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On the asymptotic period of powers of a fuzzy matrix

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Abstract

In our prior study, we have examined in depth the notion of an asymptotic period of the power sequence of an $n \times n$ fuzzy matrix with max-Archimedean-t-norms, and established a characterization for the power sequence of an $n \times n$ fuzzy matrix with an asymptotic period using analytical-decomposition methods. In this paper, by using graph-theoretical tools, we further give an alternative proof for this characterization. With the notion of an asymptotic period using graph-theoretical tools, we additionally show a new characterization for the limit behaviour, and then derive some results for the power sequence of an $n \times n$ fuzzy matrix with an asymptotic period.

Keywords: Boolean matrices; Fuzzy matrices; Max-Archimedean-t-norms

1. Introduction

The limit behaviour of consecutive powers of a fuzzy matrix has been widely discussed in the literature. In the study of the powers of a fuzzy matrix, the involvement of different algebraic operations may yield different results. In general, most papers on consecutive powers of a fuzzy matrix are under the max–min operations [1–10], the max-product operations [11–13], max-zero-t-norms [14], and max-Archimedean-t-norms [15]. As in the work of Thomason [10], he proved that the sequence of consecutive powers of a fuzzy matrix with max–min composition either converges to an idempotent matrix or oscillates in finitely many steps. Over a distributive lattice using graph-theoretical tools, Cechlárová [16] studied the powers of a fuzzy matrix. In later years, Han and Li [17] studied the power sequence of incline matrices, to which the boolean matrices, the fuzzy matrices and lattices matrices belong. Moreover, Gavalec [6,7] explored the periodicity and orbits of matrices with max–min compositions. Hashimoto then [8] assumed the transitivity for the fuzzy matrix to ensure convergence. With a clearer view, Fan and Liu [4] defined the concept of maximum principle for the fuzzy matrix to have convergence, and Kolodziejczyk [18] defined the notion of "s-transitive" to have convergence or to oscillate with a period 2. Fan and Liu [5] also explored the oscillating property for the sequence of the powers of a fuzzy matrix. Guu et al. [19], in the year of 2001, extended the study of convergence of powers of a fuzzy matrix to the products of a finite number of fuzzy matrices. In their papers, concepts of compactness and transitivity were extended to show the convergence of products of a finite number of fuzzy matrices. Guu et al. [20] further characterized the convergence of products of a finite number of fuzzy matrices

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in terms of boolean matrices. Possible applications to the products of many finite fuzzy matrices were suggested as well. In particular, Lur et al. [21,22] proposed the notion of simultaneous nilpotent for a finite set of fuzzy matrices.

We [15] characterized the limit behaviour for the sequence of consecutive powers of a fuzzy matrix with the notion of an asymptotic period under max-Archimedean-t-norms by using analytic-decomposition methods. In this paper, by using graph-theoretical tools, we focus on giving an alternative proof for this characterization. Additionally, we shall show a new characterization for the limit behaviour with the notion of an asymptotic period, and concluded with some results for the power sequence of an $n \times n$ fuzzy matrix with an asymptotic.

2. Preliminaries and results

Let $\mathbb{F}$ denote the unit interval, i.e. $\mathbb{F} = [0, 1]$. By a fuzzy matrix, A we mean $A = [a_{ij}]$ with $a_{ij} \in \mathbb{F}$. Let $\mathbb{F}^{n\times n}$ denote the set of all the $n \times n$ fuzzy matrices. We may denote a_{ij} by $[A]_{ij}$. The symbol $\mathbf{0}$ denotes the zero fuzzy matrix and I denotes the identity fuzzy matrix in $\mathbb{F}^{m\times n}$. For $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{F}^{n\times n}$,

$$[A \vee B]_{ij} := a_{ij} \vee b_{ij},$$

where $a_{ij} \vee b_{ij} := \max\{a_{ij}, b_{ij}\}$. We say $A \le B$ if $a_{ij} \le b_{ij}$ for all $1 \le i, j \le n$. Let Φ_A denote the set of all nonzero entries of A, and let $\bar{\lambda}$ denote the largest element in Φ_A. For a $\lambda \in \Phi_A$, A_λ denotes a boolean matrix $[A_\lambda]_{ij}$, where

$$[A_\lambda]_{ij} := \begin{cases} 1 & \text{if } a_{ij} \ge \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{A} = \bar{\lambda} A_{\bar{\lambda}}$, $\underline{A} = \bigvee_{\lambda\in\Phi_A\setminus\{\bar{\lambda}\}} \lambda A_\lambda$ if $\Phi_A \neq \emptyset$ and $\bar{A} = \underline{A} = \mathbf{0}$ if $\Phi_A = \emptyset$. Then we have

$$A = \bigvee_{\lambda\in\Phi_A} \lambda A_\lambda = \bar{A} \bigvee \underline{A}.$$

