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Inertially arbitrary sign patterns with no nilpotent realization ☆

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Abstract

An n by n sign pattern $\mathscr{S}$ is inertially arbitrary if each ordered triple (n_1, n_2, n_3) of nonnegative integers with $n_1 + n_2 + n_3 = n$ is the inertia of some real matrix in $Q(\mathscr{S})$, the sign pattern class of $\mathscr{S}$. If every real, monic polynomial of degree n having a positive coefficient of x^{n-2} is the characteristic polynomial of some matrix in $Q(\mathscr{S})$, then it is shown that $\mathscr{S}$ is inertially arbitrary. A new family of irreducible sign patterns $\mathscr{G}_{2k+1} (k \geqslant 2)$ is presented and proved to be inertially arbitrary, but not potentially nilpotent (and thus not spectrally arbitrary). The well-known Nilpotent-Jacobian method cannot be used to prove that $\mathscr{G}_{2k+1}$ is inertially arbitrary, since $\mathscr{G}_{2k+1}$ has no nilpotent realization. In order to prove that $Q(\mathscr{G}_{2k+1})$ allows each inertia with $n_3 \geqslant 1$, a realization of $\mathscr{G}_{2k+1}$ with only zero eigenvalues except for a conjugate pair of pure imaginary eigenvalues is identified and used with the Implicit Function Theorem. Matrices in $Q(\mathscr{G}_{2k+1})$ with inertias having $n_3 = 0$ are constructed by a recursive procedure from those of lower order. Some properties of the coefficients of the characteristic polynomial of an arbitrary matrix having certain fixed inertias are derived, and are used to show that $\mathscr{G}_5$ and $\mathscr{G}_7$ are minimal inertially arbitrary sign patterns.

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1. Introduction

Throughout all matrices are real and all polynomials are real and monic. The *inertia* of a nonzero polynomial $p(x)$ is the ordered triple (n_1, n_2, n_3), where n_1 is the number of zeros of $p(x)$ with positive real part, n_2 is the number of zeros of $p(x)$ with negative real part, and n_3 is the number of zeros of $p(x)$ with zero real part. The degree of $p(x)$ is denoted by $\deg p(x)$. Note that $n_1 + n_2 + n_3 = \deg p(x)$. For an n by n matrix A, the *inertia* of A, denoted by $i(A)$, is the inertia of the characteristic polynomial of A, namely $p_A(x) = \det(xI_n - A)$, where I_n denotes the identity matrix of order n. An n by n matrix A is *nilpotent* if $A^k = O$ for some positive integer k, or equivalently if $p_A(x) = x^n$.

An n by n *sign pattern* $\mathscr{S} = [s_{ij}]$ is an n by n matrix with entries in $\{+, -, 0\}$. A *subpattern* of $\mathscr{S}$ is an n by n sign pattern $\mathscr{U} = [u_{ij}]$ such that $u_{ij} = 0$ whenever $s_{ij} = 0$. If $\mathscr{U} \neq \mathscr{S}$, then $\mathscr{U}$ is a *proper subpattern* of $\mathscr{S}$. For a real scalar a, the sign of a is denoted by $\operatorname{sgn}(a)$, and is $+$, $-$ or 0. The *sign pattern class* $Q(\mathscr{S})$ of $\mathscr{S}$ is the set of n by n matrices $A = [a_{ij}]$ such that $\operatorname{sgn}(a_{ij}) = s_{ij}$ for all i, j. If $A \in Q(\mathscr{S})$, then A is a *realization* of $\mathscr{S}$.

The *spectrum of* $\mathscr{S}$ is the set of spectra of all matrices in $Q(\mathscr{S})$, and is denoted by $\sigma(\mathscr{S})$. Similarly, the *inertia of* $\mathscr{S}$ is the set of inertias of all matrices in $Q(\mathscr{S})$, and is denoted by $i(\mathscr{S})$. If every self-conjugate multi-set of n complex numbers is in $\sigma(\mathscr{S})$, then $\mathscr{S}$ is a *spectrally arbitrary pattern* (*SAP*). If an n by n sign pattern $\mathscr{S}$ has a nilpotent realization, then $\mathscr{S}$ is *potentially nilpotent* (*PN*). It is clear that if $\mathscr{S}$ is a SAP, then $\mathscr{S}$ is PN. If each ordered triple (n_1, n_2, n_3) of nonnegative integers with $n_1 + n_2 + n_3 = n$ is in $i(\mathscr{S})$, then $\mathscr{S}$ is an *inertially arbitrary pattern* (*IAP*). If $\mathscr{S}$ is an IAP and no proper subpattern of $\mathscr{S}$ is an IAP, then $\mathscr{S}$ is a *minimal inertially arbitrary pattern* (*MIAP*).

Several families of SAPs (which are necessarily IAPs) have been identified in the literature. For example, such families were obtained in [1,2,3,4,8] by the well-known Nilpotent-Jacobian method (see, for example, [4, Observation 10] and [2, Theorem 2.1]), and in [9] using a construction based on a Soules matrix. In addition, families of n by n IAPs with $n^2 - n + 2\,(n \geqslant 2)$ and $n^2 - n\,(n \geqslant 3)$ nonzero entries are obtained in [7] and [10], respectively, using explicit constructions. For $n = 3$, if an irreducible sign pattern $\mathscr{S}$ is an IAP, then $\mathscr{S}$ is a SAP and $\mathscr{S}$ is PN; see [1]. In [3], an irreducible 4 by 4 IAP that is PN but not a SAP is given. In [3], a 7 by 7 reducible IAP that is not PN is given that is a direct sum of an irreducible 2 by 2 IAP and an irreducible 5 by 5 sign pattern that is not an IAP. However, this does not imply the existence of a family of irreducible n by n sign patterns that are inertially arbitrary but not PN. To find an irreducible sign pattern $\mathscr{S}$ that is an IAP but not PN, the Nilpotent-Jacobian method cannot be used since $\mathscr{S}$ does not have any nilpotent realization.

For odd $n \geqslant 5$, we present a family of irreducible IAPs that is not PN (and thus is not spectrally arbitrary). For inertia (n_1, n_2, n_3) with $n_3 \geqslant 1$, we do this by a new technique that uses the Implicit Function Theorem and evaluates a Jacobian at a matrix realization having a certain characteristic polynomial with two nonzero coefficients, and when $n_3 = 0$ we use a constructive, recursive procedure. As far as we know, this family is the first family of irreducible sign patterns demonstrated to be inertially arbitrary but not PN.

In order to accomplish our goals, in Section 2 we provide some general results on the inertia of a polynomial and a matrix. In particular, we identify a certain subset of the set of all polynomials of degree $n \geqslant 4$ that attains all possible inertias, and interpret this for the inertia of a sign pattern; see Theorem 1. In Section 3 we introduce a family of irreducible sign patterns and show that none of these sign patterns of odd order is PN. In Section 4 we show that every sign pattern of odd order in the family allows each inertia (n_1, n_2, n_3) with $n_3 \geqslant 1$, and in Section 5 that these sign patterns of odd order allow each inertia $(n_1, n_2, 0)$. Thus for odd order, every member of the family is inertially arbitrary, but not PN; see Theorem 19. In Section 6 it is shown that every sign pattern of even order in the family is not an IAP. We conclude with some discussion on minimality of these sign patterns of odd order.

2. General results on inertia

Let $p(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$. The coefficient r_k of x^{n-k} in $p(x)$ is also denoted by $r_k(p(x))$ when we need to explicitly identify the polynomial $p(x)$. Let

$$P_n = \{p(x)|\deg p(x) = n \quad \text{and} \quad r_2 > 0\}.$$

For a nonnegative integer h, we use $x^h P_n$ to denote the set $\{x^h p(x)|p(x) \in P_n\}$. Thus $x^h P_n$ is a subset of P_{n+h}.

Theorem 1. *For $n \geqslant 4$, let $\mathscr{S}$ be an n by n sign pattern. If $P_n \subseteq \{p_A(x)|A \in Q(\mathscr{S})\}$, then $\mathscr{S}$ is inertially arbitrary. Moreover, if $0 \leqslant h \leqslant n-4$ and $x^h P_{n-h} \subseteq \{p_A(x)|A \in Q(\mathscr{S})\}$, then $\mathscr{S}$ allows each inertia (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = n$ and $n_3 \geqslant h$.*

Proof. By induction on $n \geqslant 4$, it is first shown that each triple (n_1, n_2, n_3) of nonnegative integers with $n_1 + n_2 + n_3 = n$ is the inertia of some polynomial in P_n. When $n = 4$, the following table shows that every possible inertia is realized by a polynomial in P_4.

