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Spherical functions on Euclidean space

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Abstract

We study spherical functions on Euclidean spaces from the viewpoint of Riemannian symmetric spaces. Here the Euclidean space $\mathbb{E}^n = G/K$ where G is the semidirect product $\mathbb{R}^n \cdot K$ of the translation group with a closed subgroup K of the orthogonal group $O(n)$. We give exact parameterizations of the space of (G, K)—spherical functions by a certain affine algebraic variety, and of the positive definite ones by a real form of that variety. We give exact formulae for the spherical functions in the case where K is transitive on the unit sphere in $\mathbb{E}^n$.

Keywords: Spherical function; Symmetric space; Euclidean space; Bessel function

1. Introduction

Riemannian symmetric spaces carry a distinguished class of functions, the *spherical functions*, that generalize the notion of characters on a commutative group. In particular, if $X = G/K$ is a Riemannian symmetric space, then $L^2(X)$ is a continuous direct sum (direct integral) of Hilbert spaces E_φ generated by positive definite spherical functions $\varphi : X \to \mathbb{C}$, G acts naturally on E_φ by an irreducible unitary representation π_φ, and the left regular representation of G on $L^2(X)$ is a continuous direct sum of the π_φ. This circle of ideas has been studied in detail for a number of years.

With one "small" exception, the symmetric space X determines a standard choice of (G, K), as follows: G is the group $\mathcal{I}(X)$ of all isometries of X, or the subgroup generated by all geodesic

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symmetries of X, or the identity component $\mathcal{I}(X)^0$, and the latter is the group generated by all products of an even number of geodesic symmetries; in any case K is the isotropy subgroup of G at a "base point" $x_0 = 1K$. The exception is when X is of the form $X' \times \mathbb{E}^n$ where $\mathbb{E}^n$ is an Euclidean space of dimension $n > 0$ and X' has no Euclidean factor. Then $G = G' \times J$ where G' is determined by X' as above and where J is any closed subgroup of the Euclidean group $E(n)$ that contains the translations. There are infinitely many natural choices of J, and the spherical functions have only been studied in a serious way for two of those choices. In this note we hope to close that gap by presenting a general theory of spherical functions on Euclidean spaces. The main results are Theorems 4.2 and 4.5 concerning spherical functions in general and their parameterization by a certain affine variety, Theorem 5.5 which parameterizes the positive definite spherical functions, and Theorems 6.3 and 6.6 concerning the case where the isotropy group is transitive on the unit sphere in the tangent space.

2. Background on spherical functions

Spherical functions on a Riemannian symmetric space $X = G/K$ can be characterized in a number of ways. We recall the relevant characterizations from [2,4,5,11].

The basic definition, which does not use differentiability, is this. A function $\varphi : G \to \mathbb{C}$ is spherical relative to (G, K) if:

(i) it is continuous,
(ii) it is bi-K-invariant (in other words, is a K-invariant function on $X = G/K$),
(iii) it is normalized by $\varphi(1) = 1$, and
(iv) if $f \in C_c(K\backslash G/K)$ there exists $\lambda_f \in \mathbb{C}$ such that $f * \varphi = \lambda_f \varphi$.

Here $C_c(K\backslash G/K)$ denotes the space of all compactly supported bi-K-invariant continuous functions on G, in other words continuous compactly supported K-invariant functions on X.

Spherical functions are also characterized as the continuous functions $\varphi : G \to \mathbb{C}$, not identically zero, that satisfy the functional equation: if $g_1, g_2 \in G$ then the integral $\varphi(g_1)\varphi(g_2) = \int_K \varphi(g_1 k g_2)\, d\mu_K(k)$. We will make serious use of the functional equation.

Finally, spherical functions $\varphi : G \to \mathbb{C}$ are characterized as the joint eigenfunctions of the algebra $\mathcal{D}(G, K)$ of G-invariant differential operators on X, normalized to take the value 1 at the base point $x_0 = 1K$. Note that the joint eigenvalue is an associative algebra homomorphism $\chi_\varphi : \mathcal{D}(G, K) \to \mathbb{C}$. We will need the fact that if φ and φ' are spherical functions, and if the joint eigenvalues $\chi_\varphi = \chi_{\varphi'}$, then $\varphi = \varphi'$.

The notion of induced spherical function mirrors the notion of induced representation. Let $Q \subset G$ be a closed subgroup such that K is transitive on G/Q, i.e. $G = KQ$, i.e. $G = QK$, i.e. Q is transitive on G/K. Let $\zeta : Q \to \mathbb{C}$ be spherical for $(Q, Q \cap K)$. The *induced spherical function* is

$$\big[\mathrm{Ind}_Q^G(\zeta)\big](g) = \int_K \widetilde{\zeta}(gk)\, d\mu_K(k) \quad \text{where } \widetilde{\zeta}(kq) = \zeta(q)\Delta_{G/Q}(q)^{-1/2}. \tag{2.1}$$

Here $\Delta_{G/Q} : Q \to \mathbb{R}$ is the quotient of modular functions, $\Delta_{G/Q}(q) = \Delta_G(q)/\Delta_Q(q) = \Delta_Q(q)^{-1}$.

