overline{\Omega}$.

An associated energy functional of the corresponding plate is given by, see, for example, [14],

$$I_\lambda(v) = f(v) - \lambda g(v)$$

with $v \in V$ and $\lambda = P/D$ where the bending stiffness is given by $D = Eh^3/12(1-\nu^2)$, h the thickness of the plate and E the modulus of elasticity.

Again, a necessary condition for u to be a stable equilibrium state is the variational inequality

$$u \in V:\ I_\lambda'(u)(v-u) \geqslant 0 \tag{5.3}$$

for all $v \in V$ or, since f and g are quadratic functionals,

$$u \in V\colon\ a(u, v-u) \geqslant \lambda b(u, v-u)$$

for all $v \in V$.

Concerning the regularity, $u \in H^{3,2}_{\mathrm{loc}}(\Omega)$ was shown in [9]. That $u \in C^2(\Omega)$ holds in the case of the homogeneous biharmonic inequality was proved in [3]. In fact, this last regularity property holds for solutions to (5.1), too, compare [37, Satz 2.2, p.53]. About the shape and the regularity of the boundary of the coincidence set $C = \{x \in \Omega;\ u(x) = \psi(x)\}$ almost nothing is known, see also the remarks in [29, Section 4].

Now we consider two cases of contact of the plate with the obstacle, for other cases see [29, Section 4].

Let (u, λ) be a solution to the variational inequality (5.3) with the coincidence set

$$C = \mathscr{A} \cup \partial\mathscr{A},$$

where $\mathscr{A}$ is a domain with a sufficiently smooth boundary, say, piecewise smooth. Integration by parts yields

$$I_\lambda'(u)(w) = -\int_{\partial\mathscr{A}} A_1(s)w(s)\ \mathrm{d}s + \int_{\mathscr{A}} wL_\lambda\psi\ \mathrm{d}x, \tag{5.4}$$

where $w = v - u$, $L_\lambda\psi \equiv \Delta^2\psi + \lambda\ \Delta\psi$ and $A_1 = \partial/\partial n(\Delta\psi - \Delta u)$. Here n denotes the outer unit normal on $\partial\mathscr{A}$. For simplicity we assume that $A_1 \in C(\partial\mathscr{A})$ holds.

Since (u, λ) solves (5.3) by assumption, we see from (5.4) that $A_1 \geqslant 0$ on $\partial\mathscr{A}$ is satisfied and that

$$\mathscr{A} \cap \{x \in \Omega;\ L_\lambda\psi > 0\}$$

has no interior points. Thus, we assume that $L_\lambda\psi \leqslant 0$ holds on $\mathscr{A}$.

By $\mu(\Omega \setminus C)$ we denote the lowest eigenvalue of

$$u \in H\colon\ a(u, v) = \mu b(u, v) \tag{5.5}$$

for all $v \in H$ with

$$H = \left\{ v \in H_0^1(\Omega\setminus C) \cap H^2(\Omega\setminus C); \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathscr{A} \right\}.$$

The bilinear forms a, b are defined by (5.1) and (5.2). By the same argument which yields Proposition 4.2, one shows the following proposition.

Proposition 5.1. *Suppose that $L_\lambda\psi < 0$ holds on $\mathscr{A}$. Then (u, λ) is stable if $\lambda < \mu(\Omega\setminus C)$ is satisfied.*

In the next proposition we consider the case that $L_\lambda\psi = 0$ holds on the coincidence set C. By $\mu(\mathscr{A})$ we denote the smallest eigenvalue of (5.5) with $H = H_0^2(\mathscr{A})$.

Proposition 5.2. *Assume that $L_\lambda\psi = 0$ on $\mathscr{A}$ and $A_1(s) > 0$ on $\partial\mathscr{A}$ hold. Then (u, λ) is stable if*

$$\lambda < \min\{\mu(\Omega\setminus C), \mu(\mathscr{A})\}$$

is satisfied.

This assertion follows from Theorem 2.1 and because under the assumptions of the previous proposition the weak limit w of the hypothesis (H) in Theorem 2.1 satisfies $w = \partial w/\partial n = 0$ on $\partial\mathscr{A}$, see [29, Section 4].