Inertia	Polynomial $p(x)$	r_2
(0, 3, 1)	$x(x+1)^3$	3
(3, 0, 1)	$x(x-1)^3$	3
(2, 1, 1)	$x(x+1)(x^2-2x+3)$	1
(1, 2, 1)	$x(x-1)(x^2+2x+3)$	1
(1, 1, 2)	$(x-1)(x+2)(x^2+3)$	1
(2, 0, 2)	$x^2(x^2-x+2)$	2
(0, 2, 2)	$x^2(x^2+x+2)$	2
(0, 1, 3)	$x(x+1)(x^2+2)$	2
(1, 0, 3)	$x(x-1)(x^2+2)$	2
(0, 0, 4)	$(x^2+1)(x^2+2)$	3
(1, 3, 0)	$(x-1)(x^3+2x^2+3x+2)$	1
(3, 1, 0)	$(x+1)(x^3-2x^2+3x-2)$	1
(2, 2, 0)	$(x^2+x+2)(x^2-3x+2)$	1
(0, 4, 0)	$(x+1)(x^3+2x^2+3x+2)$	5
(4, 0, 0)	$(x-1)(x^3-2x^2+3x-2)$	5

Assume that $n \geqslant 5$ and the result is true for $n-1$, that is, each triple (n_1, n_2, n_3) satisfying $n_1 + n_2 + n_3 = n-1$ is the inertia of some polynomial in P_{n-1}. Then each triple (n_1, n_2, n_3) satisfying $n_1 + n_2 + n_3 = n$ and $n_3 \geqslant 1$ is the inertia of some polynomial in xP_{n-1}, which is a proper subset of P_n.

Next, we consider the inertias $(n_1, n_2, 0)$ with $n_1 + n_2 = n$. By the induction hypothesis, for each $\ell \in \{0, 1, \ldots, n-1\}$, there exists $p(x) \in P_{n-1}$ with inertia $(n-1-\ell, \ell, 0)$. Consider $(x-a)p(x)$ and $(x+a)p(x)$ where $a > 0$, and note that $(x-a)p(x)$ has inertia $(n-\ell, \ell, 0)$, and $(x+a)p(x)$ has inertia $(n-1-\ell, \ell+1, 0)$. In order to complete the proof it is sufficient to show that there exists a value of a so that $r_2((x-a)p(x)) > 0$ and $r_2((x+a)p(x)) > 0$. Since the coefficient of x^{n-2} in $(x-a)p(x)$ is equal to the coefficient of x^{n-3} in $p(x)$ minus the product of a and the coefficient of x^{n-2} in $p(x)$, it follows that

$$r_2((x-a)p(x)) = r_2(p(x)) - ar_1(p(x)).$$

Similarly,

$$r_2((x+a)p(x)) = r_2(p(x)) + ar_1(p(x)).$$

The term $r_2(p(x))$ is positive since $p(x) \in P_{n-1}$. Hence, $r_2((x-a)p(x))$ and $r_2((x+a)p(x))$ are positive for sufficiently small $a > 0$. Therefore, if $P_n \subseteq \{p_A(x) | A \in Q(\mathscr{S})\}$, then $\mathscr{S}$ is inertially arbitrary.

The more general statement follows since $x^h P_{n-h}$ is a set of polynomials of degree n having at least h eigenvalues with zero real part. $\square$

The above result identifies a certain subset of the set of all polynomials of degree $n \geqslant 4$ that attains all possible inertias, and thus can be used to show that an n by n sign pattern is an IAP without necessarily being a SAP. We remark that Theorem 1 does not hold for $n = 2, 3$. It can be easily verified that for $n = 2$, a polynomial in P_2 cannot have, for example, the inertia $(1, 0, 1)$. When $n = 3$, a polynomial in P_3 cannot have the inertia $(1, 1, 1)$.

Let A be an n by n matrix, and let α, β be nonempty subsets of $\{1, 2, \ldots, n\}$. Then $A(\alpha, \beta)$ (resp. $A[\alpha, \beta]$) denotes the submatrix of A obtained by removing (resp. retaining) rows indexed by α and columns indexed by β. When $\alpha = \beta$, we use $A(\alpha)$ and $A[\alpha]$, respectively. The determinant of $A[\alpha]$ is a *principal minor* of A. If α is a singleton set $\{i\}$, then $A(i)$ denotes $A(\{i\})$. Let D be a diagonal matrix. If every diagonal entry of D is positive, then D is a *positive diagonal* matrix.

Theorem 2. *Let A be an n by n matrix. Suppose that* $\det(A) \neq 0$ *and* $i(A(n)) = (p, q, 0)$ *with* $p + q = n - 1$. *Then*

(a) *if* $\det(A)\det(A(n)) > 0$, *there exists a positive diagonal matrix* $D = I_{n-1} \oplus [\epsilon]$ *such that* $i(DA) = (p+1, q, 0)$; *and*
(b) *if* $\det(A)\det(A(n)) < 0$, *there exists a positive diagonal matrix* $D = I_{n-1} \oplus [\epsilon]$ *such that* $i(DA) = (p, q+1, 0)$.

Proof. Let $D' = I_{n-1} \oplus [0]$ and $D_\epsilon = I_{n-1} \oplus [\epsilon]$. Then the eigenvalues of $D'A$ are 0 together with the eigenvalues of $A(n)$. Hence, for a sufficiently small positive number ϵ, $D_\epsilon A$ has at least p eigenvalues with positive real part, and at least q eigenvalues with negative real part. Since $D_\epsilon A$ is nonsingular, $i(D_\epsilon A)$ is either $(p+1, q, 0)$ or $(p, q+1, 0)$.

The parity of the determinant of a square matrix is equal to the parity of the number of eigenvalues with negative real part. Hence, if $\det(A)$ and $\det(A(n))$ have the same sign, then $i(D_\epsilon A) = (p+1, q, 0)$, and if $\det(A)$ and $\det(A(n))$ have different signs, then $i(D_\epsilon A) = (p, q+1, 0)$. $\square$

Note that the well-known Fisher–Fuller Theorem in [6] follows by repeated application of Theorem 2(b).

3. Sign pattern $\mathscr{G}_{2k+1}$ is not PN

For $k \geqslant 2$, define the sign pattern of order $2k+1$ with $5k+1$ nonzero entries

$$\mathscr{G}_{2k+1} = \begin{bmatrix} \alpha_1 & \beta_1 & \delta_1 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & & & \vdots \\ 0 & 0 & 0 & \gamma_3 & \delta_3 & 0 & & \\ 0 & \beta_4 & 0 & 0 & \gamma_4 & \ddots & \ddots & \vdots \\ \alpha_5 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \gamma_{2k-1} & \delta_{2k-1} \\ 0 & \beta_{2k} & 0 & 0 & & & \ddots & \gamma_{2k} \\ \alpha_{2k+1} & 0 & 0 & 0 & \cdots & & 0 & 0 \end{bmatrix},$$

where $\alpha_1 = -, \alpha_2 = +, \alpha_{2j+1} = -$ for $j = 2, \ldots, k$; $\beta_1 = -, \beta_2 = +, \beta_{2j} = -$ for $j = 2, \ldots, k$; $\gamma_2 = +, \gamma_j = -$ for $j = 3, 4, \ldots, 2k$; and $\delta_{2j+1} = -$ for $j = 0, 1, \ldots, k-1$. We define $\mathscr{G}_{2k}$ to be $\mathscr{G}_{2k+1}(2k+1)$. For example,

$$\mathscr{G}_7 = \begin{bmatrix} - & - & - & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 & 0 \\ 0 & - & 0 & 0 & - & 0 & 0 \\ - & 0 & 0 & 0 & 0 & - & - \\ 0 & - & 0 & 0 & 0 & 0 & - \\ - & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathscr{G}_6 = \begin{bmatrix} - & - & - & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 \\ 0 & - & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & 0 & - \\ 0 & - & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} -a_1 & -b_1 & -d_1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & & & \vdots \\ 0 & 0 & 0 & -c_3 & -d_3 & 0 & & \\ 0 & -b_4 & 0 & 0 & -c_4 & \ddots & \ddots & \vdots \\ -a_5 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & -c_{2k-1} & -d_{2k-1} \\ 0 & -b_{2k} & 0 & 0 & & & \ddots & -c_{2k} \\ -a_{2k+1} & 0 & 0 & 0 & \cdots & & 0 & 0 \end{bmatrix} \in Q(\mathscr{G}_{2k+1}), \tag{1}$$

where a_j, b_j, c_j, and d_j are positive.

The following well known result describes the coefficients of the characteristic polynomial of a square matrix A in terms of principal minors of A.

Proposition 3 (Theorem 1.31, [5]). *Let A be an n by n matrix and let $p_A(x) = x^n + r_1x^{n-1} + \cdots + r_{n-1}x + r_n$. Then $(-1)^k r_k$ is the sum of all k by k principal minors of A.*

If u is a 1 by n vector, then the jth entry of uA is denoted by $(uA)_j$.

Theorem 4. *For $k \geqslant 2$, the sign pattern $\mathscr{G}_{2k+1}$ is not potentially nilpotent (and thus not spectrally arbitrary).*

Proof. Let $n = 2k + 1$. Note that $\mathscr{G}_n(\{1\}, \{2\})$ is permutation equivalent to an upper triangular sign pattern with nonzero diagonal, and hence every matrix in $Q(\mathscr{G}_n)$ has rank at least $n - 1$. Suppose to the contrary that $\mathscr{G}_n$ is potentially nilpotent, and let N be a nilpotent realization of $\mathscr{G}_n$. Since N is singular and its rank is at least $n - 1$, N has nullity 1.