3. Background on Euclidean space

Let K be any closed subgroup of the orthogonal group $O(n)$ and consider the semidirect product group $G = \mathbb{R}^n \cdot K$. That gives an expression of Euclidean space as a Riemannian symmetric coset space, $\mathbb{E}^n = G/K$. Here K is identified with the isotropy subgroup of G at the base point $(0, K)$ corresponding to a choice of origin. The extreme cases are $K = \{1\}$, where G is just the group of Euclidean translations of $\mathbb{E}^n$, and $K = O(n)$ or $SO(n)$, where G is the full Euclidean group $E(n)$ or its identity component $E(n)^0$.

The group K acts on $\mathbb{C}^n$ by complexification of its natural action on $\mathbb{R}^n$ as a subgroup of $O(n)$. It preserves the $\mathbb{C}$-bilinear form $b(\xi, \eta)$ that is the complex extension of the $O(n)$-invariant inner product on $\mathbb{R}^n$. If $\xi \in \mathbb{C}^n$ we have the quasicharacter $\varphi_\xi : \mathbb{R}^n \to \mathbb{C}$ given by $\varphi_\xi(x) = e^{ib(x,\xi)}$. In the identification of $\mathbb{R}^n$ with $\mathbb{E}^n$ this gives a function which we also write as $\varphi_\xi : \mathbb{E}^n \to \mathbb{C}$. The compact group K rotates that function and we average it over K to obtain

$$\varphi_\xi^K(x) := \int_K \varphi_{k(\xi)}(x)\, d\mu_K = \int_K e^{ib(x,k(\xi))}\, d\mu_K. \tag{3.1}$$

Lemma 3.2. *If $\xi \in \mathbb{C}^n$ then the lift of φ_ξ^K from $\mathbb{E}^n$ to G is a (G, K)-spherical function.*

This is standard and can be found in any of [2,4,5,11]. Curiously, the fact that every (G, K)-spherical function is one of the φ_ξ^K seems only to be in the literature for the extreme cases mentioned above. We give a general proof below.

In the special case $K = \{1\}$ we have $\varphi_\xi^K = \varphi_\xi$, resulting in all the quasicharacters on $\mathbb{R}^n$ and thus all the spherical functions on $\mathbb{E}^n$.

In the trivial case $n = 1$, either $K = \{1\}$ and the matter is described above, or $K = \{\pm 1\}$ and $\mathcal{D}(G, K)$ is the algebra of polynomials in $\Delta = -d^2/dx^2$. In the latter case, the solutions to $\Delta f = \lambda^2 f$ on $\mathbb{R}$ are the linear combinations of $e^{\pm i\lambda x}$, so the K-invariant ones are the multiples of $\cosh(i\lambda x)$. Thus, if $K = \{\pm 1\}$ we have $\varphi_\xi^K(x) = \cosh(i\sqrt{\xi}x)$.

Now suppose $n > 1$. In the special case $K = O(n)$ or $SO(n)$, the φ_ξ^K are radial functions. Here $\mathcal{D}(G, K)$ is the algebra of polynomials in the Laplace–Beltrami operator $\Delta = -\sum \partial^2/\partial x_i^2$, so the joint eigenvalue of φ_ξ^K is specified by its Δ-eigenvalue. That eigenvalue is $b(\xi, \xi)$, because $\Delta(\varphi_\xi^K)(x) = \Delta_x \int_K e^{ib(x,k(\xi))}\, d\mu_K = \int_K \Delta_x e^{ib(x,k(\xi))}\, d\mu_K = \int_K b(k(\xi), k(\xi)) e^{ib(x,k(\xi))}\, d\mu_K = \int_K b(\xi, \xi) e^{ib(x,k(\xi))}\, d\mu_K = b(\xi, \xi)\varphi_\xi^K(x)$. Note that φ_ξ^K is a radial function and the radial part of $-\Delta$ is $d^2/dr^2 + ((n-1)/r)(d/dr)$. Comparing this with the Bessel equation

$$t^2 \frac{d^2 f}{dt^2} + t\frac{df}{dt} + \left(t^2 - \nu^2\right) f = 0$$

of order $\nu = (n-2)/2$, we see from [8, Chapter III] that the radial eigenfunctions of Δ for eigenvalue λ^2 are the multiples of

$$(\lambda r)^{-\nu} J_\nu(\lambda r) = (\lambda r)^{-\nu} \cdot \frac{(\frac{1}{2}\lambda r)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(\lambda r \cos\theta) \sin^{2\nu}\theta\, d\theta$$

$$= \frac{1}{2^\nu \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos(\lambda r \cos\theta) \sin^{2\nu}\theta \, d\theta, \quad \nu = \frac{n-2}{2},$$

where J_ν is the Bessel function of order ν. Note from the last expression that this is even in λ. Given $\xi \in \mathbb{C}^n$ we take λ to be either of $\pm\sqrt{b(\xi,\xi)}$. Taking account of $\varphi_\xi^K(0) = 1$ we have

$$\begin{aligned}\varphi_\xi^K(x) &= c(n)\big(\lambda\|x\|\big)^{-(n-2)/2} J_{(n-2)/2}\big(\lambda\|x\|\big)\\ &= \frac{c(n)}{2^{(n-2)/2}\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \cos\big(\lambda\|x\|\cos\theta\big) \sin^{n-2}\theta \, d\theta,\end{aligned}$$

$$\text{where } \lambda = \pm\sqrt{b(\xi,\xi)} \text{ and } c(n)^{-1} = \frac{1}{2^{(n-2)/2}\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_0^\pi \sin^{n-2}\theta \, d\theta.$$