Definition 1 (*[23]*). Let $T(x, y)$ be a real-valued function on $[0, 1] \times [0, 1]$ with $0 \le T(x, y) \le 1$. T is called a *t-norm* if T satisfies the following conditions:

(a) $T(T(x, y), z) = T(x, T(y, z))$ for all $x, y, z \in [0, 1]$.
(b) $T(x, y) = T(y, x)$ for all $x, y \in [0, 1]$.
(c) $T(x, y) \le T(x_1, y_1)$ for all $0 \le x \le x_1 \le 1$ and $0 \le y \le y_1 \le 1$.
(d) $T(x, 1) = x$ for all $x \in [0, 1]$.

Definition 2. Let T be a t-norm. Let us denote for $k \ge 2$: $T^k(x) = T(T^{k-1}(x), x)$. T is called *Archimedean* if $\lim_{k\to\infty} T^k(x) = 0$ for all $x \in (0, 1)$.

Each Archimedean-t-norm satisfies $T(x, x) < x$ for all $x \in (0, 1)$, but the converse implication holds only with an additional assumption that is upper semicontinuous (see [24, pp. 27–29]. For two fuzzy matrices $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{F}^{n\times n}$, their product is denoted by $A \otimes B$, where $[A \otimes B]_{ij} = \vee_{m=1}^{n} T(a_{im}, b_{mj})$ and T is an Archimedean-t-norm. The notation $A^2_\otimes$ means $A \otimes A$, $A^k_\otimes$ means kth power of A. We say the power sequence of A is *convergent* if the sequence $\{A^k_\otimes : k \in \mathbb{N}\}$ converges; that is, $\lim_{k\to\infty} a^k_{ij}$ exists for all $i, j = 1, 2, \ldots, n$. Let C be an $n \times n$ boolean matrix. Note that $C^2_\otimes = CC$, where CC is the product of boolean matrices in boolean algebra. It is well known that the sequence of consecutive powers of a boolean matrix in max-Archimedean-t-norms either converges in finitely many steps or oscillates with a finite period (see, e.g. [25]). Precisely, we say that the power sequence of C is *p-periodic* if there exist l_0, p such that

$$C^l_\otimes = C^{l+kp}_\otimes, \quad k \in \mathbb{N}, l \ge l_0 \ge 1.$$

The minimal such p is called the *period*. If $p = 1$, the powers of C are convergent.

Let $A = [a_{ij}] \in \mathbb{F}^{n\times n}$ and let T be an Archimedean-t-norm. For $x, y \in [0, 1]$, we denote $xTy = T(x, y)$. The *weighted directed graph* $\mathcal{G}(A)$ associated with A has vertex set $\{1, 2, \ldots, n\}$ and an arc (i, j) from i to j with the weighted a_{ij} if $a_{ij} > 0$. A *directed path* $\gamma(i, i_1, \ldots, i_{k-1}, j)$ with the length k is a sequence of k

arcs $(i, i_1), (i_1, i_2), \ldots, (i_{k-1}, j)$. We may say γ is a k-directed path from i to j. The *weight of a directed path* $\gamma(i_0, i_1, \ldots, i_k)$, as denoted by $w(\gamma(i_0, i_1, \ldots, i_k))$ or simply by $w(\gamma)$, is defined by

$$w(\gamma(i_0, i_1, \ldots, i_k)) := a_{i_0 i_1} T a_{i_1 i_2} T \cdots T a_{i_{k-1} i_k}.$$

A *directed circuit* of the length k is a directed path $\gamma(i_0, i_1, \cdots, i_k)$ with $i_0 = i_k$. The maximum weight of a directed circuit in $\mathcal{G}(A)$ is denoted by $\mu(A)$. A directed circuit with the weight equal to $\mu(A)$ is called a *critical directed circuit*, and vertices on critical directed circuit are called *critical vertices*. Associated with A, we define the *critical fuzzy matrix* A^c of A as

$$[A^c]_{ij} := \begin{cases} a_{ij} & \text{if } a_{ij} \text{ lies on a critical directed circuit,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\mu(A) = 1$, then the critical fuzzy matrix A^c, $\bar{A}$ are boolean matrices and $A^c \leq \bar{A} \leq A$. If there are no directed circuits in $\mathcal{G}(A)$, then we let $\mu(A) = 0$.