Because this result is important for applications, we will give a sketch of the proof. More precisely, we show that

$$\int_{\partial\mathscr{A}} A_1(s)w(s)\,\mathrm{d}s = 0 \quad \text{and} \quad \int_{\partial\mathscr{A}} A_1(s)\left|\frac{\partial w}{\partial n}\right|\,\mathrm{d}s = 0$$

hold for the weak limit w of the sequence w_n of hypothesis (H). For the proof we set $z = \psi - u$. Then by definition of $V_{t_n}(u)$ we have

$$t_n w_n(x + hn) \leqslant z(x + hn). \tag{5.6}$$

If $x \in \Gamma$, $\Gamma = \partial\mathscr{A}$, then the Taylor expansion implies that, see [29, Section 4],

$$w(x + hn) = w(x) + h\frac{\partial w}{\partial n}(x) + R_1(w,\ x,\ h)$$

with

$$\left|\int_S R_1\,\mathrm{d}s\right| \leqslant c\,|S|^{1/2}\,|h|^{3/2}\,\|w\|_{H^{2,2}(\Omega)}$$

for each measurable subset $S \subset \Gamma$ and

$$z(x + hn) = z(x) + h\frac{\partial z}{\partial n}(x) + \tfrac{1}{2}h^2 u_{x_i x_j}(x) n_i n_j + R_2$$

holds for almost all $x \in \Gamma$ with

$$\left|\int_S R_2\,\mathrm{d}s\right| \leqslant c\,|S|^{1/2}\,|h|^{5/2}\,\|u\|_{H^{3,2}(\Omega')}$$

for a fixed Ω' such that

$$A \subset\subset \Omega' \subset\subset \Omega,$$

$h > 0$ sufficiently small.

Combining these expansions with (5.6) we arrive at

$$t_n\left[w_n(x)A_1 + hA_1\frac{\partial w_n}{\partial n}(x) + R_1 A_1\right] \leqslant R_2 A_1,$$

almost everywhere on $\partial\mathscr{A}$ since

$$z = \nabla z = \nabla^2 z = 0 \quad \text{on } \partial\mathscr{A}.$$

Inserting $h = k \operatorname{sign} \partial w_n/\partial_n(x)$ into this inequality for a constant k, $|k| \leqslant h_0$, we obtain

$$t_n\left[-\alpha_n + k\int_{\partial\mathscr{A}} A_1(s)\left|\frac{\partial w_n}{\partial n}\right|\,\mathrm{d}s - C_1\,|k|^{3/2}\right] \leqslant C_2\,|k|^{5/2}, \tag{5.7}$$

where

$$\alpha_n = -\int_{\partial\mathscr{A}} A_1(s) w_n(s)\,\mathrm{d}s.$$

If $\alpha_n = 0$ holds, then (5.7) implies that

$$\int_{\partial\mathscr{A}} A_1(s) \left| \frac{\partial w_n}{\partial n} \right| \mathrm{d}s = 0.$$

Let $\alpha_n > 0$ for a subsequence and set

$$\beta_n = \int_{\partial\mathscr{A}} A_1(s) \left| \frac{\partial w_n}{\partial n} \right| \mathrm{d}s.$$

If $\beta_n \geqslant \beta > 0$ was satisfied for a subsequence, then we would obtain a contradiction from (5.7) as follows.

Let $k = 2\alpha_n/\beta$ in (5.7). Then

$$t_n \left[\alpha_n - C_1 \left(\frac{2\alpha_n}{\beta} \right)^{3/2} \right] \leqslant C_2 \left(\frac{2\alpha_n}{\beta} \right)^{5/2}. \tag{5.8}$$

We recall that $\alpha_n > 0$ and $\alpha_n \to 0$ hold since

$$\alpha_n \leqslant A t_n, \tag{5.9}$$

see hypothesis (H) of Section 2. From (5.8) and (5.9) we see that

$$\alpha_n \leqslant A \frac{C_2 (2\alpha_n/\beta)^{5/2}}{\alpha_n - C_1 (2\alpha_n/\beta)^{3/2}},$$

which yields a contradiction if α_n tends to zero.

Remark 5.3. In [31] the results mentioned here are extended to the nonlinear plate governed by the von Karman equations. This nonlinear approach yields a good stability criterion from the mechanical point of view. In contrast to the above-mentioned papers variations of the displacement vector in the base domain are considered, too.

5.1. The circular plate

Let Ω be the disc

$$\Omega = B_1(0) = \{ x \in \mathbb{R}^2;\ x_1^2 + x_2^2 < 1 \}$$

and $\psi(x) = d = \text{const.} > 0$. In this case we can define a radially symmetric branch u of solutions to the variational inequality (5.3) explicitly as follows, see also [25, Section 2].