Since

$$\int_0^\pi \sin^{n-2}\theta \, d\theta = \sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right) \Big/ \Gamma\left(\frac{n}{2}\right) \quad \text{for } n > 1$$

we have $c(n) = \pi^{-1/2} 2^{(n-2)/2} \Gamma(1/2)\Gamma(n/2)$, and

$$\varphi_\xi^K(x) = \frac{\pi^{-1/2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \int_0^\pi \cos\big(\sqrt{b(\xi,\xi)}\|x\|\cos\theta\big) \sin^{n-2}\theta \, d\theta \quad \text{for } n > 1. \tag{3.3}$$

Of course, in general one cannot expect an explicit formula such as (3.3) for arbitrary cases of K, but there is a certain structure, and now we examine it.

4. General spherical functions on Euclidean space

In this section, K is an arbitrary closed subgroup of the orthogonal group $O(n)$, and G is the semidirect product $\mathbb{R}^n \cdot K$ consisting of all translations of $\mathbb{E}^n$ with those rotations that are given by elements of K.

As a closed subgroup of the Lie group $O(n)$, K is a compact linear Lie group. Thus we can define its complexification $K_\mathbb{C}$ as follows. The identity component $K_\mathbb{C}^0$ is the analytic subgroup of $GL(n;\mathbb{C})$ with Lie algebra $\mathfrak{k}_\mathbb{C}$. Since $Ad(K)\mathfrak{k} = \mathfrak{k}$ we have $Ad(K)\mathfrak{k}_\mathbb{C} = \mathfrak{k}_\mathbb{C}$, so K normalizes $K_\mathbb{C}^0$. Thus $K_\mathbb{C} := K K_\mathbb{C}^0$ is well defined and acts on $\mathbb{C}^n$ as a closed complex subgroup of the complex orthogonal group $O(n;\mathbb{C})$. Note that K is a maximal compact subgroup of $K_\mathbb{C}$.

Recall the categorical quotient $\mathbb{C}^n /\!/ K_\mathbb{C}$. If $\mathcal{P}(\mathbb{C}^n)^K$ denotes the algebra of all $K_\mathbb{C}$-invariant polynomials on $\mathbb{C}^n$, we have the equivalence relation that $\xi \sim \xi'$ if $p(\xi) = p(\xi')$ for all $p \in \mathcal{P}(\mathbb{C}^n)^K$. Then $\mathbb{C}^n /\!/ K_\mathbb{C}$ is the space of equivalence classes, and it has the structure of affine variety for which $\mathcal{P}(\mathbb{C}^n)^K$ is the algebra of rational functions. See [1] for a good quick development of this material, [6] and [3] for complete treatments.

Lemma 4.1. *Fourier transform gives a K-equivariant from the space $\mathcal{D}(\mathbb{C}^n)$ of constant coefficient differential operators on $\mathbb{R}^n$ onto the space $\mathcal{P}(\mathbb{C}^n)$ of polynomials on $\mathbb{C}^n$. In particular it gives an isomorphism of $\mathcal{D}(G, K)$ onto $\mathcal{P}(\mathbb{C}^n)^K$.*

Without K-equivariance, this is a standard basic fact from Fourier analysis, and the K-equivariance is clear because b is K-invariant.

Theorem 4.2. *Let $\xi, \xi' \in \mathbb{C}^n$. Then the following conditions are equivalent*:

1. *The (G, K)-spherical functions $\varphi^K_\xi = \varphi^K_{\xi'}$.*
2. *The orbit closure $\mathrm{cl}\, K_\mathbb{C}(\xi)$ meets $\mathrm{cl}\, K_\mathbb{C}(\xi')$.*
3. *If p is a $K_\mathbb{C}$-invariant polynomial on $\mathbb{C}^n$ then $p(\xi) = p(\xi')$.*
4. *The vectors $\xi, \xi' \in \mathbb{C}^n$ have the same image under the projection $\pi : \mathbb{C}^n \to \mathbb{C}^n /\!/ K_\mathbb{C}$ to the categorical quotient.*

Proof. First note that $\xi' \mapsto \varphi^K_{\xi'}(x)$ is constant on every K-orbit in $\mathbb{C}^n$, for if $\xi' = k'(\xi'')$ then

$$\begin{aligned}\varphi^K_{\xi'}(x) &= \int_K e^{ib(x,kk'(\xi''))}\, d\mu_K(k) = \int_K e^{ib(k'^{-1}k^{-1}(x),\xi'')}\, d\mu_K(k) \\ &= \int_K e^{ib(k^{-1}(x),\xi'')}\, d\mu_K(k) = \int_K e^{ib(x,k(\xi''))}\, d\mu_K(k) = \varphi^K_{\xi''}(x).\end{aligned}$$