For all $k \in \mathbb{N}$, $1 \leq r, s \leq n$, let $\mathcal{L}_k$ denote the set of all k-directed path in $\mathcal{G}(A)$ and let $\mathcal{L}_k^{rs}$ denote the set of all k-directed path from r to s in $\mathcal{G}(A)$. For $\gamma(i_0, i_1, \ldots, i_k) \in \mathcal{L}_k$, the number of arcs (i_t, i_{t+1}) with $0 < a_{i_t i_{t+1}} < 1$ for all $t = 0, 1, \ldots, k-1$, is denoted by $\#\gamma$. For all $i = 0, 1, 2, \ldots, k$, let $\Gamma_i(k) = \{\gamma \in \mathcal{L}_k : \#\gamma = i\}$, and let $\Gamma_i^{rs}(k) = \{\gamma \in \mathcal{L}_k^{rs} : \#\gamma = i\}$. For any subset S of directed paths in $\mathcal{G}(A)$, we denote $w(S) := \max\{w(\gamma) : \gamma \in S\}$, if $S = \emptyset$, then $w(S) = 0$. For any real number x, let us denote $\lfloor x \rfloor$ the largest integer which is less than or equal to x.

Definition 3 (*[15]*). Let A be an $n \times n$ fuzzy matrix. The power sequence $\{A_{\otimes}^l : l \in \mathbb{N}\}$ of fuzzy matrices in $\mathbb{F}^{n \times n}$ is *asymptotically p-periodic* if $\lim_{k\to\infty} A_{\otimes}^{i+kp}$ exists for all $i = 1, 2, \ldots, p$. The minimal such p is called the *asymptotic period p*. If $p = 1$, we have a convergent sequence.

Theorem 1. *Let A be an $n \times n$ fuzzy matrix. Then the following statements are mutually equivalent.*

(i) *The sequence $\{A_{\otimes}^k : k \in \mathbb{N}\}$ has an asymptotic period p.*
(ii) *The powers of $\bar{A}$ have a period p.*
(iii) *The powers of A^c have a period p.*

The equivalence of the two statements (i) and (ii) of Theorem 1 was established by Pang [15] using analytic-decomposition methods. In this article, we give an alternative proof using the graph-theoretical tools. The following lemmas will be needed in the proof of Theorem 1.

Lemma 1. *Let A be an $n \times n$ fuzzy matrix. Then*

(i) *If $\mu(A) = 0$, then $A_{\otimes}^n = \mathbf{0}$.*
(ii) *If $0 < \mu(A) < 1$, then $\lim_{k\to\infty} A_{\otimes}^k = 0$.*

Proof. (i) Assume $A_{\otimes}^n \neq \mathbf{0}$. Then there exists a n-directed path $\gamma(i, i_1, \ldots, i_{n-1}, j)$ for some $1 \leq i, j \leq n$ with $w(\gamma(i, i_1, \ldots, i_{n-1}, j)) \neq 0$. Let $i_0 = i$ and $i_n = j$. By the pigeonhole principle, we have $i_r = i_s$ for some $0 \leq r < s \leq n$. It follows that $\hat{\gamma}(i_r, i_{r+1}, \ldots, i_s)$ is a directed circuit with $w(\hat{\gamma}) \neq 0$. Then $\mu(A) \neq 0$, which leads to a contradiction. Therefore, $A_{\otimes}^n = \mathbf{0}$.

(ii) Let $\alpha = \max\{a_{ij} \mid 0 < a_{ij} < 1\}$ and let m be large enough. For all $1 \leq r, s \leq n$, we have

$$\mathcal{L}_m^{rs} = \Gamma_0^{rs}(m) \cup \left(\bigcup_{j=1}^{m} \Gamma_j^{rs}(m) \right).$$

Note that if $\gamma(i, i_1, \ldots, i_{n-1}, j)$ is a n-directed path with $w(\gamma) = 1$ for some $1 \leq i, j \leq n$, then there exist $0 \leq r < s \leq n$ such that $\hat{\gamma}(i_r, i_{r+1}, \ldots, i_s)$ is a directed circuit with $w(\hat{\gamma}) = 1$, where $i_0 = i$, $i_n = j$. Let $k = \lfloor \frac{m-2n+1}{n} \rfloor$. Then for all $j \leq k$, we have $n - 1 < \frac{m-j}{j+1}$. Since $\mu(A) < 1$, we have

$$\Gamma_0^{rs}(m) = \emptyset \qquad \text{and} \qquad \bigcup_{j=1}^{k} \Gamma_j^{rs}(m) = \emptyset.$$