We first show that $\mathscr{G}_n$ has a nilpotent realization A, similar to N, that has a nonzero left nullvector with each entry in $\{1, -1, 0\}$. Let $u = (u_1, \ldots, u_n)$ be a nonzero left nullvector of N, i.e., $uN = 0$, and let $e_j = \frac{1}{|u_j|}$ if $u_j \neq 0$ and $e_j = 1$ if $u_j = 0$. Then the diagonal matrix $D = \operatorname{diag}(e_1, \ldots, e_n)$ is positive, and $(uD)_j \in \{1, -1, 0\}$ for each $j = 1, \ldots, n$. From $uN = 0$, it follows that $uDD^{-1}ND = 0$. Let $v = uD$ with $v = (v_1, \ldots, v_n)$, and $A = D^{-1}ND$. Then $A \in Q(\mathscr{G}_n)$, $vA = 0$ ($v_j \in \{1, -1, 0\}$), and A is nilpotent.

Let A be of the form (1). Since $-c_j$ is the only nonzero entry in column $j + 1$ for each $j = 3, 5, \ldots, 2k - 1$, it follows that $v_j = 0$ for each $j = 3, 5, \ldots, 2k - 1$. This implies, by considering $(vA)_{j+1}$ for $j = 4, 6, \ldots, 2k$, that $v_j = 0$ for each $j = 4, 6, \ldots, 2k$. Since $v_1 = 0$ implies that $v_2 = 0$ and hence $v_n = 0$, it follows that $v_1 \in \{1, -1\}$. Suppose that $v_1 = 1$. Then $v_1(-d_1) < 0$ and $(vA)_3 = 0$ imply that $v_2 = 1$. Therefore, without loss of generality, the nonzero nullvector v of A is one of the following vectors (i), (ii), (iii):

$$\text{(i) } (1, 1, 0, \ldots, 0, 1), \quad \text{(ii) } (1, 1, 0, \ldots, 0, -1), \quad \text{(iii) } (1, 1, 0, \ldots, 0).$$

If the nullvector v is (i), (ii) or (iii), respectively, then the entries a_2, b_2, c_2 in the matrix A of the form (1) with $n = 2k + 1$ are as follows:

$$\begin{aligned} &\text{(i) } a_2 = a_1 + a_n, b_2 = b_1, c_2 = d_1; \quad \text{(ii) } a_2 = a_1 - a_n, b_2 = b_1, c_2 = d_1; \\ &\text{(iii) } a_2 = a_1, b_2 = b_1, c_2 = d_1. \end{aligned} \tag{2}$$

Let $p_A(x) = x^n + r_1 x^{n-1} + \cdots + r_{n-1}x + r_n$. For cases (i) and (ii), by Proposition 3, as $\det(A[\{1, 2\}])$ is the only nonzero 2 by 2 principal minor,

$$\text{(i) } r_2 = \det \begin{bmatrix} -a_1 & -b_1 \\ a_1 + a_n & b_1 \end{bmatrix} = a_n b_1 \neq 0,$$

$$\text{(ii) } r_2 = \det \begin{bmatrix} -a_1 & -b_1 \\ a_1 - a_n & b_1 \end{bmatrix} = -a_n b_1 \neq 0.$$

For case (iii), consider $(-1)^{n-1} r_{n-1}$. Note that the last row of $A(1)$ is zero, and the first two rows of $A(j)$ for $j = 3, \ldots, n$ are multiples of each other. Thus, by Proposition 3,

$$\begin{aligned} \text{(iii) } (-1)^{n-1} r_{n-1} = \det(A(2)) &= (a_n)(-d_1) \det \begin{bmatrix} -c_3 & -d_3 & 0 & \cdots & 0 \\ 0 & -c_4 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \vdots & & -c_{2k-1} & -d_{2k-1} \\ 0 & 0 & \cdots & 0 & -c_{2k} \end{bmatrix} \\ &= -a_n d_1 \prod_{i=3}^{2k} c_i \neq 0. \end{aligned}$$

Therefore, for each case in (2), A is not nilpotent, which contradicts the assumption. $\square$

4. Inertia (n_1, n_2, n_3) with $n_3 \geqslant 1$ for $\mathscr{G}_{2k+1}$

In this section it is shown that $\mathscr{G}_{2k+1}$ for each integer $k \geqslant 2$ allows each inertia (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = 2k + 1$ and $n_3 \geqslant 1$. We begin by showing in detail that $\mathscr{G}_5$ allows each inertia (n_1, n_2, n_3) with $n_1 + n_3 + n_3 = 5$ and $n_3 \geqslant 1$. We do this by using Theorem 1 and the Implicit Function Theorem for a realization having characteristic polynomial $x^5 + r_2x^3$ with $r_2 > 0$, and a similar argument as was used in [4, Theorem 9] for a nilpotent realization. Then we use this same method for odd $n \geqslant 7$.

Lemma 5. *For any r_1, r_3, r_4 and any positive r_2, there exists a matrix $A \in Q(\mathscr{G}_5)$ such that*

$$p_A(x) = x^5 + r_1x^4 + r_2x^3 + r_3x^2 + r_4x.$$

Proof. For $c > 0$, since $A \in Q(\mathscr{G}_5)$ if and only if $cA \in Q(\mathscr{G}_5)$, and since

$$\det(xI - cA) = x^5 + cr_1x^4 + c^2r_2x^3 + c^3r_3x^2 + c^4r_4x,$$

it suffices to show that the theorem holds for (r_1, r_2, r_3, r_4) arbitrarily close to $(0, 0, 0, 0)$ with $r_2 > 0$. Consider the following matrix in $Q(\mathscr{G}_5)$ of the form (i) in (2) with a written for a_5, and $a_1 = c_3 = d_1 = d_3 = 1$:

$$A = \begin{bmatrix} -1 & -b_1 & -1 & 0 & 0 \\ 1+a & b_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & -b_4 & 0 & 0 & -c_4 \\ -a & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$p_A(x) = x^5 + (1 - b_1)x^4 + ab_1x^3 + (a - b_4)x^2 + (ab_4 - ac_4)x.$$

We seek positive numbers a, b_1, b_4 and c_4 such that

$$\begin{aligned} &1 - b_1 - r_1 = 0, \\ &ab_1 - r_2 = 0, \\ &a - b_4 - r_3 = 0, \\ &ab_4 - ac_4 - r_4 = 0. \end{aligned}$$

Setting $a = \frac{r_2}{b_1}$ gives

$$\begin{aligned} &1 - b_1 - r_1 = 0, \\ &\frac{r_2}{b_1} - b_4 - r_3 = 0, \\ &\frac{r_2}{b_1}b_4 - \frac{r_2}{b_1}c_4 - r_4 = 0. \end{aligned} \tag{3}$$

If $r_1 = r_3 = r_4 = 0$, it is easy to verify that the solution to (3) is

$$\hat{b}_1 = 1; \quad \hat{b}_4 = r_2; \quad \hat{c}_4 = r_2.$$

Let $f_1 = 1 - b_1$, $f_3 = \frac{r_2}{b_1} - b_4$, and $f_4 = \frac{r_2}{b_1}b_4 - \frac{r_2}{b_1}c_4$. Then, using the Implicit Function Theorem, it is sufficient to show that $\frac{\partial(f_1,f_3,f_4)}{\partial(b_1,b_4,c_4)}$ is nonzero at $(\hat{b}_1, \hat{b}_4, \hat{c}_4) = (1, r_2, r_2)$ in order to complete the proof. From (3),

$$\left.\frac{\partial(f_1, f_3, f_4)}{\partial(b_1, b_4, c_4)}\right|_{(1,r_2,r_2)} = \det\begin{bmatrix} -1 & 0 & 0 \\ -r_2 & -1 & 0 \\ -r_2^2 & r_2 & -r_2 \end{bmatrix} = -r_2 \neq 0.$$

Thus, for any r_1, r_3, r_4 and any positive r_2 sufficiently close to 0, there exist positive values a, b_1, b_4, c_4 such that $p_A(x) = x^5 + r_1x^4 + r_2x^3 + r_3x^2 + r_4x$ with $r_2 > 0$. $\square$

Since, by the above result, xP_4 is a subset of $\{p_A(x) | A \in Q(\mathscr{G}_5)\}$, the following is a direct consequence of Theorem 1 with $n = 5$ and $h = 1$.

Theorem 6. *The sign pattern $\mathscr{G}_5$ allows each inertia (n_1, n_2, n_3) with $n_1 + n_3 + n_3 = 5$ and $n_3 \geqslant 1$.*

We now proceed to a proof of the corresponding result for $n = 2k + 1 \geqslant 7$. Consider the following matrix in $Q(\mathscr{G}_n)$ of the form (i) in (2) with a written for $a_5 = \cdots = a_{2k+1}$, $d_{2i-1} = 1$ for $i = 1, 2, \ldots, k$, and $c_{2j+1} = 1$ for $j = 1, 2, \ldots, k - 1$:

$$A_{2k+1} = \begin{bmatrix} -1 & -b_1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1+a & b_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & & \vdots \\ 0 & -b_4 & 0 & 0 & -c_4 & 0 & \ddots & \\ -a & 0 & 0 & & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & & 0 & -1 & -1 \\ 0 & -b_{2k} & & & & & 0 & -c_{2k} \\ -a & 0 & 0 & \cdots & & \cdots & 0 & 0 \end{bmatrix}.$$

Let p_{2k+1} be the characteristic polynomial of A_{2k+1}. Then

$$p_{2k+1} = \det(xI - A_{2k+1}) = \det\begin{bmatrix} x+1 & b_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1-a & x-b_1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 1 & 1 & 0 & & \vdots \\ 0 & b_4 & 0 & x & c_4 & 0 & \ddots & \\ a & 0 & 0 & & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & & x & 1 & 1 \\ 0 & b_{2k} & & & & & x & c_{2k} \\ a & 0 & 0 & \cdots & & \cdots & 0 & x \end{bmatrix}.$$