Let q be given, satisfying $0 < q < 1$. We seek a radially symmetric eigenfunction to

$$\Delta^2 u + \lambda\, \Delta u = 0 \quad \text{in } B_1 \setminus \overline{B}_q, \quad u'(q) = u''(q) = 0,$$

and

$$u(1) = 0, \quad u''(1) + \nu u'(1) = 0,$$

i.e., for the simply supported plate, under the side condition

$$u(q) = d.$$

In the case of the clamped plate the above boundary condition at $|x| = 1$ has to be replaced by

$$u(1) = u'(1) = 0.$$

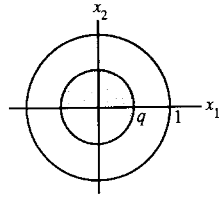

Fig. 5.1. Circular plate, loaded radially along perimeter, contact region shaded.

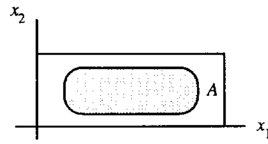

Fig. 5.2. Rectangular plate, load normal on all sides, A contact region.

We set $u \equiv d$ on B_q and

$$u(r) = d\left(1 - \frac{\frac{1}{2}\pi[J_0(kq)N_0(kr) - N_0(kq)J_0(kr)] - \ln(r/q)}{\frac{1}{2}\pi[J_0(kq)N_0(k) - N_0(kq)J_0(k)] + \ln(q)}\right)$$

on $B_1 \setminus B_q$, see Fig. 5.1.

For the simply supported plate we denote by $k = k(\nu, q)$ the smallest zero of the equation

$$\tfrac{1}{2}\pi(1-\nu)[N_0(kq)J_1(k) - J_0(kq)N_1(k)] + \tfrac{1}{2}\pi k[J_0(kq)N_0(k) - N_0(kq)J_0(k)] - \frac{1}{k}(1-\nu) = 0.$$

And in the clamped case $k = k(q)$ is the smallest zero of

$$\tfrac{1}{2}\pi[J_0(kq)N_1(q) - N_0(kq)J_1(k)] + \frac{1}{k} = 0.$$

By J_n and N_n we denote Bessel functions of the first and second kind and nth order.

The above defined u are the solutions of (5.3) to the eigenvalue $\lambda = \lambda(\nu, q) = k^2(\nu, q)$ in the simply supported case and $\lambda = \lambda(q) = k^2(q)$ for the clamped plate.

Let τ_0 be the smallest zero of $J_1(\tau) = 0$, that is, $\tau_0 = 3.831\,705\,97$.

From the eigenvalue criterion Proposition 5.2 we obtain in [25] that for the simply supported plate u is stable if

$$\lambda(\nu, q) \leqslant \left(\frac{\tau_0}{q}\right)^2$$

holds, and in the clamped case if

$$\lambda(q) < \left(\frac{\tau_0}{q}\right)^2$$

is satisfied.

The critical load λ_{crit} is obtained from the case that equality holds in the above inequalities. For the clamped case computations yield

$$\lambda_{crit} = 84.195\,935\,52.$$

For the simply supported plate λ_{crit} depends on ν, see [25] for values of $\lambda_{crit}(\nu)$. For example,

$$\lambda_{crit}(0.318) = 46.001\,616\,30.$$

We mention that one has to test the stability of the constructed radially symmetric solution with perturbations $v \in V$, $\|v - u\|$ small, possibly without radial symmetry, see [25].

5.2. The rectangular plate

For the rectangular plate an explicit solution of the associated variational inequality as in Section 5.1 is not known. Therefore, continuation has to be applied to the variational inequality in order to determine the contact region and evaluate the stability criterion, see Fig. 5.2. A numerical method was developed in [28] for a discretization of the problem and is used to compute the critical load both in the simply supported and the clamped case.