Second, note that the function $\varphi^K_{\xi'}(x)$ is holomorphic in ξ'. Thus $f(\xi') = \varphi^K_{\xi'}(x)$ is a holomorphic function on the complex manifold $K_\mathbb{C}(\xi)$ that is constant on each of the totally real submanifolds $K(\xi')$, $\xi' \in K_\mathbb{C}(\xi)$. If J is the almost complex structure operator on $K_\mathbb{C}(\xi)$ then the real tangent spaces satisfy $T_{\xi'}(K_\mathbb{C}(\xi')) = T_{\xi'}(K(\xi')) + J(T_{\xi'}(K(\xi')))$. Thus $df = 0$ and f is constant on every topological component of $K_\mathbb{C}(\xi)$. But $K(\xi)$ meet every topological component of $K_\mathbb{C}(\xi)$, and f is constant on $K(\xi)$, so now f is constant on $K_\mathbb{C}(\xi)$. As f is continuous it is constant on $\mathrm{cl}(K_\mathbb{C}(\xi))$. Similarly f is constant on $\mathrm{cl}(K_\mathbb{C}(\xi'))$. Thus, if $\mathrm{cl}(K_\mathbb{C}(\xi))$ meets $\mathrm{cl}(K_\mathbb{C}(\xi'))$ then $\varphi^K_{\xi'} = \varphi^K_\xi$.

Conversely suppose that $\mathrm{cl}(K_\mathbb{C}(\xi))$ does not meet $\mathrm{cl}(K_\mathbb{C}(\xi'))$. Then we have distinct points $z, z' \in \mathbb{C}^n /\!/ K_\mathbb{C}$ such that $\mathrm{cl}(K_\mathbb{C}(\xi)) = \pi^{-1}(z)$ and $\mathrm{cl}(K_\mathbb{C}(\xi')) = \pi^{-1}(z')$, so there is a rational function on $\mathbb{C}^n /\!/ K_\mathbb{C}$ with value 0 at z and 1 at z'. That function lifts to a $K_\mathbb{C}$-invariant polynomial p on $\mathbb{C}^n$, by Lemma 4.1. Here $p(\xi) = 0$ and $p(\xi') = 1$. The inverse Fourier transform of multiplication by p is a K-invariant constant coefficient differential operator $D \in \mathcal{D}(G/K)$. By construction $D\varphi^K_\xi = 0$ and $D\varphi^K_{\xi'} = \varphi^K_{\xi'} \neq 0$, so $\varphi^K_\xi \neq \varphi^K_{\xi'}$.

We have proved that assertions 1 and 2 are equivalent. Equivalence of assertions 2, 3 and 4 is standard from invariant theory; see [1] or [3]. □

Remark 4.3. One could also prove Theorem 4.2 by computing the Fourier transforms of φ^K_ξ and $\varphi^K_{\xi'}$ and using Lemma 4.1 to show equivalence of assertions 1 and 3, and then using invariant theory for equivalence of assertions 2, 3 and 4.

Now we know exactly when two of the (G, K)-spherical functions φ^K_ξ are equal, and we prove that every (G, K)-spherical function is one of them.

Theorem 4.4. *If $\varphi : \mathbb{E}^n \to \mathbb{C}$ is (G, K)-spherical, then there exists $\xi \in \mathbb{C}^n$ such that $\varphi = \varphi_\xi^K$.*

Proof. Let $\chi : \mathcal{D}(G, K) \to \mathbb{C}$ denote the joint eigenvalue of φ as a joint eigenfunction of $\mathcal{D}(G, K)$. Under Lemma 4.1 χ corresponds to an algebra homomorphism $\widehat{\chi} : \mathcal{P}(\mathbb{C}^n)^K \to \mathbb{C}$. We interpret $\widehat{\chi}$ as an element of Spec $\mathcal{P}(\mathbb{C}^n)^K$. Thus $\widehat{\chi}$ is evaluation at some point $[\xi]$ of $\mathbb{C}^n /\!/ K_{\mathbb{C}}$. Let $\xi \in \pi^{-1}[\xi]$. Now φ_ξ^K is a joint eigenfunction of $\mathcal{D}(G, K)$ that also has joint eigenvalue χ. As both φ and φ_ξ^K are (G, K)-spherical, they are equal. □

We now have all the (G, K)-spherical functions parameterized in the form φ_ξ^K, and we know exactly when two of them are equal. In effect, the affine variety $\mathbb{C}^n /\!/ K_{\mathbb{C}}$ serves as their parameter space. We make this a little bit more precise.