Considering the cofactor expansion of $\det(xI - A_{2k+1})$ along the last row, since

$$\begin{aligned} &\det[(xI - A_{2k+1})(\{2k+1\}, \{1\})] \\ &\quad = \det\begin{bmatrix} b_1 & 1 \\ x - b_1 & -1 \end{bmatrix} \det\begin{bmatrix} 1 & 1 \\ x & c_4 \end{bmatrix} \cdots \det\begin{bmatrix} 1 & 1 \\ x & c_{2k} \end{bmatrix} \end{aligned}$$

and

$$\det[(xI - A_{2k+1})(2k+1)] \\ = xp_{2k-1} + b_{2k}\det\begin{bmatrix} x+1 & 1 \\ -1-a & -1 \end{bmatrix}\det\begin{bmatrix} 1 & 1 \\ x & c_4 \end{bmatrix}\cdots\det\begin{bmatrix} 1 & 1 \\ x & c_{2k-2} \end{bmatrix}\det[1],$$

it follows that

$$p_{2k+1} = (a)(-x)(c_4 - x)(c_6 - x)\cdots(c_{2k} - x) + x^2 p_{2k-1} \\ + x(b_{2k})(a - x)(c_4 - x)\cdots(c_{2k-2} - x).$$

Let

$$g_{2k+1} = -ax\prod_{i=2}^{k}(c_{2i} - x) + (b_{2k})x(a - x)\prod_{i=2}^{k-1}(c_{2i} - x). \quad (4)$$

Note that the degree of each summand of g_{2k+1} is k, and hence $\deg(g_{2k+1}) \leqslant k$. For $k \geqslant 3$,

$$p_{2k+1} = x^2 p_{2k-1} + g_{2k+1}. \quad (5)$$

Using the above notation, the following result identifies a realization of $\mathscr{G}_{2k+1}$ having $2k-1$ zero eigenvalues and a conjugate pair of pure imaginary eigenvalues.

Proposition 7. *Let $k \geqslant 2$. If $b_1 = 1$ and $a = b_{2j} = c_{2j}$ for each $j = 2, 3, \ldots, k$, then*

$$p_{2k+1} = x^{2k+1} + ax^{2k-1}.$$

Proof. The proof is by induction on k. When $k = 2$, the proof of Lemma 5 gives $p_5 = x^5 + ax^3$. Assume that $k \geqslant 3$ and proceed by induction. By (5) and the induction hypothesis,

$$p_{2k+1} = x^2(x^{2k-1} + ax^{2k-3}) + g_{2k+1}.$$

Since $b_1 = 1$ and $a = b_{2j} = c_{2j}$ for each $j = 2, 3, \ldots, k$, it follows from (4) that $g_{2k+1} = 0$. Hence, the result follows. □

For consistency with the notation in the proof of Lemma 5, let $f_i(A_{2k+1})$ be the coefficient of $x^{(2k+1)-i}$ in p_{2k+1} for $i = 1, 2, \ldots, 2k$. Define $f_i(A_{2k+1}) = 0$ for $i \geqslant 2k+1$.

Proposition 8. *For any $k \geqslant 2$,*

$$f_1(A_{2k+1}) = 1 - b_1, \quad f_2(A_{2k+1}) = ab_1, \quad f_3(A_{2k+1}) = a - b_4.$$

Proof. By Proposition 3, it follows that $f_1(A_{2k+1}) = 1 - b_1$ and $f_2(A_{2k+1}) = ab_1$. From Lemma 5, $f_3(A_5) = a - b_4$. Assume that $k \geqslant 3$ and proceed by induction. Since $p_{2k+1} = x^2 p_{2k-1} + g_{2k+1}$, and $\deg(g_{2k+1}) \leqslant k$, it follows that $f_3(A_{2k+1}) = f_3(A_{2k-1})$. Thus, by the induction hypothesis, $f_3(A_{2k+1}) = f_3(A_{2k-1}) = a - b_4$. □

From here until the end of Lemma 13, let $r > 0$ and

$$a = \frac{r}{b_1}. \quad (6)$$

It is our goal to show that

$$\left.\frac{\partial(f_1(A_{2k+1}), f_3(A_{2k+1}), f_4(A_{2k+1}), \ldots, f_{2k}(A_{2k+1}))}{\partial(b_1, b_4, c_4, \ldots, b_{2k}, c_{2k})}\right|_{(1,r,r,\ldots,r)} \neq 0.$$

Consider p_{2k+1} and g_{2k+1} as functions of $b_1, b_4, c_4, \ldots, b_{2k}, c_{2k}$. Let $z \in \{b_1, b_4, c_4, \ldots, b_{2k}, c_{2k}\}$. Then $\frac{\partial f_j(A_{2k+1})}{\partial z}$ is equal to the coefficient of x^{2k+1-j} in $\frac{\partial p_{2k+1}}{\partial z}$.

Remark 9. By (5), $\frac{\partial f_j(A_{2k+1})}{\partial z}$ is the sum of $\frac{\partial f_j(A_{2k-1})}{\partial z}$ and the coefficient of x^{2k+1-j} in $\frac{\partial g_{2k+1}}{\partial z}$.

Proposition 10. *Let $k \geqslant 3$ and $w \in \{b_4, c_4, \ldots, b_{2k-2}, c_{2k-2}\}$. Then*

$$\left.\frac{\partial g_{2k+1}}{\partial w}\right|_{(1,r,\ldots,r)} = 0.$$

Moreover,

$$\left.\frac{\partial g_{2k+1}}{\partial b_{2k}}\right|_{(1,r,\ldots,r)} = x(r-x)^{k-1} \quad \textit{and} \quad \left.\frac{\partial g_{2k+1}}{\partial c_{2k}}\right|_{(1,r,\ldots,r)} = -rx(r-x)^{k-2}.$$

Proof. Suppose that $w \in \{b_4, \ldots, b_{2k-2}\}$. Since w does not appear in (4), $\left.\frac{\partial g_{2k+1}}{\partial w}\right|_{(1,r,\ldots,r)} = 0$. Suppose that $w \in \{c_4, \ldots, c_{2k-2}\}$. Then

$$\frac{\partial g_{2k+1}}{\partial w} = -ax\left[\prod_{i=2}^{k}(c_{2i}-x)\right]\frac{1}{w-x} + b_{2k}x(a-x)\left[\prod_{i=2}^{k-1}(c_{2i}-x)\right]\frac{1}{w-x}.$$

Thus, by (6),

$$\left.\frac{\partial g_{2k+1}}{\partial w}\right|_{(1,r,\ldots,r)} = -rx(r-x)^{k-2} + rx(r-x)^{k-2} = 0.$$

Next, a simple computation shows that

$$\frac{\partial g_{2k+1}}{\partial b_{2k}} = x(a-x)\left[\prod_{i=2}^{k-1}(c_{2i}-x)\right] \quad \text{and} \quad \frac{\partial g_{2k+1}}{\partial c_{2k}} = -ax\left[\prod_{i=2}^{k-1}(c_{2i}-x)\right],$$

from which the result follows. □

Proposition 11. *Let $k \geqslant 3$. Then*

$$\left.\frac{\partial f_j(A_{2k+1})}{\partial c_4}\right|_{(1,r,\ldots,r)} = 0$$

$$\textit{for } j = 1, 3, 5, 6, \ldots, 2k-1, 2k, \textit{ and } \left.\frac{\partial f_4(A_{2k+1})}{\partial c_4}\right|_{(1,r,\ldots,r)} = -r.$$

Proof. Consider the first statement. When $j = 1$ or 3, Proposition 8 shows that c_4 does not appear in $f_1(A_{2k+1})$ and $f_3(A_{2k+1})$, giving the result. For $j = 5, 6, \ldots, 2k$, the proof is by induction on k. When $k = 3$, it follows that $p_7 = x^2 p_5 + g_7$. Note that by the definition of $f_j(A_{2k+1})$, $f_5(A_5) = f_6(A_5) = 0$. Since Proposition 10 implies that $\left.\frac{\partial g_7}{\partial c_4}\right|_{(1,r,\ldots,r)} = 0$, by Remark 9, it follows that $\left.\frac{\partial f_j(A_7)}{\partial c_4}\right|_{(1,r,\ldots,r)} = 0$ for $j = 5, 6$.

Assume that $k \geqslant 4$, and proceed by induction. By Remark 9 and the induction hypothesis, $\frac{\partial f_j(A_{2k+1})}{\partial c_4}\Big|_{(1,r,\ldots,r)}$ is equal to the coefficient of x^{2k+1-j} in $\frac{\partial g_{2k+1}}{\partial c_4}\Big|_{(1,r,\ldots,r)}$. Since Proposition 10 implies that $\frac{\partial g_{2k+1}}{\partial c_4}\Big|_{(1,r,\ldots,r)} = 0$, the first result follows.