Numerical results may be found in [28] for $\psi(x) \equiv d = \text{const.} > 0$. For example, we have obtained for the square plate $\Omega = (0, 1) \times (0, 1)$ in the case of the simply supported plate that $\lambda_{crit} = 164.97$ and for the clampled plate $\lambda_{crit} = 355.32$. In [29, Section 5] two cases of a nonconstant obstacle over the simply supported square plate $\Omega = (0, 1) \times (0, 1)$ were considered:

$$\psi(x, y) = d\left[1 + (x + y)\frac{\tan \alpha}{\sqrt{2}}\right], \qquad d = \text{const.} > 0, \tag{5.10}$$

$$\psi(x, y) = d\left[1 + a(x - 0.5)^2(y - 0.5)^2\right], \qquad d = \text{const.} > 0, \quad a = \text{const. with } -16 < a < 0. \tag{5.11}$$

The obstacle (5.11) suits Proposition 5.1, since $L_\lambda \psi < 0$ holds on Ω, and (5.10) suits Proposition 5.2, because $L_\lambda \psi \equiv 0$ is satisfied on Ω.

It turned out that the numerically constructed solutions of the variational inequality with an obstacle of the type (5.11) were stable for all values λ considered, see [29]. For the first example (5.10) results were presented in [29] for $d = 0.025$, $\alpha = 10°$. For larger values of α, computations were performed for α up to $60°$, the stability bound stayed the same and it is also identical to that of the case $\psi \equiv \text{const.}$ (cf. [28]), that is, $\lambda_{crit} = 164.97$.

6. The computation of stability bounds

In this sector a survey of some of the numerical methods will be given that have been used to compute stability bounds. While, in general, a continuation technique has to be employed to approach the critical parameter values, in some cases a direct computation is possible. This was done in [25] using the results presented in Section 5.1. The computations were rather straightforward and we refer to this paper for details.

For the computation of the stability bounds in nonlinear models and, due to the nonlinearity of the admissible sets, in general also for linear models, continuation is necessary. Starting from an initial state, solutions corresponding to a sequence of values of a naturally occurring or an artificially introduced parameter are computed. Conditions characterizing the stability bounds are evaluated and, if necessary, additional iterative techniques are used to determine these bounds accurately.

6.1. Continuation for equations

In the following we introduce some basic techniques for the continuation along paths of solutions to nonlinear problems. As in the beginning of the paper, we first consider equations.

Let

$$G(u, \lambda)=0, \quad u \in X, \ \lambda \in \mathbb{R}, \tag{6.1}$$

be such an equation. Here X is, for simplicity, assumed to be a Hilbert space and G maps into X. There may be more than the one real parameter λ but the others are considered to be fixed at certain values leaving λ as the only variable parameter. As long as the derivative $G_u = \mathrm{d}G/\mathrm{d}u$ is invertible, the *implicit function theorem* guarantees that a *path* $u(\lambda)$ of solutions of (6.1) exists. If a point (u_0, λ_0) on this branch is known and if $G_u^0 = G_u(u_0, \lambda_0)$, $G_\lambda^0 = G_\lambda(u_0, \lambda_0)$, then a first-order *Euler predictor* step for a solution at $\lambda_1 = x_0 + \delta\lambda$ is given by

$$u_1 = u_0 + \delta u, \qquad G_u^0 \delta u + G_\lambda^0 = 0. \tag{6.2a,b}$$

There are basically two different *singular points* of (6.1); i.e., points (u_0, λ_0) in which G_u^0 is not invertible. They are distinguished by

$$G_\lambda^0 \notin \operatorname{Range}(G_u^0), \qquad G_\lambda^0 \in \operatorname{Range}(G_u^0). \tag{6.3a,b}$$

In the first case, they are called *turning* or *fold points* while in the second case they are called *bifurcation points*. In the first case, locally there are two solutions for either $\lambda < \lambda_0$ or $\lambda > \lambda_0$ and no solutions on the other side of λ_0. In the second case, solutions bifurcate from the given branch. Although the stability bounds to be computed, in general, correspond to such a bifurcation point, it is not necessary for the following to treat this case. For more details, see, for example, [7,35] and the references therein.

In order to overcome fold points, it is necessary to reparametrize the solution branch. If s denotes arclength along the branch and $(u_0, \lambda_0) = (u(s_0), \lambda(s_0))$, then (6.2) is replaced by

$$\begin{aligned} &u_1 = u_0 + \delta s \dot{u}_0, && \lambda_1 = \lambda_0 + \delta s \dot{\lambda}_0, \\ &G_u^0 \dot{u}_0 + G_\lambda^0 \dot{\lambda}_0 = 0, && \| \dot{u}_0 \|^2 + \dot{\lambda}_0^2 = 1. \end{aligned} \tag{6.4}$$

Here δs is a suitably chosen arclength-step and the tangent $(\dot{u}_0, \dot{\lambda}_0) = (\mathrm{d}u/\mathrm{d}s(s_0), \mathrm{d}\lambda/\mathrm{d}s(s_0))$ is defined through (6.4) up to a sign.