Then

$$\begin{aligned}[A_\otimes^m]_{rs} &= w(\mathcal{L}_m^{rs}) \\ &= w\left(\Gamma_0^{rs}(m) \cup \left(\bigcup_{j=1}^{k} \Gamma_j^{rs}(m)\right) \cup \left(\bigcup_{j=k+1}^{m} \Gamma_j^{rs}(m)\right)\right) \\ &= w\left(\bigcup_{j=k+1}^{m} \Gamma_j^{rs}(m)\right) \le T^{k+1}(\alpha) \to 0 \qquad \text{as } m \to \infty.\end{aligned}$$

This implies that the sequence of $\{A_\otimes^k : k \in \mathbb{N}\}$ converges to $\mathbf{0}$. ■

Lemma 2. *Let A be an $n \times n$ fuzzy matrix with $\mu(A) = 1$. If the powers of A^c are p-periodic, then for all $1 \le r, s \le n$, $i = 0, 1, 2, \ldots$, there exists a positive integer N_i^{rs} such that*

$$w(\Gamma_i^{rs}(m)) = w(\Gamma_i^{rs}(m + kp)) \quad \textit{for all } m > N_i^{rs}, k = 1, 2, \ldots.$$

Proof. For all $1 \le r, s \le n$, $i = 0, 1, 2, \ldots$. It suffices to show that $w(\Gamma_i^{rs}(m)) = w(\Gamma_i^{rs}(m + p))$. Since the powers of A^c are p-periodic, there exists l_0 such that $(A^c)_\otimes^l = (A^c)_\otimes^{l+kp}$ for all $k \in \mathbb{N}, l \ge l_0 \ge 1$. Let $N_i^{rs} = n(i+1)(l_0+p) - 1 - p$. Then $m > N_i^{rs}$ is equivalent to $(m+p-i)/(i+1) > n(l_0+p) - 1$, which implies by a simple counting argument that any directed path $\gamma(r = i_0, i_1, \ldots, i_{m+p} = s) \in \Gamma_i^{rs}(m+p)$ contains one t-directed path $\gamma'(i_h, i_{h+1}, \ldots, i_{h+t})$ with $w(\gamma') = 1$, where $t \ge n(l_0 + p), 0 \le h \le h + t \le m + p$.

Claim. The directed path $\gamma'(i_h, i_{h+1}, \ldots, i_{h+t})$ contains a critical directed circuit with the length greater than or equal to $l_0 + p$.

Put $n_1 = h$, and let s_1 be the maximum integer such that $n_1 \le s_1 \le h + t$ and $i_{n_1} = i_{s_1}$. Put $n_2 = s_1 + 1$, and let s_2 be the maximum integer such that $n_2 \le s_2 \le h + t$ and $i_{n_2} = i_{s_2}$. Following the continuity, we have a sequence of

$$h = n_1 \le s_1 < n_2 \le s_2 < \cdots < n_{\hat{j}} \le s_{\hat{j}}$$

with $\hat{j} \le n$ and $i_{n_k} = i_{s_k}$ for all $k = 1, 2, \ldots, \hat{j}$. Then

$$\sum_{k=1}^{\hat{j}} |s_k - n_k| + (\hat{j} - 1) \ge n(l_0 + p),$$

so that there exists $1 \le \hat{i} \le \hat{j}$ such that $|s_{\hat{i}} - n_{\hat{i}}| \ge l_0 + p$. Then the directed path $\gamma'(i_h, i_{h+1}, \ldots, i_{h+t})$ contains a critical directed circuit with the length greater than or equal to $l_0 + p$. Without loss of generality, we assume that $|s_1 - n_1| \ge l_0 + p$. Since the powers of A^c are p-periodic, there exists a critical directed circuit $\gamma''(i_{n_1}, i_{\hat{r}}, \ldots, i_{\hat{r}+|s_1-n_1|-p-2}, i_{s_1})$ with the length $|s_1 - n_1| - p$. Then the directed path