For the second statement, note from the proof of Lemma 5 that $\frac{\partial f_4(A_5)}{\partial c_4}\Big|_{(1,r,\ldots,r)} = -r$. Hence, Remark 9 and Proposition 10 imply that $\frac{\partial f_4(A_7)}{\partial c_4}\Big|_{(1,r,\ldots,r)} = -r$. Assume that $k \geqslant 4$, and proceed by induction. By Remark 9 and the induction hypothesis, $\frac{\partial f_4(A_{2k+1})}{\partial c_4}\Big|_{(1,r,\ldots,r)}$ is equal to the sum of $-r$ and the coefficient of x^{2k+1-4} in $\frac{\partial g_{2k+1}}{\partial c_4}\Big|_{(1,r,\ldots,r)}$. Since Proposition 10 implies that $\frac{\partial g_{2k+1}}{\partial c_4}\Big|_{(1,r,\ldots,r)} = 0$, the result follows. □

Proposition 12

(a) *For* $k \geqslant 4$, $\frac{\partial f_j(A_{2k+1})}{\partial b_i}\Big|_{(1,r,\ldots,r)} = \frac{\partial f_j(A_{2k+1})}{\partial c_i}\Big|_{(1,r,\ldots,r)} = 0$ *for* $i = 6, 8, \ldots, 2k-2$ *and* $j = i+1, i+2, \ldots, 2k$.

(b) *For* $k \geqslant 3$, $\frac{\partial f_{2k-1}(A_{2k+1})}{\partial b_{2k}}\Big|_{(1,r,\ldots,r)} = -(k-1)r^{k-2}$, $\frac{\partial f_{2k-1}(A_{2k+1})}{\partial c_{2k}}\Big|_{(1,r,\ldots,r)} = (k-2)r^{k-2}$, $\frac{\partial f_{2k}(A_{2k+1})}{\partial b_{2k}}\Big|_{(1,r,\ldots,r)} = r^{k-1}$ *and* $\frac{\partial f_{2k}(A_{2k+1})}{\partial c_{2k}}\Big|_{(1,r,\ldots,r)} = -r^{k-1}$.

Proof. (a) The proof is by induction on k. For $k = 4$, Remark 9 and Proposition 10 imply that $\frac{\partial f_j(A_9)}{\partial b_6}\Big|_{(1,r,\ldots,r)}$ for $j = 7, 8$ is the sum of $\frac{\partial f_j(A_7)}{\partial b_6}\Big|_{(1,r,\ldots,r)}$ and the coefficient of x^{9-j} in $\frac{\partial g_9}{\partial b_6}\Big|_{(1,r,\ldots,r)}$, both of which are zero.

Assume that $k \geqslant 5$ and proceed by induction. By Proposition 10 and the induction hypothesis, it follows that $\frac{\partial f_j(A_{2k+1})}{\partial b_{2i}}\Big|_{(1,r,\ldots,r)} = \frac{\partial f_j(A_{2k-1})}{\partial b_{2i}}\Big|_{(1,r,\ldots,r)} = 0$ for $i = 6, 8, \ldots, 2k-2$ and $j = i+1, i+2, \ldots, 2k$.

Similarly, $\frac{\partial f_j(A_{2k+1})}{\partial c_{2i}}\Big|_{(1,r,\ldots,r)} = 0$ for $i = 6, 8, \ldots, 2k-2$ and $j = i+1, i+2, \ldots, 2k$.

(b) Note that by definition, $\frac{\partial f_{2k-1}(A_{2k-1})}{\partial b_{2k}}\Big|_{(1,r,\ldots,r)} = \frac{\partial f_{2k-1}(A_{2k-1})}{\partial c_{2k}}\Big|_{(1,r,\ldots,r)} = 0$. Thus, by Remark 9, $\frac{\partial f_{2k-1}(A_{2k+1})}{\partial b_{2k}}\Big|_{(1,r,\ldots,r)}$ $\left(\text{resp. } \frac{\partial f_{2k}(A_{2k+1})}{\partial b_{2k}}\Big|_{(1,r,\ldots,r)}\right)$ is the coefficient of x^2 (resp. x) in $\frac{\partial g_{2k+1}}{\partial b_{2k}}\Big|_{(1,r,\ldots,r)}$. By Proposition 10, the result follows, and the proofs of the derivatives with respect to c_{2k} are similar. □

We now abbreviate $f_j(A_{2k+1})$ to f_j.

Lemma 13. *For* $k \geqslant 3$,

$$\frac{\partial(f_1, f_3, f_4, \ldots, f_{2k})}{\partial(b_1, b_4, c_4, b_6, c_6, \ldots, b_{2k}, c_{2k})}\Bigg|_{(1,r,\ldots,r)} \neq 0.$$

Proof. By Prop. 8, $\left.\frac{\partial(f_1,f_3,f_4,\ldots,f_{2k})}{\partial(b_1,b_4,c_4,b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r,\ldots,r)} = \left.\frac{\partial(f_1,f_3)}{\partial(b_1,b_4)}\right|_{(1,r,\ldots,r)} \left.\frac{\partial(f_4,f_5,\ldots,f_{2k})}{\partial(c_4,b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r,\ldots,r)} = \left.\frac{\partial(f_4,f_5,\ldots,f_{2k})}{\partial(c_4,b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r,\ldots,r)}$. Proposition 11 implies that

$$\left.\frac{\partial(f_4,f_5,\ldots,f_{2k})}{\partial(c_4,b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r,\ldots,r)} = (-r)\left.\frac{\partial(f_5,f_6,f_7,\ldots,f_{2k})}{\partial(b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r,\ldots,r)}.$$

By Proposition 12, $\left.\frac{\partial(f_5,f_6,f_7,\ldots,f_{2k})}{\partial(b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r,\ldots,r)}$ is the determinant of the following block upper triangular matrix:

$$\begin{bmatrix} -2r & r & & & & & \\ r^2 & -r^2 & & & & & \\ & & -3r^2 & 2r^2 & & * & \\ & & r^3 & -r^3 & & & \\ & & & & \ddots & & \\ & & O & & & -(k-1)r^{k-2} & (k-2)r^{k-2} \\ & & & & & r^{k-1} & -r^{k-1} \end{bmatrix}.$$

Since each 2 by 2 block of this block upper triangular matrix is nonsingular, the result follows. □

Theorem 14. *For $k \geqslant 2$, the sign pattern $\mathcal{G}_{2k+1}$ allows each inertia (n_1,n_2,n_3) with $n_1+n_3+n_3 = 2k+1$ and $n_3 \geqslant 1$.*

Proof. The case $k=2$ is proved in Theorem 6. For $k \geqslant 3$, consider $A_{2k+1} \in Q(\mathcal{G}_{2k+1})$, and let $a = \frac{r_2}{b_1}$ for positive r_2. By Proposition 7, if $b_1 = 1, a = b_{2j} = c_{2j} = r_2$ for $j = 2,3,\ldots,k$, then $p_{2k+1} = x^{2k+1} + r_2x^{2k-1}$, and Lemma 13 implies that

$$\left.\frac{\partial(f_1,f_3,f_4,\ldots,f_{2k})}{\partial(b_1,b_4,c_4,b_6,c_6,\ldots,b_{2k},c_{2k})}\right|_{(1,r_2,\ldots,r_2)} \neq 0.$$

Thus, a similar argument as in the proof of Lemma 5 shows that for any r_j for $j = 1,3,4,\ldots,2k-1,2k$, there exists a matrix A in $Q(\mathcal{G}_n)$ such that

$$p_A(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-2}x^2 + r_{n-1}x.$$

Theorem 1 with $h=1$ now gives the result. □

5. Inertia $(n_1, n_2, 0)$ for $\mathcal{G}_{2k+1}$

In this section it is shown that for $k \geqslant 2$, the sign pattern $\mathcal{G}_{2k+1}$ allows each inertia $(n_1,n_2,0)$ with $n_1+n_2 = 2k+1$. We begin by providing a useful result on the signs of the determinants of matrices in $Q(\mathcal{G}_{2k+\ell})$ for $\ell \geqslant 0$.

Lemma 15. *For $k \geqslant 2$ and $\ell \geqslant 0$, let A be a matrix of the form (1) in $Q(\mathcal{G}_{2k+1})$ and $A' \in Q(\mathcal{G}_{2k+\ell})$ such that $A'[\{1,2,3\}] = A[\{1,2,3\}]$. If ℓ is odd, then*

$$\operatorname{sgn}(\det(A')) = \operatorname{sgn}(-\det(A[\{1,2\},\{2,3\}])).$$

If ℓ is even, then

$$\operatorname{sgn}(\det(A')) = \operatorname{sgn}(\det(A[\{1,2\},\{1,3\}])).$$

Proof. Using notation similar to that in (1), let the nonzero entries of A' be written using positive a'_j, b'_j, c'_j and d'_j. If ℓ is odd, by the cofactor expansion of the determinant along the last row of A',

$$\begin{aligned}\det(A') &= (-a'_{2k+\ell})(-1)^{2k+\ell-3}\det(A'[\{1,2\},\{2,3\}])\prod_{i=3}^{2k+\ell-1} c'_i\\ &= -\left[a'_{2k+\ell}\prod_{i=3}^{2k+\ell-1} c'_i\right]\det(A[\{1,2\},\{2,3\}]).\end{aligned}$$

If ℓ is even, by the cofactor expansion of the determinant along the last row of A',

$$\begin{aligned}\det(A') &= (-b'_{2k+\ell})(-1)^{2k+\ell-3}\det(A'[\{1,2\},\{1,3\}])\prod_{i=3}^{2k+\ell-1} c'_i\\ &= \left[b'_{2k+\ell}\prod_{i=3}^{2k+\ell-1} c'_i\right]\det(A[\{1,2\},\{1,3\}]).\end{aligned}$$