While the arclength s is an artificial parameter generally not of physical relevance in the given problem (6.1), other parameters have more direct significance or may even be part of the problem formulation. Frequently, these parameters are linear or nonlinear functionals of the function u. In the following, we will only consider one such functional, namely $\| u \|$ where $\| \cdot \|$ is the norm in X.

After predicting a new solution point in the neighborhood of (u_0, λ_0) a *corrector iteration* is needed to obtain this point to the required degree of accuracy. The standard procedure for this purpose is to solve an *augmented system* with Newton's method starting from the predicted point,

$$G(u, \lambda) = 0, \qquad N(u, \lambda) = 0. \tag{6.5}$$

Here N denotes a normalization condition. In the case of λ-continuation (6.2), only $G(u, \lambda) = 0$ needs to be solved while in the other cases, a suitable N has to be added. The pseudo-arclength normalization is given by

$$N(u, \lambda) = \dot{u}_0(u - u_0) + \dot{\lambda}_0(\lambda - \lambda_0) - \delta s. \tag{6.6}$$

Solutions satisfying (6.5), (6.6) lie on the hyperplane through (u_0, λ_0) which is orthogonal to $(\dot{u}_0, \dot{\lambda}_0)$.

If a point on the branch with given norm $\| u \| = r$ is to be computed, then

$$N(u, \lambda) = \tfrac{1}{2}(\| u \|^2 - r^2) \tag{6.7}$$

may be used. In each case, the Newton iteration for (6.5) is $(k = 1, 2, \ldots)$

$$\begin{pmatrix} u_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} u_k + \delta u_k \\ \lambda_k + \delta \lambda_k \end{pmatrix}, \qquad \begin{pmatrix} G_u^k & G_\lambda^k \\ N_u^k & N_\lambda^k \end{pmatrix} \begin{pmatrix} \delta u_k \\ \delta \lambda_k \end{pmatrix} = - \begin{pmatrix} G^k \\ N^k \end{pmatrix}. \tag{6.8}$$

It should be noted that the matrix in this system is regular in fold points.

In order to generalize the techniques outlined above for equations to variational inequalities, it is clear that the most important ingredient will be a *tangent* along a branch, i.e., a quantity corresponding to u_λ or $(\dot{u}_0, \dot{\lambda}_0)$ above. In other words, the first term in a Taylor expansion of the solution will be needed for a predictor-corrector scheme. There is, however, an important difference between the cases of infinite- and finite-dimensional X. If X is finite-dimensional corresponding, for example, to a discretization of a continuous problem, then, in the obstacle problems considered above, the number of *contact points* may only change at a finite number of, in general, isolated points along a solution branch.

In the following section, an overview will be given on some of the techniques that have been used to continue along solution branches of variational inequalities and to compute stability bounds.

6.2. *Continuation for variational inequalities*

In this section, some recent results on continuation for variational inequalities and the computation of stability branches will be surveyed. One of the first papers in which the first of these issues was addressed is [34]. Here, a finite-dimensional variational inequality, typically a discretization of an obstacle problem, was solved as constrained optimization problem. The proposed algorithm uses a norm as continuation parameter. Explicit normalization, an active set strategy for a projected Newton method, and regularization are some of its features. A complete convergence proof was given. The continuation technique employed in [34] was simple. In a zeroth-order prediction, the solution at some point on the branch was just renormalized to yield a predicted solution for a nearby norm-level. The overall method, however, was more efficient that that of [33] where a gradient projection method had been applied.

The key to obtain improved continuation algorithms is to find the analogues of the *tangents* along the branches, see Section 6.1 for equations. One of the first papers in which this question was considered, at least for a special class of semilinear variational inequalities, is [5]. Based on the results in this work, numerical continuation methods have been developed and successfully applied to model problems, see [4] and the references therein. Substantial generalizations, both in the class of problems as well as the type of continuation considered, were given in our recent work, some of which was quoted in Section 3.