Let $S(G, K)$ denote the space of all (G, K)-spherical functions on $\mathbb{E}^n$. Recall the spherical transform $f \mapsto \hat{f}$ from $L^1(K \backslash G/K)$ to functions on $S(G, K)$ given by $\hat{f}(\varphi) = \int_G f(g)\varphi(g^{-1})\, d\mu_G$. Then $S(G, K)$ carries the weak topology from the maps $\hat{f} : S(G, K) \to \mathbb{C}$, $f \in L^1(K \backslash G/K)$, or equivalently from the $\hat{f} : S(G, K) \to \mathbb{C}$ with $f \in C_c(K \backslash G/K)$. Further, $S(G, K)$ is a locally compact Hausdorff topological space.

Theorem 4.5. *The map $\varphi_\xi^K \mapsto \pi(\xi)$ is a homeomorphism of $S(G, K)$ onto $\mathbb{C}^n /\!/ K_{\mathbb{C}}$. Thus $S(G, K)$ is parametrized topologically by the affine variety $\mathbb{C}^n /\!/ K_{\mathbb{C}}$.*

5. Positive definite spherical functions on Euclidean space

For questions of (G, K)-harmonic analysis one must know just which (G, K)-spherical functions φ are positive definite, in other words satisfy

$$\sum \varphi\big(g_j^{-1} g_i\big)\overline{c_j} c_i \geqq 0 \quad \text{whenever } m > 0,\ \{c_1, \ldots, c_m\} \subset \mathbb{C} \text{ and } \{g_1, \ldots, g_m\} \subset G.$$

Then φ leads to a unitary representation π_φ on a Hilbert space $\mathcal{H}_\varphi$ constructed as follows. Start with the vector space of all functions $f : G \to \mathbb{C}$ of finite support. Give it the G-invariant positive semidefinite Hermitian form $\langle f, f' \rangle = \sum \varphi(g^{-1}g') f(g)\overline{f'(g')}$. Dividing out the kernel of the form one has a pre-Hilbert space, $\mathcal{H}_\varphi$ is the completion of that space, and π_φ is the natural action of G on $\mathcal{H}_\varphi$. See [2, §(XXII)9] or [11, §8.6] for the details.

The representation π_φ is an irreducible unitary representation and its space of K-fixed vectors is of dimension 1. Such representations are called (G, K)-spherical representations of G. Let u_φ be a K-fixed unit vector. Then $\mathcal{H}_\varphi$ determines φ (or more precisely its lift to G) by the formula $\varphi(g) = \langle u, \pi_\varphi(g)u \rangle$. Again see [2, §(XXII)9] or [11, Chapter 8] for details.

If $\varphi = \varphi_\xi^K$ we write π_ξ^K for π_φ and $\mathcal{H}_{\pi_\xi^K}$ for $\mathcal{H}_\varphi$.

Proposition 5.1. *Let $\xi \in \mathbb{C}^n$. Then φ_ξ^K is the induced spherical function* $\mathrm{Ind}_{\mathbb{R}^n}^G(\varphi_\xi)$.

Proof. Apply formula (2.1) to φ_ξ with $Q = \mathbb{R}^n$. Since G and $\mathbb{R}^n$ are unimodular, it says that the induced spherical function is given by

$$\mathrm{Ind}_{\mathbb{R}^n}^G(\varphi_\xi)(xk) = \int_K \varphi_\xi\big(k_1^{-1}(x)\big)\, d\mu_K(k_1)$$

$$= \int_K \varphi_{k_1(\xi)}(x)\, d\mu_K(k_1) = \varphi_\xi^K(x) = \varphi_\xi^K(xk)$$

for $x \in \mathbb{R}^n$ and $k \in K$. □

We apply the Mackey little group method to G relative to its normal subgroup $\mathbb{R}^n$: if ψ is an irreducible unitary representation of G then it can be constructed (up to unitary equivalence) as follows. If χ is a unitary character on $\mathbb{R}^n$ let K_χ denote its K-normalizer, so $G_\chi := \mathbb{R}^n \cdot K_\chi$ is the G-normalizer of χ. Write $\widetilde{\chi}$ for the extension of χ to G_χ given by $\widetilde{\chi}(xk) = \chi(x)$; it is a well defined unitary character on G_χ. If γ is an irreducible unitary representation of K_χ let $\widetilde{\gamma}$ denote its extension of G_χ given by $\widetilde{\gamma}(xk) = \gamma(k)$. Denote $\psi_{\chi,\gamma} = \mathrm{Ind}_{G_\chi}^G(\widetilde{\chi} \otimes \widetilde{\gamma})$. Then there exist choices of χ and γ such that $\psi = \psi_{\chi,\gamma}$.

Lemma 5.2. *In the notation just above, $\psi_{\chi,\gamma}$ has a K-fixed vector if and only if γ is the trivial 1-dimensional representation of K_χ. In that case $\psi_{\chi,\gamma} = \mathrm{Ind}_{G_\chi}^G(\widetilde{\chi})$ and the K-fixed vector is given (up to scalar multiple) by $u(xk) = \chi(x)^{-1}$.*

Proof. The representation space $\mathcal{H}_\psi$ of $\psi = \psi_{\chi,\gamma}$ consists of all L^2 functions $f : G \to \mathcal{H}_\gamma$ such that $f(g'x'k') = \gamma(k')^{-1} \cdot \chi(x')^{-1} f(g')$ for $g' \in G$, $x' \in \mathbb{R}^n$ and $k' \in K_\chi$, and ψ acts by $(\psi(g)f)(g') = f(g^{-1}g')$.