The results now follow. □

Theorem 16. *For $k \geqslant 2$, let $A \in Q(\mathscr{G}_{2k+1})$ and $i(A) = (p, q, 0)$ with $p + q = 2k + 1$. Suppose that $\det(A)\det(A[\{1,2\},\{1,3\}]) > 0$ (resp. < 0). Then, for each $n \geqslant 2k + 1$, there exists $M \in Q(\mathscr{G}_n)$ such that $M[\{1,\ldots,2k+1\}] = A$ and $i(M) = (n - q, q, 0)$ (resp. $(p, n - p, 0)$).*

Proof. Let $A \in Q(\mathscr{G}_{2k+1})$ satisfy $\det(A)\det(A[\{1,2\},\{1,3\}]) > 0$ (resp. < 0) and $i(A) = (p, q, 0)$ with $p + q = 2k + 1$. Take $A' \in Q(\mathscr{G}_{2k+2})$ so that $A'(2k+2) = A$. Since $\operatorname{sgn}(\det(A')) = \operatorname{sgn}(\det(A[\{1,2\},\{1,3\}]))$ by Lemma 15 with $\ell = 2$, the theorem assumption implies that $\det(A)\det(A') > 0$ (resp. < 0). By Theorem 2(a) (resp. (b)), there exists a positive diagonal matrix $D = I_{2k+1} \oplus [\epsilon]$ such that $(DA')(2k+2) = A$ and $i(DA') = (p+1, q, 0)$ (resp. $(p, q+1, 0)$). Since $DA' \in Q(\mathscr{G}_{2k+2})$, the result for $n = 2k+2$ follows by taking $M = DA'$. Repeated application of Lemma 15 and Theorem 2 gives the result for all $n \geqslant 2k+3$. □

The following result along with Theorems 4 and 6 shows that $\mathscr{G}_5$ is an IAP that is not PN; we believe that this is the first identified such sign pattern that is irreducible.

Lemma 17. *The sign pattern $\mathscr{G}_5$ allows every inertia $(n_1, n_2, 0)$ with $n_1 + n_2 = 5$. If $n_1 > 0$ and $n_2 > 0$, there exist matrices $A, B \in Q(\mathscr{G}_5)$ such that $i(A) = i(B) = (n_1, n_2, 0)$, $\det(A[\{1,2\},\{1,3\}]) > 0$ and $\det(B[\{1,2\},\{1,3\}]) < 0$. If either $n_1 = 0$ or $n_2 = 0$, there exists a matrix $A \in Q(\mathscr{G}_5)$ such that $i(A) = (n_1, n_2, 0)$ and $\det(A[\{1,2\},\{1,3\}]) > 0$.*

Proof. The following are matrices in $Q(\mathscr{G}_5)$ with the inertias of the form $(n_1, n_2, 0)$:

$$(5,0,0)\ A=\begin{bmatrix}-2&-5&-1&0&0\\4&5&1.1&0&0\\0&0&0&-2&-1\\0&-3&0&0&-1\\-1&0&0&0&0\end{bmatrix};\quad (0,5,0)\ A=\begin{bmatrix}-5&-2&-2.1&0&0\\9&2&2&0&0\\0&0&0&-1&-3\\0&-3&0&0&-2\\-4&0&0&0&0\end{bmatrix};$$

$$(4,1,0)\ A=\begin{bmatrix}-1&-1&-2&0&0\\1&1&1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}-1&-1&-2&0&0\\0.5&1&1.1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix};$$

$$(1,4,0)\ A=\begin{bmatrix}-3&-2&-1.6&0&0\\2&1&1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}-3&-2&-1&0&0\\2&1&1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix};$$

$$(3,2,0)\ A=\begin{bmatrix}-1&-2&-1.1&0&0\\1&1&1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}-1&-2&-1&0&0\\1&1&1.1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix};$$

$$(2,3,0)\ A=\begin{bmatrix}-2&-1&-1.1&0&0\\1&1&0.5&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}-2&-1&-1.1&0&0\\1&1&1&0&0\\0&0&0&-1&-1\\0&-1&0&0&-1\\-1&0&0&0&0\end{bmatrix}.$$

Note that matrices A and B above satisfy $\det(A[\{1,2\},\{1,3\}]) > 0$ and $\det(B[\{1,2\},\{1,3\}]) < 0$. $\square$

By Theorem 16 with the matrices in the proof of Lemma 17, it follows that $\mathscr{G}_n$ for each $n \geqslant 5$ allows the following inertias:

$$\begin{array}{llllll}
\mathscr{G}_5 & \mathscr{G}_6 & \mathscr{G}_7 & \mathscr{G}_8 & \mathscr{G}_9 & \qquad\qquad \mathscr{G}_n \\[2ex]
(5,0,0)A \to & (6,0,0) \to & (7,0,0) \to & (8,0,0) \to & (9,0,0) \to \cdots \to & \left(n, \boxed{0}, 0\right) \\[2ex]
(0,5,0)A \to & (0,6,0) \to & (0,7,0) \to & (0,8,0) \to & (0,9,0) \to \cdots \to & \left(\boxed{0}, n, 0\right) \\[2ex]
(4,1,0)A \to & (4,2,0) \to & (4,3,0) \to & (4,4,0) \to & (4,5,0) \to \cdots \to & \left(\boxed{4}, n-4, 0\right) \\
(4,1,0)B \to & (5,1,0) \to & (6,1,0) \to & (7,1,0) \to & (8,1,0) \to \cdots \to & \left(n-1, \boxed{1}, 0\right) \\[2ex]
(1,4,0)A \to & (2,4,0) \to & (3,4,0) \to & (4,4,0) \to & (5,4,0) \to \cdots \to & \left(n-4, \boxed{4}, 0\right) \\
(1,4,0)B \to & (1,5,0) \to & (1,6,0) \to & (1,7,0) \to & (1,8,0) \to \cdots \to & \left(\boxed{1}, n-1, 0\right)
\end{array}$$

$$\begin{array}{l}
(3,2,0)A \to (4,2,0) \to (5,2,0) \to (6,2,0) \to (7,2,0) \to \cdots \to \big(n-2, \boxed{2}, 0\big) \\
(3,2,0)B \to (3,3,0) \to (3,4,0) \to (3,5,0) \to (3,6,0) \to \cdots \to \big(\boxed{3}, n-3, 0\big) \\
\\
(2,3,0)A \to (2,4,0) \to (2,5,0) \to (2,6,0) \to (2,7,0) \to \cdots \to \big(\boxed{2}, n-2, 0\big) \\
(2,3,0)B \to (3,3,0) \to (4,3,0) \to (5,3,0) \to (6,3,0) \to \cdots \to \big(n-3, \boxed{3}, 0\big).
\end{array} \tag{7}$$

The boxed numbers are invariant in each row of the above table. Note that Table (7) implies that, for $k \in \{3, 4\}$, the sign pattern $\mathscr{G}_{2k+1}$ allows each inertia $(n_1, n_2, 0)$ with $n_1 + n_2 = 2k + 1$.

In the proof of the following lemma, we use a $(2k + 1)$ by $(2k + 1)$ matrix

$$\begin{bmatrix}
0 & -b_1 & -d_1 & 0 & \cdots & 0 \\
0 & b_2 & c_2 & 0 & \cdots & \vdots \\
0 & 0 & 0 & -1 & \ddots & \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & & -1 \\
-1 & 0 & 0 & & \cdots & 0
\end{bmatrix}, \tag{8}$$

which has characteristic polynomial

$$x^{2k+1} - b_2 x^{2k} - d_1 x + (b_2 d_1 - b_1 c_2). \tag{9}$$

Lemma 18. *For $k \geqslant 3$, the sign pattern $\mathscr{G}_{2k+1}$ allows each inertia $(n_1, n_2, 0)$ with $n_1 + n_2 = 2k + 1$.*

Proof. The cases for $k = 3$ and 4 follow from (7). For $k \geqslant 5$, we first show that for $k = 5$ and 6, $\mathscr{G}_{2k+1}$ allows each inertia $(n_1, n_2, 0)$ with $n_1 + n_2 = 2k + 1$. Consider the 11 by 11 matrix M_1 of the form (8) with $b_2 = d_1 = 0$ and $b_1 = c_2 = 1$. Then, by (9), $p_{M_1}(x) = x^{11} - 1$ and thus $i(M_1) = (5, 6, 0)$.

Let $A_1 \in Q(\mathscr{G}_{11})$ be of the form (1) such that $a_{11} = b_1 = c_2 = c_3 = \cdots = c_{10} = 1$, $a_1 = \epsilon^2$, $a_2 = 2\epsilon$, and the other $a_j, b_j, d_j = \epsilon$, where $\epsilon > 0$. It can be easily checked that A_1 is obtained by replacing some zero entries of M_1 by some multiple of ϵ or ϵ^2. Thus, by continuity of eigenvalues, $i(A_1) = i(M_1) = (5, 6, 0)$ for sufficiently small ϵ. Note that $\operatorname{sgn}(\det(A_1[\{1, 2\}, \{1, 3\}])) = \operatorname{sgn}(-\epsilon^2 + 2\epsilon^2) = +$.