As in the case of equations, cf. (6.2), the first candidate for continuation is λ-continuation. The theory for this case was given in [26]. Corresponding numerical methods have been used, for example, in [24,28,29,31]; of these, [24] is special since the underlying problem is the nonlinear beam of Section 4.2, i.e., a one-dimensional problem. Just as the theoretical analysis of these problems permits one to obtain results, cf. (4.9), which could not be obtained in similarly explicit form for higher-dimensional problems, in [24] a special reformulation of the problem (4.6) was used, which reduced it to a sequence of boundary value problems.

A free boundary problem

Let f, g and V_0 be as in Section 4.2. Consider the equation

$$f'(u)(v) - \lambda g'(u)(v) = 0 \tag{6.9}$$

for all $v \in V_0$ and the boundary conditions

$$u(0) = 0, \quad u(k) = d, \quad u'(k) = u''(k) = 0.$$

In general, of course, the point k, $0 < k < \frac{1}{2}$, of first contact of this simply supported beam with the obstacle is unknown. At this point *three* boundary conditions are prescribed. A standard finite-element discretization was used in [24] to solve (6.9) with the exception of the last boundary condition which is used to adjust k. The discretized form of (6.9) was solved by a damped Newton method. For details, see [24]. It turned out that for $\lambda \in ((2\pi)^2, (4\pi)^2)$, $\lambda < \mu_0$ always holds, μ_0 as in Proposition 4.5. A simple bisection algorithm then was used to find the λ such that λ was equal to the second quantity in the stability criterion, yielding the desired stability bound.

One point, however, which will be addressed here in more detail, is the solution of the eigenvalue problem (4.8). As the theory surveyed above shows, such eigenvalue problems have frequently to be solved in this context. For the continuation of this *extreme* eigenvalue, the standard *inverse iteration method* was successfully applied.

Let $a_h(v_h)$, $b_h(v_h)$ be discretizations of the bilinear form f'', g'' in (4.8) at an approximate solution u_h, h the discretization parameter. Let further $y^{(0)}$ be a nonzero vector satisfying $\| y^{(0)} \| = 1$ where $\| \cdot \|$ is the norm induced by b_h. The iteration

$$a_h(u_h)\tilde{y}^{(k+1)} = b_h(u_h)y^{(k)}, \quad k = 0, 1, \ldots,$$

$$\mu_0^{(k+1)} = \frac{\tilde{y}^{(k+1)\mathrm{T}} a_h(u_h)\tilde{y}^{(k+1)}}{\| \tilde{y}^{(k+1)} \|^2}, \qquad y^{(k+1)} = \frac{\tilde{y}^{(k+1)}}{\| \tilde{y}^{(k+1)} \|}$$

will in general converge to the desired eigenvalue μ_0.

The numerical results reported in [24] were based on relatively coarse discretizations. In [18] this problem was solved as an optimal control problem exploiting even further the one-dimensionality of the problem. In particular, in this work the question could be answered for symmetric two-sided obstacles to which states the solution jumps at the critical value. Numerical computations confirmed the relatively large discretization error present in the results of [24]. Thus, these computations were redone in a rather refined manner which included gradual increasing of the obstacle distance d to avoid divergence. First improved results were reported in [18]. Here, we present more details and additional results for the nonlinear simply supported beam. We confine the results to the cases $d = 0.05$, 0.025 and 0.0125 already considered in [24]. Table 6.1 lists the critical values k_{crit} for three values of the discretization parameter $h = 1/n$, cf. [24], and a value obtained by extrapolation to $h = 0$ $(n \to \infty)$ from a quadratic interpolation polynomial through these values.

Table 6.1
Computed and extrapolated critical values k_{crit} for the simply supported beam with constant obstacle

n	$d=0.05$	$d=0.025$	$d=0.0125$
25	0.2481995	0.2489380	0.2491174
30	0.2483341	0.2490758	0.2492554
35	0.2484334	0.2491711	0.2493512
∞	0.249085	0.249688	0.24987

While the extrapolated values are somewhat smaller than those obtained from the asymptotic formula (4.9), showing that the higher-order terms in this formula are not negligible for the d's used, their behavior for $d \to 0$ clearly shows a superlinear convergence.

The solution methods used in [28,29] are of different types. As mentioned above, both methods are based on λ-continuation. This is appropriate where no turning or fold points are expected along the solution branch of interest. In both cases, the algorithms are discrete in nature, i.e., they are applied to a discretization of the continuous problem and they have no obvious continuous analogue.