$$\hat{\gamma}(i_0, i_1, \ldots, i_{n_1}, i_{\hat{r}}, \ldots, i_{\hat{r}+|s_1-n_1|-p-2}, i_{s_1}, \ldots, i_{m+p}) \in \Gamma_i^{rs}(m),$$

and $w(\gamma) = w(\hat{\gamma})$. This implies that

$$w(\Gamma_i^{rs}(m + p)) \le w(\Gamma_i^{rs}(m)). \tag{1}$$

On the other hand, since $m > N_i^{rs}$ is equivalent to

$$\frac{m - i}{i + 1} > nl_0 - 1 + \frac{(ni + n - 1)p}{i + 1} \ge nl_0 - 1,$$

which implies by a simple counting argument that for any directed path $\gamma(i_0, i_1, \ldots, i_m) \in \Gamma_i^{rs}(m)$ contains a t-directed path $\gamma'(i_h, i_{h+1}, \ldots, i_{h+t})$ with $w(\gamma') = 1$, where $t \ge nl_0, 0 \le h \le h + t \le m$.

Claim. The directed path $\gamma'(i_l, i_{l+1}, \ldots, i_{l+t})$ contains a critical directed circuit with the length greater than or equal to l_0.

Put $n_1 = h$, and let s_1 be the maximum integer such that $n_1 \le s_1 \le h + t$ and $i_{n_1} = i_{s_1}$. Put $n_2 = s_1 + 1$, and let s_2 be the maximum integer such that $n_2 \le s_2 \le h + t$ and $i_{n_2} = i_{s_2}$. Following the continuity, we have a sequence

$$h = n_1 \le s_1 < n_2 \le s_2 < \cdots < n_{\hat{j}} \le s_{\hat{j}}$$

with $\hat{j} \le n$ and $i_{n_k} = i_{s_k}$ for all $k = 1, 2, \ldots, \hat{j}$. Then

$$\sum_{k=1}^{\hat{j}} |s_k - n_k| + (\hat{j} - 1) \ge nl_0,$$

so that there is $1 \le \hat{i} \le \hat{j}$ such that $|s_{\hat{i}} - n_{\hat{i}}| \ge l_0$. Then the directed path $\gamma'(i_h, i_{h+1}, \ldots, i_{h+t})$ contains a directed circuit with the length greater than or equal to l_0. Without loss of generality, we assume that $|s_1 - n_1| \ge l_0$. Since the powers of A^c are p-periodic, there exists a critical directed circuit $\gamma''(i_{n_1}, i_{\hat{r}}, \ldots, i_{\hat{r}+|s_1-n_1|+p-2}, i_{s_1})$ with the length $|s_1 - n_1| + p$. Then the path

$$\hat{\gamma}(i_0, \ldots, i_{n_1}, i_{\hat{r}}, \ldots, i_{\hat{r}+|s_1-n_1|+p-2}, i_{s_1}, \ldots, i_m) \in \Gamma_i^{rs}(m + p),$$

and clearly $w(\gamma) = w(\hat{\gamma})$. This implies that

$$w(\Gamma_i^{rs}(m)) \le w(\Gamma_i^{rs}(m + p)). \tag{2}$$

Hence by (1) and (2), we have $w(\Gamma_i^{rs}(m + p)) = w(\Gamma_i^{rs}(m))$. This completes the proof. ■

We proceed now to prove Theorem 1. We first prove that the following statements are mutually equivalent:

(i)′ The sequence $\{A_\otimes^k : k \in \mathbb{N}\}$ is asymptotically p-periodic;
(ii)′ The powers of $\bar{A}$ are p-periodic;
(iii)′ The powers of A^c are p-periodic.

If $\mu(A) < 1$, then (i)′ ⇔(ii)′ ⇔ ow(iii)′ follows from Lemma 1 and $A^c \le A$ and $\bar{A} \le A$. Next, we consider the case $\mu(A) = 1$.