Let B_1 be the matrix obtained by replacing the $(1, 1)$- and $(2, 1)$-entries of A_1 by $-\epsilon$ and ϵ, respectively. Similarly, $i(B_1) = (5, 6, 0)$ and $\operatorname{sgn}(\det(B_1[\{1, 2\}, \{1, 3\}])) = \operatorname{sgn}(-\epsilon + \epsilon^2) = -$. By Theorem 16, $\mathscr{G}_n$ for $n \geqslant 11$ allows the following inertias:

$$\begin{array}{ccccccccc}
\mathscr{G}_{11} & & \mathscr{G}_{12} & & \mathscr{G}_{13} & & & & \mathscr{G}_n \\
\\
(5,6,0)A_1 & \to & (6,6,0) & \to & (7,6,0) & \to & \cdots & \to & \big(n-6, \boxed{6}, 0\big) \\
\\
(5,6,0)B_1 & \to & (5,7,0) & \to & (5,8,0) & \to & \cdots & \to & \big(\boxed{5}, n-5, 0\big).
\end{array} \tag{10}$$

As in (7), the boxed numbers are invariant in each row of (10).

Next, consider the 11 by 11 matrix M_2 of the form (8) with $c_2 = 0$ and $b_1 = b_2 = d_1 = 1$. Then, by (9), $p_{M_2}(x) = x^{11} - x^{10} - x + 1 = (x^{10} - 1)(x - 1)$. Since the inertia of $x^{10} - 1$ is $(5, 5, 0)$, it follows that $i(M_2) = (6, 5, 0)$. Let $A_2 \in Q(\mathscr{G}_{11})$ be of the form (1) such that $a_{11} = b_1 = b_2 = d_1 = c_3 = \cdots = c_{10} = 1$, and the other $a_j, b_j, c_j, d_j = \epsilon$, where $\epsilon > 0$. It can be easily checked that A_2 is obtained by replacing some zero entries of M_2 by ϵ. Thus, by continuity of eigenvalues, $i(A_2) = i(M_2) = (6, 5, 0)$ for sufficiently small ϵ. Note that $\operatorname{sgn}(\det(A_2[\{1, 2\}, \{1, 3\}])) = \operatorname{sgn}(-\epsilon^2 + \epsilon) = +$.

Let B_2 be the matrix obtained by replacing the $(2, 1)$- and $(2, 3)$-entries of A_2 by ϵ^2 and 2ϵ, respectively. Similarly, $i(B_2) = (6, 5, 0)$ and $\operatorname{sgn}(\det(B_2[\{1, 2\}, \{1, 3\}])) = \operatorname{sgn}(-2\epsilon^2 + \epsilon^2) = -$. By Theorem 16, $\mathscr{G}_n$ for $n \geqslant 11$ allows the following inertias:

$$\begin{array}{ccccccccc}
\mathscr{G}_{11} & & \mathscr{G}_{12} & & \mathscr{G}_{13} & & & & \mathscr{G}_n \\
(6,5,0)A_2 & \to & (6,6,0) & \to & (6,7,0) & \to & \cdots & \to & \left(\boxed{6}, n-6, 0\right) \\
(6,5,0)B_2 & \to & (7,5,0) & \to & (8,5,0) & \to & \cdots & \to & \left(n-5, \boxed{5}, 0\right).
\end{array} \tag{11}$$

Therefore, by (7), (10) and (11), $\mathscr{G}_{2k+1}$ for $k = 5$ and 6 allows each inertia $(n_1, n_2, 0)$ with $n_1 + n_2 = 2k + 1$.

By continuing this procedure for each pair $\mathscr{G}_{4\ell-1}$ and $\mathscr{G}_{4\ell+1}$ ($\ell \geqslant 4$), and noting that the inertia of $x^{4\ell-1} - 1$ is $(2\ell - 1, 2\ell, 0)$ and the inertia of $x^{4\ell-2} - 1$ is $(2\ell - 1, 2\ell - 1, 0)$, the result follows. □

Theorem 14 and Lemmas 17, 18 combined with Theorem 4 give our main result.

Theorem 19. *For $k \geqslant 2$, the irreducible sign pattern $\mathscr{G}_{2k+1}$ is an inertially arbitrary pattern that is not potentially nilpotent.*

6. Discussion of $\mathscr{G}_{2k}$ and minimality of $\mathscr{G}_{2k+1}$

This section begins with some results on the signs of coefficients of $p_A(x)$ that depend on the inertia of A. These results are used to show that for $k \geqslant 2$, $\mathscr{G}_{2k}$ is not inertially arbitrary, and that $\mathscr{G}_5, \mathscr{G}_7$ are MIAPs.

Let A be an n by n matrix and let $p_A(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_{n-1}x + r_n$. For real a, b with $b \neq 0$, a is an eigenvalue of A if and only if $(x - a) \mid p_A(x)$, and $a \pm bi$ are eigenvalues of A if and only if $(x^2 - 2ax + a^2 + b^2) \mid p_A(x)$. In the following it is shown that a certain inertia determines $\operatorname{sgn}(r_j)$ for some $j \in \{1, \ldots, n\}$.

Lemma 20. *Let A be an n by n matrix.*

(a) *If $i(A) = (0, n, 0)$, then $r_j > 0$ for each $j = 1, \ldots, n$.*
(b) *If $i(A) = (n, 0, 0)$, then $r_j < 0$ for each j odd, and $r_j > 0$ for each j even.*
(c) *If $i(A) = (0, 0, n)$, then $r_j = 0$ for each j odd.*
(d) *If $i(A) = (1, n - 2, 1)$, then $r_{n-1} < 0$.*
(e) *If $i(A) = (n - 2, 1, 1)$, then $r_{n-1} < 0$ for n odd and $r_{n-1} > 0$ for n even.*

Proof

(a) Since $i(A) = (0, n, 0)$, each linear factor of $p_A(x)$ is of the form $x + a$ for some $a > 0$, and each irreducible quadratic factor of $p_A(x)$ is of the form $x^2 + bx + c$ for some positive b, c. Since $p_A(x)$ is the product of such factors, the result follows.

(b) Since $i(A) = (n, 0, 0)$, each linear factor of $p_A(x)$ is of the form $x - a$ for some $a > 0$, and each irreducible quadratic factor of $p_A(x)$ is of the form $x^2 - bx + c$ for some positive b, c. Thus, as in (a), the result follows.

(c) Suppose that $n = 2k$ is even. If A has 2ℓ pure imaginary eigenvalues, then A has $2(k - \ell)$ zero eigenvalues. Thus, $p_A(x)$ is of the form $x^{2(k-\ell)}(x^2 + a_1)\cdots(x^2 + a_\ell)$ where $a_j > 0$, and hence the result follows.
Suppose that $n = 2k - 1$ is odd. If A has 2ℓ pure imaginary eigenvalues, then A has $2(k - \ell) - 1$ zero eigenvalues. Thus, $p_A(x)$ is of the form $x^{2(k-\ell)-1}(x^2 + a_1)\cdots(x^2 + a_\ell)$ where $a_j > 0$, and hence the result follows.

(d) Since $i(A) = (1, n - 2, 1)$, A has a unique positive eigenvalue a and exactly one zero eigenvalue. Thus, by (a), $p_A(x)$ is of the form $x(x - a)(x^{n-2} + a_1x^{n-3} + \cdots + a_{n-3}x + a_{n-2})$ where $a_j > 0$. Since the coefficient of x is $-aa_{n-2} < 0$, the result follows.

(e) Since $i(A) = (n - 2, 1, 1)$, it follows that $p_A(x)$ is of the form $x(x + a)(x^{n-2} - b_1x^{n-3} + \cdots + (-1)^{n-3}b_{n-3}x + (-1)^{n-2}b_{n-2})$ where $a, b_1, \ldots, b_{n-2} > 0$. Thus, $r_{n-1} = (-1)^{n-2} ab_{n-2}$, and the result follows. $\square$

Remark 21. Let $\mathscr{S}$ be an n by n sign pattern, $1 \leqslant k \leqslant n$ and $\tau \in \{+, -, 0\}$. For k odd, (a), (b) and (c) in Lemma 20 imply that if there does not exist any matrix $A \in Q(\mathscr{S})$ such that the sign of the coefficient of x^{n-k} in $p_A(x)$ is τ, then $\mathscr{S}$ is not an IAP. This is in contrast with the case $k = 2$, since $P_n \subseteq \{p_A(x)|A \in Q(\mathscr{S})\}$ suffices to show that $\mathscr{S}$ is inertially arbitrary; see Theorem 1.

In the following theorem it is shown that for $k \geqslant 2$, $\mathscr{G}_{2k}$ is not an IAP. For $k \geqslant 2$, let

$$A = \begin{bmatrix} -a_1 & -b_1 & -d_1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & & & \vdots \\ 0 & 0 & 0 & -c_3 & -d_3 & 0 & & \\ 0 & -b_4 & 0 & 0 & -c_4 & \ddots & \ddots & \vdots \\ -a_5 & 0 & 0 & 0 & 0 & \ddots & -d_{2k-3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & -c_{2k-2} & 0 \\ -a_{2k-1} & 0 & 0 & 0 & & & \ddots & -c_{2k-1} \\ 0 & -b_{2k} & 0 & 0 & \cdots & & 0 & 0 \end{bmatrix} \in Q(\mathscr{G}_{2k}), \tag{12}$$

where a_j, b_j, c_j and $d_j > 0$.