Consider the finite-dimensional variational inequality

$$u \in V\colon \ f'(u)(v-u) \geqslant \lambda g'(u)(v-u) \tag{6.10}$$

for all $v \in V$ of the form (3.1) but where now V is a closed convex subset of $\mathbb{R}^n$, $n = n(h)$, h the discretization parameter. For simplicity, we confine ourselves to the case

$$V = \{u \in \mathbb{R}^n,\ u \leqslant d\},$$

$d \equiv$ const., where inequalities are to be understood componentwise. Starting from a known solution (u_0, λ_0), $u_0 \in V$, a sequence of iterates $\{u_k\}$, $u_k \in V$, will be generated which converge to a solution of (6.10) to the parameter value $\lambda = \lambda_0 + \epsilon$. For each k the set of active constraints is defined by

$$I_k = I(u_k) = \{i \in \{1, \ldots, n\},\ u_{ki} = d_i\}.$$

Two matrices P_k and Q_k are given by

$$P_k = (e_i)_{i \in I_k}, \qquad Q_k = E_n - P_k P_k^{\mathrm{T}},$$

where $e_i \in \mathbb{R}^n$ is the ith unit vector and E_n the $n \times n$ identity matrix. Let K be as in Section 3.2. The first iterate u_1 is obtained through a predictor step. Let, locally near (u_0, λ_0), the solution branch be parametrized by $(u(\epsilon), \lambda_0 + \epsilon)$, $|\epsilon| \leqslant \epsilon_0$, and let $u(\epsilon)$ have the form

$$u(\epsilon) = u_0 + \epsilon \bar{u}_0 + o(\epsilon),$$

where $\bar{u}_0 \in K$ is the unique solution of (3.2). A possible predictor step thus is given by the following.

Predictor.

$$\lambda = \lambda_0 + \epsilon, \qquad u_{1i} = \min(u_{0i} + \epsilon \bar{u}_{0i},\ d_i), \quad i = 1, \ldots, n.$$

An iterative corrector procedure is now needed to compute a solution $u(\epsilon)$ starting with u_1.

Corrector. Set $k = 1$.

(i) Compute $q_k = f'(u_k) - \lambda g'(u_k)$ and terminate the iteration if $P_k^{\mathrm{T}} q_k \leqslant 0$, $\| Q_k q_k \| = 0$.

(ii) Set $|q_{kj}| := \max\{ |q_{ki}|, (P_k^{\mathrm{T}} q_k)_i > 0\}$. If possible, deactivate the jth constraint, $\tilde{I}_k = I_k - \{j\}$ and determine $\tilde{Q}_k$, otherwise let $\tilde{I}_k = I_k$, $\tilde{Q}_k = Q_k$.

(iii) Solve the linear system

$$[f''(u_k) - \lambda g''(u_k)]\, \delta u_k = \lambda g'(u_k) - f'(u_k),$$

but in the "free" variables ($i \notin \tilde{I}_k$) only.

(iv) Determine the maximal admissible steplength $\bar{\alpha}_k$ from

$$\bar{\alpha}_k = \min_{i \notin \tilde{I}_k} \left(\frac{d_i - u_{ki}}{\delta u_{ki}},\ \delta u_{ki} > 0 \right),$$

and set $\alpha_k = \min(\bar{\alpha}_k, 1)$, $u_{k+1} = u_k + \alpha_k \delta u_k$. Set $k = k + 1$ and go to (i).

The corrector is a projected Newton step. The notation, in particular the matrices P_k, Q_k, was introduced to include more general variational inequalities. For further comments on this type of corrector, in particular a specific deactivation strategy for (ii), see [36]. A simple deactivation condition would be $\| Q_k g_k \| = 0$.

While this method may be analyzed along the lines of the proof in [34] and will, in general, exhibit fast asymptotic convergence, the following relaxation-based method of [29,31] is conceptually simpler.

The predictor is the same as above while the corrector is replaced by the following.

Corrector. $u^{(0)} = u_1$. For $k = 0, 1, 2, \ldots$ do

$$u_i^{(k+1)} = \min\left(u_i^{(k)} - \omega \frac{I'_\lambda(z_i^{(k)})}{\partial I'_\lambda / \partial u_{ki}(z_i^{(k)})},\ \psi_i \right), \quad i = 1, \ldots, n,$$

where $z_i^{(k)} = (u_1^{(k+1)}, \ldots, u_{i-1}^{(k+1)}, u_i^{(k)}, \ldots, u_n^{(k)})$ and $0 < \omega < 2$ is a suitably chosen relaxation parameter. Here we have considered as, in fact, also in the examples of [29], an, in general, nonconstant obstacle function ψ. The convergence of this method was shown in [32].