(i)′ ⇒ (ii)′. Since the sequence $\{A_\otimes^k : k \in \mathbb{N}\}$ is asymptotically p-periodic, then for all $1 \le \hat{i} \le p$, we let $\tilde{A}_\otimes^{\hat{i}} = \lim_{k\to\infty} A_\otimes^{\hat{i}+kp}$. Let $1 \le r, s \le n$. For all $\hat{i} = 1, 2, \ldots, p$, $k = 1, 2, \ldots$, we have

$$\begin{aligned}[A_\otimes^{\hat{i}+kp}]_{rs} &= w(\mathcal{L}_{\hat{i}+kp}^{rs}) \\ &= w(\Gamma_0^{rs}(\hat{i} + kp)) \bigvee w\left(\bigcup_{j=1}^{\hat{i}+kp} \Gamma_j^{rs}(\hat{i} + kp)\right).\end{aligned}$$

Moreover,

$$w(\Gamma_0^{rs}(\hat{i} + kp)) \in \{0, 1\} \quad \text{and} \quad w\left(\bigcup_{j=1}^{\hat{i}+kp} \Gamma_j^{rs}(\hat{i} + kp)\right) \le \alpha < 1,$$

where $\alpha = \max\{a_{ij} : 0 \le a_{ij} < 1\}$. Then we have for all $k = 1, 2, \ldots$,

$$w(\mathcal{L}_{\hat{i}+kp}^{rs}) = 1 \quad \text{if and only if } w(\Gamma_0^{rs}(\hat{i} + kp)) = 1$$

and

$$w(\mathcal{L}_{\hat{i}+kp}^{rs}) \le \alpha \quad \text{if and only if } w(\Gamma_0^{rs}(\hat{i} + kp)) = 0.$$

We distinguish two cases:

Case 1. If $[\tilde{A}_\otimes^{\hat{i}}]_{rs} = 1$, then there exists a positive integer $N_{\hat{i}}$ such that $[A_\otimes^{\hat{i}+kp}]_{rs} = 1$ for all $k \ge N_{\hat{i}}$, so that

$$w(\mathrm{L}_{\hat{i}+kp}^{rs}) = 1 \quad \text{and} \quad w(\Gamma_0^{rs}(\hat{i} + kp)) = 1 \quad \text{for } k \ge N_{\hat{i}}.$$

Hence, $[\bar{A}_\otimes^{\hat{i}+kp}]_{rs} = 1$ for all $k \ge N_{\hat{i}}$.

Case 2. If $[\tilde{A}_\otimes^{\hat{i}}]_{rs} \ne 1$, then there exists a positive integer N such that $[A_\otimes^{\hat{i}+kp}]_{rs} \le \alpha < 1$ for $k \ge N_{\hat{i}}$, so that

$$w(\mathcal{L}_{\hat{i}+kp}^{rs}) \le \alpha \quad \text{and} \quad w(\Gamma_0^{rs}(\hat{i} + kp)) = 0 \quad \text{for } k \ge N_{\hat{i}}.$$

Hence, $[\bar{A}_{\otimes}^{\hat{i}+kp}]_{rs} = 0$ for all $k \geq N_{\hat{i}}$.

Let $N = \max_{1\leq \hat{i} \leq p}(\hat{i} + pN_{\hat{i}})$. Then we have

$$[\bar{A}_{\otimes}^{l}]_{rs} = [\bar{A}_{\otimes}^{l+kp}]_{rs} \quad \text{for all } k \in \mathbb{N}, l \geq N.$$

Therefore, the powers of $\bar{A}$ are p-periodic.

(ii)$'$ $\Rightarrow$ (iii)$'$. Since $\mu(A) = 1$, there exists a critical directed circuit in $\mathcal{G}(A)$ with the weight equal to 1, so that each entries of A^c is either 0 or 1. For all $1 \leq i, j \leq n$,

$$\begin{aligned}[\bar{A}^c]_{ij} = 1 &\Leftrightarrow \text{ there exist } 1 \leq i, j, i_1, \ldots, i_k, i \leq n \text{ such that } \bar{a}_{ij} T \bar{a}_{ji_1} T \cdots T \bar{a}_{i_k i} = 1\\ &\Leftrightarrow \bar{a}_{ij} = \bar{a}_{ji_1} = \cdots = \bar{a}_{i_k i} = 1\\ &\Leftrightarrow a_{ij} = a_{ji_1} = \cdots = a_{i_k i} = 1\\ &\Leftrightarrow \text{ there exist } 1 \leq i, j, i_1, \ldots, i_k, i \leq n \text{ such that } a_{ij} T a_{ji_1} T \cdots T a_{i_k i} = 1\\ &\Leftrightarrow [A^c]_{ij} = 1.\end{aligned}$$

Then we have $\bar{A}^c = A^c$. Since the powers of $\bar{A}$ are p-periodic, the powers of $\bar{A}^c$ are p-periodic (by Theorem 5.4.25 (3) in [25]). Therefore, the powers of A^c are p-periodic.