Theorem 22. *For $k \geqslant 2$, the sign pattern $\mathscr{G}_{2k}$ is not an inertially arbitrary pattern.*

Proof. We first show that $\mathscr{G}_4$ is not an IAP. Let $A \in Q(\mathscr{G}_4)$ be of the form (12). Then

$$p_A(x) = x^4 + (a_1 - b_2)x^3 + (a_2b_1 - a_1b_2)x^2 - b_4c_2c_3x + a_2b_4c_3d_1 - a_1b_4c_2c_3.$$

Note that $r_3 < 0$. Thus, by Lemma 20(a), there is no matrix in $Q(\mathscr{G}_4)$ having inertia $(0, 4, 0)$.

Let $k \geqslant 3$. We show that there is no matrix in $Q(\mathscr{G}_{2k})$ having inertia $(0, 2k-1, 1)$. Suppose to the contrary that there exists a matrix $N \in Q(\mathscr{G}_{2k})$ with $i(N) = (0, 2k-1, 1)$. Since the nullity of N is 1, by a similar argument as in the proof of Theorem 4, it can be shown that there exists a singular matrix $A \in Q(\mathscr{G}_{2k})$ of the form (12), similar to N, that has a nonzero left nullvector $v = (v_1, \ldots, v_{2k})$ with $v_j \in \{1, -1, 0\}$ for each $j \in \{1, \ldots, 2k\}$, and this nonzero left nullvector v of A is, without loss of generality, one of the following vectors (i), (ii), (iii):

$$\text{(i) } (1, 1, 0, \ldots, 0, 1), \quad \text{(ii) } (1, 1, 0, \ldots, 0, -1), \quad \text{(iii) } (1, 1, 0, \ldots, 0).$$

If the nullvector v is (i), (ii) or (iii), respectively, then the entries a_1, b_1, d_1 in the matrix A of the form (12) are as follows:

$$\text{(i) } a_1 = a_2, b_1 = b_2 - b_{2k}, d_1 = c_2; \quad \text{(ii) } a_1 = a_2, b_1 = b_2 + b_{2k}, d_1 = c_2;$$
$$\text{(iii) } a_1 = a_2, b_1 = b_2, d_1 = c_2.$$

Now, it is shown that in each case, $p_A(x)$ has a negative coefficient. For case (i), by Proposition 3, as $\det(A[\{1, 2\}])$ is the only nonzero 2 by 2 principal minor,

$$\text{(i) } r_2 = \det \begin{bmatrix} -a_2 & -b_2 + b_{2k} \\ a_2 & b_2 \end{bmatrix} = -a_2 b_{2k} < 0.$$

For case (iii), consider r_{2k-1}. Note that the last row of $A(2)$ is zero, and the first two rows of $A(j)$ for $j = 3, \ldots, 2k$ are multiples of each other. Thus, by Proposition 3,

$$\text{(iii) } r_{2k-1} = (-1)^{2k-1}(-b_{2k})c_2 \prod_{i=3}^{2k-1}(-c_i) = -b_{2k} \prod_{i=2}^{2k-1} c_i < 0.$$

For case (ii), consider r_{2k-1}. Since the last row of $A(2)$ is zero, $\det(A(2)) = 0$. Since

$$\det(A(3)) = \det \begin{bmatrix} -a_2 & -b_2 - b_{2k} \\ a_2 & b_2 \end{bmatrix} \det \begin{bmatrix} 0 & -c_4 & 0 & \cdots & 0 \\ 0 & 0 & -c_5 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ & & & 0 & -c_{2k-1} \\ 0 & \cdots & & \cdots & 0 \end{bmatrix},$$

it follows that $\det(A(3)) = 0$. Note that by the cofactor expansion of the determinant along the last row,

$$\det(A(j)) = b_{2k} \det \begin{bmatrix} -a_2 & -c_2 \\ a_2 & c_2 \end{bmatrix} \det(A(\{1, 2, j, 2k\}, \{1, 2, 3, j\})) = 0$$

for $j = 4, \ldots, 2k-1$. Therefore,

$$\text{(ii) } r_{2k-1} = (-1)^{2k-1}(\det(A(1)) + \det(A(2k))) = -b_{2k} \prod_{i=2}^{2k-2} c_i (c_{2k-1} + a_{2k-1}) < 0.$$

By Lemma 20(a), the existence of a negative coefficient in the characteristic polynomial of each case contradicts the assumption that $i(A) = (0, 2k-1, 1)$. $\square$

From the above proof, any $A \in Q(\mathscr{G}_4)$ has $r_3 < 0$ in $p_A(x)$, thus $\mathscr{G}_4$ is not PN. Since $\mathscr{G}_{2k}(1)$ is permutation equivalent to an upper triangular sign pattern with nonzero diagonal, every singular realization of $\mathscr{G}_{2k}$ has nullity 1. By a similar argument as in the proof of Theorem 22, $\mathscr{G}_{2k}$ is not PN for $k \geqslant 3$.

Theorem 23. *The sign patterns $\mathscr{G}_5$ and $\mathscr{G}_7$ are MIAPs.*

Proof. Let $A \in Q(\mathscr{G}_5)$ be of the form (1). Consider the following coefficients of $p_A(x)$:

$$\begin{aligned} r_1 &= a_1 - b_2, \\ r_2 &= a_2 b_1 - a_1 b_2, \\ r_3 &= a_5 d_1 d_3 - b_4 c_2 c_3, \\ r_5 &= a_5 b_2 c_3 c_4 d_1 - a_5 b_1 c_2 c_3 c_4. \end{aligned}$$

Let $j \in \{1, 2\}$ and let $\hat{A} \in Q(\mathscr{U})$, where $\mathscr{U}$ is a proper subpattern of $\mathscr{G}_5$ with $\alpha_j = 0$. Then the coefficient r_j for $p_{\hat{A}}(x)$ is nonpositive for every such $\hat{A}$. Hence, by Lemma 20(a), $\mathscr{U}$ does not allow inertia (0, 5, 0) and hence is not an IAP. If any one of $\alpha_5, \beta_1, \beta_2, \gamma_2, \gamma_3, \gamma_4$ and δ_1 is 0, then r_5 for each realization of the resultant proper subpattern of $\mathscr{G}_5$ has a fixed sign (one of $+$, $-$ and 0); if one of β_4 and δ_3 is 0, then r_3 for each realization of the resultant proper subpattern of $\mathscr{G}_5$ has a fixed sign. Hence, by Remark 21, no such proper subpattern of $\mathscr{G}_5$ is an IAP.

Next, let $A \in Q(\mathscr{G}_7)$ be of the form (1). The polynomial $p_A(x)$ has r_1, r_2, r_3 as above and

$$r_7 = a_7 b_2 c_3 c_4 c_5 c_6 d_1 - a_7 b_1 c_2 c_3 c_4 c_5 c_6$$

By a similar argument as above, it can be shown that a proper subpattern of $\mathscr{G}_7$ with one of α_i, γ_i, β_1, β_2, β_4, δ_1 and δ_3 equal to 0 is not an IAP.

Assume that none of α_i, γ_i, β_1, β_2, β_4, δ_1 and δ_3 is 0. Now, it is shown that a proper subpattern $\mathscr{U}$ of $\mathscr{G}_7$ with one of δ_5 and β_6 equal to 0 does not allow inertia $(0, 6, 1)$. Note that if a matrix has inertia $(0, 6, 1)$, then $r_7 = 0$ and Lemma 20(a) implies that $r_j > 0$ for each $j = 1, \ldots, 6$.

Let $\mathscr{U}$ be a proper subpattern of $\mathscr{G}_7$ with $\delta_5 = 0$. Suppose that there exists a matrix $\hat{A} \in Q(\mathscr{U})$ with $i(\hat{A}) = (0, 6, 1)$. As in the proof of Theorem 4, since $\hat{A}$ has nullity 1, it follows that $\hat{A}$ is of the form (i), (ii) or (iii) in that proof. However, since the matrix $\hat{A}$ of the form (ii) has $r_2 < 0$ and the matrix $\hat{A}$ of the form (iii) has $r_2 = 0$, $\hat{A}$ must be of the form (i) with $d_5 = 0$. A simple computation gives

$$\begin{aligned} r_5 &= a_7 c_5 d_1 d_3 (c_6 - b_6) - b_6 c_3 c_4 c_5 d_1 \\ r_6 &= a_7 c_3 c_4 c_5 d_1 (b_6 - c_6). \end{aligned}$$

Since $r_6 > 0$, it follows that $b_6 > c_6$. This implies that $r_5 < 0$, contradicting the assumption.

Let $\mathscr{U}$ be a proper subpattern of $\mathscr{G}_7$ with $\beta_6 = 0$. Suppose that there exists a matrix $\hat{A} \in Q(\mathscr{U})$ with $i(\hat{A}) = (0, 6, 1)$. As above, it follows that $\hat{A}$ is of the form (i) with $b_6 = 0$. A simple computation gives

$$r_6 = -a_7 c_3 c_4 c_5 c_6 d_1 < 0,$$

contradicting the assumption. □

Based on the above result and some additional computations, we conclude with the conjecture that for $k \geqslant 4$, the sign pattern $\mathscr{G}_{2k+1}$ is a minimal inertially arbitrary pattern.

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