Now suppose that $0 \neq f \in \mathcal{H}_\psi$ is fixed under $\psi(K)$. If $k' \in K_\chi$ then $\gamma(k') \cdot f(1) = f(1)$. If $f(1) = 0$ then $f(G_\chi) = 0$ and K-invariance says $f = 0$, contrary to assumption. Thus $f(1) \neq 0$ and irreducibility of γ forces γ to be trivial.

Conversely if γ is trivial then $f(xk) = \chi(x)^{-1}$ is a nonzero K-fixed vector in $\mathcal{H}_\psi$. And it is the only one, up to scalar multiple, because any two K-fixed vectors must be proportional. □

Lemma 5.3. *In the notation above, $\mathrm{Ind}_{G_\chi}^G(\widetilde{\chi})$ is unitarily equivalent to the subrepresentation of $\mathrm{Ind}_{\mathbb{R}^n}^G(\widetilde{\chi})$ generated by the K-fixed unit vector $u(xk) = \chi(x)^{-1}$.*

Theorem 5.4. *Let φ be a (G,K)-spherical function. Then φ is positive definite if and only if it is of the form φ_ξ^K for some $\xi \in \mathbb{R}^n$. Further, if $\xi, \xi' \in \mathbb{R}^n$ then $\varphi_\xi^K = \varphi_{\xi'}^K$ if and only if $\xi' \in K(\xi)$.*

Proof. Let $\xi \in \mathbb{R}^n$. The formula (3.1) exhibits φ_ξ^K as a limit of non-negative linear combinations of positive definite functions on $\mathbb{R}^n$, so it is positive definite.

Now let φ be a positive definite (G,K)-spherical function. Let π_φ be the associated irreducible unitary representation, and $\mathcal{H}_\varphi$ the representation space, such that there is a K-fixed unit vector $u_\varphi \in \mathcal{H}_\varphi$ such that $\varphi(g) = \langle u_\varphi, \pi_\varphi(g)u_\varphi \rangle$ for all $g \in G$. Following the discussion of the Mackey little group method, and Lemma 5.2, we have a unitary character χ on $\mathbb{R}^n$ such that π_φ is unitarily equivalent to $\mathrm{Ind}_{G_\chi}^G(\widetilde{\chi})$. Making the identification, one K-fixed unit vector in $\mathcal{H}_\varphi$ is given by $u(xk) = \chi(x)^{-1}$. We have $\xi \in \mathbb{R}^n$ such that $\chi = \varphi_\xi$, so now $u(xk) = \varphi_\xi(x)^{-1}$, and we compute

$$\begin{aligned} \varphi(g) &= \langle u, \pi_\varphi(g)u \rangle && \text{(construction of } \pi_\varphi) \\ &= \langle u, \mathrm{Ind}_{G_{\varphi_\xi}}^G(\widetilde{\varphi_\xi})(g)u \rangle && \text{(Lemma 5.2)} \end{aligned}$$

$$= \langle u, \mathrm{Ind}_{\mathbb{R}^n}^G(\varphi_\xi)(g)u\rangle \qquad \text{(construction of } u)$$
$$= \mathrm{Ind}_{\mathbb{R}^n}^G(\varphi_\xi)(g) \qquad \text{(Lemma 5.3)}$$
$$= \varphi_\xi^K(g) \qquad \text{(Proposition 5.1).}$$

That completes the proof of the first assertion.

For the second, suppose that $\xi, \xi' \in \mathbb{R}^n$ with $\varphi_\xi^K = \varphi_{\xi'}^K$. Then (up to unitary equivalence) $\mathrm{Ind}_{G_{\varphi_\xi}}^G(\widetilde{\varphi_\xi}) = \mathrm{Ind}_{G_{\varphi_{\xi'}}}^G(\widetilde{\varphi_{\xi'}})$. That gives us direct integral decompositions

$$\int_K^{\oplus} \varphi_{k(\xi)}\, d\mu_K(k) = \mathrm{Ind}_{G_{\varphi_\xi}}^G(\widetilde{\varphi_\xi})\big|_{\mathbb{R}^n} = \mathrm{Ind}_{G_{\varphi_{\xi'}}}^G(\widetilde{\varphi_{\xi'}})\big|_{\mathbb{R}^n} = \int_K^{\oplus} \varphi_{k(\xi')}\, d\mu_K(k).$$

All our groups are of Type I, so it follows that $K(\xi) = K(\xi')$. □

We now combine Theorems 4.5 and 5.4. Let $P(G,K)$ denote the set of all positive definite (G,K)-spherical functions with the subspace topology from $S(G,K)$. Then $P(G,K)$ is known to be a locally compact Hausdorff space, whose topology is the subspace topology from $P(G,K) \subset D^*$ where D^* is the closed unit ball in the dual space $L^1(K\backslash G/K)^*$. See, for example, [11, Proposition 9.2.7].