As was first observed in [36], certain discrete parameter-dependent variational inequalities possess a large number of fold points and, additionally, so-called (spurious) transition points. These are points where the (discrete) coincidence or contact set changes. For these problems, λ-continuation is not appropriate. One possibility is continuation with respect to a norm for which theoretical results were quoted from [30] in Section 3.1.

For simplicity, we consider the same discrete problem as in [36], namely (6.10) with $f'' \equiv A =$ const., $g' = b(u)$, $g'' = B(u)$. We also exclude the case that the point (u_0, λ_0) is a singular point. Then we define

$$\dot{u}_0 = \bar{u}_0 \cdot \dot{\lambda}_0, \quad \| \dot{u}_0 \|^2 + \dot{\lambda}_0^2 = 1,$$

where $\bar{u}_0$ is the solution of (3.2) and $\| \cdot \|$ the norm induced by A.

In the following algorithm we attempt to continue from a given solution (u_0, λ_0) with $\| u_0 \| = r$ to a solution (u_t, λ_t) with $\| u_t \| = r_t \neq r$.

Predictor. Let u_0, λ_0, $\dot{u}_0$, $\dot{\lambda}_0$ be given. Compute $\overline{\delta s}$ from $\| u_0 + \overline{\delta s}\dot{u}_0 \| = r$, and δs from

$$\delta s := \min\left\{ \overline{\delta s},\ \min_{i \notin I_k} \left(\frac{\psi_i - u_{0i}}{\dot{u}_{0i}},\ \dot{u}_{0i} > 0 \right) \right\}.$$

Set

$$u_1 = u_0 + \delta s \dot{u}_0, \qquad \lambda_1 = \lambda_0 + \delta s \dot{\lambda}_0, \qquad I_1 = I(u_1).$$

Corrector. Set $k = 1$.

(i) Compute $g_k = A_k u_k - \lambda_k b_k$ and terminate the iteration if $G_k^{\mathrm{T}} g_k \leqslant 0$, $\| Q_k g_k \| = 0$.

(ii) Set $| g_{kj} | := \max\{ | g_{ki} |, (G_k^{\mathrm{T}} g_k)_i > 0 \}$. If possible, deactivate the jth constraint, $\tilde{I}_k = I_k - \{ j \}$ and determine $\tilde{Q}_k$, otherwise let $\tilde{I}_k = I_k$, $\tilde{Q}_k = Q_k$.

(iii) Solve the linear system

$$\begin{bmatrix} A_k - \lambda_k B_k & -b_k \\ u_k^{\mathrm{T}} A_k^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \delta u_k \\ \delta \lambda_k \end{bmatrix} = \begin{bmatrix} \lambda_k b_k - A_k u_k \\ \frac{1}{2}\left(r_t^2 - u_k^{\mathrm{T}} A_k u_k \right) \end{bmatrix}, \tag{6.11}$$

but in the "free" variables ($i \notin \tilde{I}_k$) only.

(iv) Determine the maximal admissible steplength $\tilde{\alpha}_k$ from

$$\tilde{\alpha}_k = \min_{i \notin I_k} \left(\frac{\psi_i - u_{ki}}{\delta u_{ki}},\ \delta u_{ki} > 0 \right),$$

and set $\alpha_k = \min(\tilde{\alpha}_k, 1)$, $u_{k+1} = u_k + \alpha_k \delta u_k$, $\lambda_{k+1} = \lambda_k + \alpha_k \delta \lambda_k$. Set $k = k + 1$ and go to (i).

We remark that the corrector is a projected Newton step for the augmented nonlinear system

$$Au - \lambda b(u) = 0, \qquad \tfrac{1}{2}\left(u^{\mathrm{T}} A u - r_t^2 \right) = 0, \tag{6.12}$$

in the space of the inactive, respectively, inactivated variables. For a different predictor and comments on this method, see [27].

The interesting solution curves first found in [36] were reproduced in [27] and even, using a multigrid method for parameter-dependent variational inequalities, in [12]. It is beyond the scope of this paper to give any details about this multi-level technique.

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