(iii)$'$ $\Rightarrow$ (i)$'$. Assume that the powers of A^c are p-periodic and let $\alpha = \max\{a_{ij} : 0 \leq a_{ij} < 1\}$. For $\varepsilon > 0$ be given. Since $\lim_{k\to\infty} T^k(\alpha) = 0$, there exists a integer $\hat{i}$ such that $T^j(\alpha) < \varepsilon/2$ for all $j \geq \hat{i}$. Let $1 \leq r, s \leq n$. For all $m = 1, 2, \ldots$ and $k = 1, 2, \ldots$, we have

$$\mathcal{L}_m^{rs} = \Gamma_0^{rs}(m) \cup \left(\bigcup_{j=1}^{\hat{i}} \Gamma_j^{rs}(m) \right) \cup \left(\bigcup_{j=\hat{i}+1}^{m} \Gamma_j^{rs}(m) \right)$$

and

$$\mathcal{L}_{m+kp}^{rs} = \Gamma_0^{rs}(m+kp) \cup \left(\bigcup_{j=1}^{\hat{i}} \Gamma_j^{rs}(m+kp) \right) \cup \left(\bigcup_{j=\hat{i}+1}^{m+kp} \Gamma_j^{rs}(m+kp) \right).$$

By Lemma 2, we may choose a positive integer N^{rs} such that for all $j = 0, 1, 2, \ldots, \hat{i}$,

$$w(\Gamma_j^{rs}(m)) = w(\Gamma_j^{rs}(m+kp)) \quad \text{for } m > N^{rs}, k \in \mathbb{N}.$$

Also, we have $k \in \mathbb{N}$,

$$|w(\Gamma_j^{rs}(m)) - w(\Gamma_j^{rs}(m+kp))| \leq 2T^{\hat{i}+1}(\alpha) < \varepsilon,$$

where $\hat{i} + 1 \leq j \leq m + kp$.

Since

$$[A_{\otimes}^{m}]_{rs} = w(\mathcal{L}_m^{rs}) \quad \text{and} \quad [A_{\otimes}^{m+kp}]_{rs} = w(\mathcal{L}_{m+kp}^{rs}),$$

then we have

$$|[A_{\otimes}^{m}]_{rs} - [A_{\otimes}^{m+kp}]_{rs}| < \varepsilon \quad \text{for } m > N^{rs}, k \in \mathbb{N}.$$

So $\lim_{k\to\infty}[A_{\otimes}^{m+kp}]_{rs}$ exists for $m > N^{rs}$. Hence, $\lim_{k\to\infty} A_{\otimes}^{i+kp}$ exists for $i = 1, 2, \ldots p$. Therefore, the sequence $\{A_{\otimes}^{k} : k \in \mathbb{N}\}$ is asymptotically p-periodic.

Next, we prove that the sequence $\{A_{\otimes}^{k} : k \in \mathbb{N}\}$, the powers of $\bar{A}$ and the powers of A^c have the same period. Assume that the sequence $\{A_{\otimes}^{k} : k \in \mathbb{N}\}$ has an asymptotic period p_1, the powers of $\bar{A}$ have a period p_2 and the powers of A^c have a period p_3. Then we have $p_2 \leq p_1$ by implication (i)$'$ $\Rightarrow$ (ii)$'$, $p_3 \leq p_2$ by implication (ii)$'$ $\Rightarrow$ (iii)$'$ and $p_1 \leq p_3$ by implication (iii)$'$ $\Rightarrow$ (i)$'$, so that $p_1 = p_2 = p_3$. This completes the proof of Theorem 1. ∎

Example 1. Consider the following 4×4 fuzzy matrix

$$A = \begin{bmatrix} 1/3 & 1 & 1/2 & 0 \\ 1 & 1/3 & 1/3 & 0 \\ 1/2 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

Then,

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that the powers of $\bar{A}$ have a period two and the powers of A^c have a period two. The directed computation verifies this assertion:

$$A_{\otimes}^{k} = \begin{bmatrix} 1 & 1/2 & 1/3 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 1 & 1/2 & 1/3 & 0 \\ 0 & 0 & 0 & T^k(1/2) \end{bmatrix}, \quad k = 4, 6, 9, \ldots$$

and

$$A_{\otimes}^{k} = \begin{bmatrix} 1/2 & 1 & 1/2 & 0 \\ 1 & 1/2 & 1/3 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & T^k(1/2) \end{bmatrix}, \quad k = 5, 7, 9, \ldots.$$

Then

$$\lim_{k\to\infty} A_{\otimes}^{2k} = \begin{bmatrix} 1 & 1/2 & 1/3 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 1 & 1/2 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\lim_{k\to\infty} A_{\otimes}^{2k+1} = \begin{bmatrix} 1/2 & 1 & 1/2 & 0 \\ 1 & 1/2 & 1/3 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the sequence $\{A_{\otimes}^{k} : k \in \mathbb{N}\}$ has an asymptotic period two.