Theorem 5.5. *The map $\varphi_\xi^K \mapsto \pi(\xi)$ is a homeomorphism of $P(G,K)$ onto $\mathbb{R}^n /\!/ K$. Thus $P(G,K)$ is parameterized topologically by the real form $\mathbb{R}^n /\!/ K$ of the complex affine variety $\mathbb{C}^n /\!/ K_{\mathbb{C}}$.*

6. The transitive case

Our results are especially interesting when K is transitive on the spheres about 0 in $\mathbb{R}^n$, for then we know the $K_{\mathbb{C}}$-orbits and the spherical functions explicitly.

Lemma 6.1. *If K is transitive on the spheres about 0 in $\mathbb{R}^n$, $n > 1$, then the $K_{\mathbb{C}}$ is transitive on each of the complex affine quadrics $Q_c = \{\xi \in \mathbb{C}^n \mid b(\xi,\xi) = c\}$, $0 \neq c \in \mathbb{C}$.*

Proof. If $b(\xi,\xi) = c$ then $K_{\mathbb{C}}(\xi) \subset Q_c$. Multiplication by a nonzero complex number t gives a $K_{\mathbb{C}}$-equivariant holomorphic diffeomorphism of Q_c onto Q_{t^2c}, so we need only prove the lemma for $0 \neq \xi \in \mathbb{R}^n$. Then $K(\xi)$ has real dimension $n-1$, by hypothesis, and is a totally real submanifold of the complex manifold $K_{\mathbb{C}}(\xi)$. It follows that $K_{\mathbb{C}}(\xi)$ has complex dimension $n-1$. Thus $K_{\mathbb{C}}(\xi)$ is open in Q_c. Now every $K_{\mathbb{C}}$-orbit in the connected manifold Q_c is open there, and it follows that $K_{\mathbb{C}}$ is transitive on Q_c. □

The situation is more complicated for the cone $Q_0 = \{\xi \in \mathbb{C}^n \mid b(\xi,\xi) = 0 \text{ and } \xi \neq 0\}$. Consider the canonical projection $p : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ Then $Q := p(Q_0)$ is the standard nondegenerate projective quadric $SO(n)/(SO(n-2) \times SO(2)) = SO(n;\mathbb{C})/P$ where P is the appropriate parabolic subgroup, and it is known (see [7,10]) that $K_{\mathbb{C}}$ is transitive on Q if and only if:

(i) $K_{\mathbb{C}}^0 = SO(n;\mathbb{C})$, or
(ii) $n = 7$ and $K_{\mathbb{C}} = G_{2,\mathbb{C}}$, or
(iii) $n = 8$ and $K_{\mathbb{C}} = Spin(7;\mathbb{C})$.

In case (i) it is obvious that $K_{\mathbb{C}}$ is transitive on Q_0, but the argument for all three goes as follows. The affine variety $K_{\mathbb{C}}(\xi)$ is not projective, so the $K_{\mathbb{C}}$-equivariant holomorphic map $p : K_{\mathbb{C}}(\xi) \to p(K_{\mathbb{C}}(\xi)) \subset \mathbb{CP}^{n-1}$ has fiber of dimension 1. The image has form $K_{\mathbb{C}}/P$ where P is a parabolic subgroup whose reductive component has center $\mathbb{C}^*$, in other words $P = P' \cdot \mathbb{C}^*$ where P' is the derived group $[P, P]$. Thus $K_{\mathbb{C}}(\xi) \cong K_{\mathbb{C}}/P'$. By dimension, now, $K_{\mathbb{C}}(\xi) = Q_0$.

Of course, it goes without saying that $K_{\mathbb{C}}$ is transitive on the remaining orbit, $\{0\}$.

Lemma 6.2. *If K is transitive on the spheres about* 0 *in $\mathbb{R}^n$, and p is a $K_{\mathbb{C}}$-invariant polynomial on $\mathbb{C}^n$, then p is constant on every quadric Q_c, $c \neq 0$, as well as on the cone $Q_0 \cup \{0\}$.*

Proof. If $c \neq 0$ then $K_{\mathbb{C}}$ is transitive on Q_c by Lemma 6.1, so p is constant on Q_c. Now let $x, y \in Q_0$. Then we have sequences $\{x_m\} \to x$ and $\{y_m\} \to y$ with $x_m, y_m \in Q_{2^{-m}}$. As p is continuous now $p(x) = \lim p(x_m) = \lim p(y_m) = p(y)$. Thus p is constant on Q_0. As 0 is in the closure of Q_0 also p is constant on $Q_0 \cup \{0\})$. □

Theorem 6.3. *Suppose that K is transitive on the spheres about* 0 *in $\mathbb{R}^n$. Then the categorical quotient $\mathbb{C}^n /\!/ K_{\mathbb{C}} \cong \mathbb{C}$, where the isomorphism is given by $Q_c \mapsto c$ for $c \neq 0$ and $Q_0 \cup \{0\} \mapsto 0$.*