The following theorem provides an extension of Fan's theorem in [14].

Corollary 1. *Let A be an $n \times n$ fuzzy matrix. Then the following statements are mutually equivalent:*

(i) *The sequence $\{A_{\otimes}^{k} : k \in \mathbb{N}\}$ is convergent.*
(ii) *The powers of $\bar{A}$ are convergent.*
(iii) *The powers of A^c are convergent.*

Proof. This is the case of $p = 1$ in Theorem 1. ■

The equivalence of the two statements (i) and (ii) of Corollary 1 were established by Fan [14].

Theorem 2. *Let A be an $n \times n$ fuzzy matrix. Then the following statements are mutually equivalent:*

(i) *The sequence $\{A_{\otimes}^{k} : k \in \mathbb{N}\}$ converges to **0**.*
(ii) *The powers of $\bar{A}$ converge to **0**.*

(iii) *The powers of* A^c *converge to* $\mathbf{0}$.

Proof. If $\mu(A) = 1$, then there exists a directed circuit $\gamma(i, i_1, \ldots, i_{t-1}, i)$ with $w(\gamma) = 1$ for some $1 \le i \le n$, so that $[(A^c)^{tk}_{\otimes}]_{ii} = [(\bar{A})^{tk}_{\otimes}]_{ii} = 1$ for all $k \in \mathbb{N}$. Then $\lim_{k\to\infty}(A^c)^k_{\otimes} \ne \mathbf{0}$ and $\lim_{k\to\infty}(\bar{A})^k_{\otimes} \ne \mathbf{0}$. Since for all $k = 1, 2, \ldots, (A^c)^k_{\otimes} \le A^k_{\otimes}$ and $(\bar{A})^k_{\otimes} \le A^k_{\otimes}$, $\lim_{k\to\infty} A^k_{\otimes} \ne \mathbf{0}$. Next, we consider the case $\mu(A) < 1$. By Lemma 1, the sequence $\{A^k_{\otimes} : k \in \mathbb{N}\}$ converges to $\mathbf{0}$. Since for all $k = 1, 2, \ldots, (A^c)^k_{\otimes} \le A^k_{\otimes}$ and $(\bar{A})^k_{\otimes} \le A^k_{\otimes}$, $\lim_{k\to\infty}(A^c)^k_{\otimes} = \lim_{k\to\infty}(\bar{A})^k_{\otimes} = \mathbf{0}$. This completes the proof. ■

The equivalence of the two statements (i) and (ii) Theorem 2 were proved by Pang [15] using analytical-decomposition methods.

Example 2. Consider the following 3×3 fuzzy matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then,

$$\bar{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $\mu(A) = 1/2$ and $\bar{A}^3_{\otimes} = \mathbf{0}$. For all $k = 3, 4, \ldots$,

$$A^{2k}_{\otimes} = \begin{bmatrix} 0 & T^{k-1}\left(\frac{1}{2}\right) & T^k\left(\frac{1}{2}\right) \\ 0 & T^k\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & T^k\left(\frac{1}{2}\right) \end{bmatrix}$$

and

$$A^{2k+1}_{\otimes} = \begin{bmatrix} 0 & T^k\left(\frac{1}{2}\right) & T^k\left(\frac{1}{2}\right) \\ 0 & 0 & T^{k+1}\left(\frac{1}{2}\right) \\ 0 & T^k\left(\frac{1}{2}\right) & 0 \end{bmatrix}$$

and for all $k = 2, 3, \ldots$,

$$(A^c)^{2k}_{\otimes} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & T^k\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & T^k\left(\frac{1}{2}\right) \end{bmatrix}$$

and

$$(A^c)^{2k+1}_{\otimes} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T^{k+1}\left(\frac{1}{2}\right) \\ 0 & T^k\left(\frac{1}{2}\right) & 0 \end{bmatrix}.$$

Then $\lim_{k\to\infty} A^k_{\otimes} = \lim_{k\to\infty}(A^c)^k_{\otimes} = \mathbf{0}$.

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