Proof. The projection $\pi : \mathbb{C}^n \to \mathbb{C}^n /\!/ K_{\mathbb{C}}$ maps Q_c to a point because Q_c is a closed $K_{\mathbb{C}}$-orbit, as noted in Lemma 6.2, and maps $Q_0 \cup \{0\}$ to a point by Lemma 6.2. Thus the map indicated in the statement of the theorem is well defined and bijective. The isomorphism follows. □

When K is transitive on the spheres about 0 in $\mathbb{R}^n$, its identity component K^0 is also transitive, and $K = K^0 F$ where F is a finite subgroup of the normalizer $N_{O(n)}(K^0)$. The possibilities for K^0 are

$$\begin{gathered} n > 1 \quad \text{and} \quad K^0 = SO(n), \\ n = 2m \quad \text{and} \quad K^0 = SU(m) \text{ or } U(m), \\ n = 4m \quad \text{and} \quad K^0 = Sp(m) \text{ or } Sp(m) \cdot U(1) \text{ or } Sp(m) \cdot Sp(1), \\ n = 7 \quad \text{and} \quad K^0 \text{ is the exceptional group } G_2, \\ n = 8 \quad \text{and} \quad K^0 = Spin(7), \\ n = 16 \quad \text{and} \quad K^0 = Spin(9). \end{gathered} \tag{6.4}$$

Given K^0 it is easy to work out all relevant possibilities for F; see [11, §11.3D]. Now we look at the spherical functions in the transitive cases.

Lemma 6.5. *If K is transitive on the spheres about* 0 *in $\mathbb{R}^n$, then $\mathcal{D}(G/K) = \mathbb{C}[\Delta]$, algebra of polynomials in the Laplace–Beltrami operator $\Delta = -\sum \partial^2/\partial x_i^2$.*

Proof. It is clear that $\mathbb{C}[\Delta] \subset \mathcal{D}(G/K)$. Now let $D \in \mathcal{D}(G/K)$ be of order m. Then the mth order symbol of D is a polynomial of pure degree m constant on spheres about 0 in $\mathbb{R}^n$, in other words a multiple cr^m with m even and $r^2 = \sum x_i^2$. Now $D - c(-\Delta)^{m/2} \in \mathcal{D}(G/K)$ and

$D - c(-\Delta)^{m/2}$ has order $< m$. By induction on the order, $D - c(-\Delta)^{m/2} \in \mathbb{C}[\Delta]$, so we have $D \in \mathbb{C}[\Delta]$. $\square$

Taking advantage of Theorems 4.5, 5.5 and 6.3 we have

Theorem 6.6. *Let K be transitive on the spheres about 0 in $\mathbb{R}^n$, $n > 1$. Then the spherical functions on $\mathbb{E}^n$ are parametrized by $\mathbb{C}$ and the positive definite spherical functions are the ones with real non-negative parameter. Here φ_ξ^K has parameter $b(\xi, \xi)$, which is its Δ-eigenvalue, and (given n) φ_ξ^K is the same for any choice of group K listed in* (6.4). *Further,*

$$\begin{aligned}\varphi_\xi^K(x) &= \big(\|\xi\| r\big)^{-(n-2)/2} J_{(n-2)/2}\big(\|\xi\| r\big) \\ &= \frac{\pi^{-1/2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \int_0^\pi \cos\big(\sqrt{b(\xi,\xi)}\|x\|\cos\theta\big) \sin^{n-2}\theta \, d\theta \quad \textit{for } n > 1,\end{aligned}$$

where $\|\xi\| = \sqrt{b(\xi,\xi)}$, $r = \sqrt{b(x,x)}$ and J_ν is the Bessel function of the first kind of order ν.

Remark 6.7. Let M be a connected flat n-dimensional Riemannian symmetric space. Then $M = \Gamma \setminus \mathbb{E}^n$ for some discrete subgroup Γ of the group $\mathbb{R}^n$ of translations of $\mathbb{E}^n$; see [9]. Let K be a closed subgroup of $O(n)$ that preserves $\Gamma(0)$. (On the group level means that K normalizes Γ, and if K is connected that forces K to centralize Γ.) Then the action of K descends to M, and M is the symmetric coset space G'/K where G' is the semidirect product $(\mathbb{R}^n/\Gamma) \cdot K$. Now $\mathcal{D}(G', K) = \mathcal{D}(G, K)$, so any (G', K)-spherical function on M lifts to a (G, K)-spherical function on $\mathbb{E}^n$, in other words the (G', K)-spherical function on M are the push-downs of the (G, K)-spherical functions φ_ξ^K on $\mathbb{E}^n$ such that $e^{ib(\xi,\gamma)} = 1$ for every $\gamma \in \Gamma$. So $S(G', K)$ is the categorical quotient $'\mathbb{C}^n /\!/ K$ where $'\mathbb{C}^n$ is given as follows. Let U denote the complex span $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ of Γ and $U^\perp$ its b-orthocomplement; then $'\mathbb{C}^n = (U^\perp \oplus U)/\Gamma$. Note that $'\mathbb{C}^n$ is the annihilator of Γ in the complexified dual space of $\mathbb{R}^n$.

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