Recurrence and transience for finite probabilistic tables

Mathieu Tracol *

LRI, Université Paris-Sud, Bat. 490 91400 Orsay, France

A R T I C L E   I N F O

Article history:
Received 16 April 2010
Received in revised form 7 December 2010
Accepted 12 December 2010
Communicated by B. Durand

Keywords:
Markov chains
Non-homogeneous Markov chains
Probabilistic Automata
Finite Probabilistic Tables
Recurrence

A B S T R A C T

A Finite Probabilistic Table, or FPT, consists of a finite state space \( S \), an initial distribution on \( S \), and a finite set of Markov matrices on \( S \), labeled by an alphabet \( \Sigma \). An infinite word on \( \Sigma \) induces a non-homogeneous Markov chain (NHMC) on \( S \).

In the context of finite homogeneous Markov chains, a state \( s \) is recurrent if with probability one a run initialized on \( s \) visits \( s \) infinitely often. Equivalently, \( s \) is recurrent if with probability one, the proportion of time a run initialized on \( s \) spends on \( s \) converges to a non-zero limit.

In this paper we introduce two natural notions of recurrence for non-homogeneous Markovian processes: a state \( s \) is weakly recurrent (resp. strongly recurrent) if with positive probability the process visits \( s \) infinitely often (resp. spends a non-zero proportion of time on \( s \)).

These notions do not coincide in the context of NHMCs, and we study the related computational problem on FPTs: given an FPT and a state \( s \), is there \( w \in \Sigma^\infty \) such that \( s \) is weakly (resp. strongly) recurrent for the associated NHMC?

We prove that the strong recurrence problem is PSPACE-complete, along with other complexity results, which contrast with previous results which showed for instance the undecidability of the weak recurrence problem.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

Markov chains have been extensively studied during the 20th century in the context of probability theory, and they are also widely used in theoretical Computer Science. Indeed, many real systems can be modeled as homogeneous Markov chains. Moreover, many efficient algorithms use Markov chains: for instance the Metropolis–Hastings algorithm implements homogeneous Markov chains, whereas the Simulated Annealing method makes use of non-homogeneous Markov chains. The distinction of homogeneous and non-homogeneous Markov chains is crucial in our study.

An important notion of homogeneous Markov chains theory is the notion of recurrence: a state \( s \) is recurrent for a homogeneous Markov chain if with probability one, when the chain is initialized on \( s \), a run will visit infinitely often \( s \). Equivalently, \( s \) is recurrent if with probability one, when initialized on \( s \), the proportion of time that a run spends on state \( s \) converges to a strictly positive limit as the length of the run goes to infinity. This limit is equal to the weight assigned to \( s \) by the stationary distribution of the irreducible Markov chain induced by the maximal irreducible component which contains \( s \).

No notion of stationary distribution can be defined in the context of non-homogeneous Markov chains, since the values of the transition probabilities depend on time. We can still consider notions of recurrence: we can define a state \( s \in S \) to be weakly recurrent if with positive probability, a run on the non-homogeneous Markov chain visits infinitely often \( s \). We can also define a state \( s \in S \) to be strongly recurrent if with positive probability, the asymptotic proportion of time a run on the
process visits the state $s$ converges to a positive limit when the length of the run goes to infinity. Changing the “infinitely often condition” by an “asymptotic positive proportion” condition allows one to express different properties which can be relevant for instance in the context of real system analysis.

These two notions of recurrence, which are equivalent in the context of homogeneous Markov chains, do not coincide in the context of non-homogeneous Markov chains. Indeed, we can easily construct non-homogeneous Markov chains such that with probability one, a run on the process visits infinitely often a state $s$, but the proportion of time spent on $s$ goes to zero as the length of the run goes to infinity.

In this paper, we define and study new notions of recurrence and transience in the context of non-homogeneous Markov chains. We focus on the particular case where all the transition matrices of the chain belong to a finite family. This model is also known as the model of Finite Probabilistic Tables, defined in [17]. We are interested in computational aspects of our new notions, and we determine the computational complexity of several associated problems. The paper is organized as follows.

- In Section 2, we recall the models of Finite Probabilistic Tables and Probabilistic Automata, we present classical problems on these models and their complexity, and we introduce the notion of support which we will use for the definition of recurrence on non-homogeneous Markov chains.
- In Section 3, we present our notions of recurrence and transience on non-homogeneous Markov chains, and nine algorithmic problems related to these notions.
- In Section 4, we introduce the notions of Loops and Filters on Finite Probabilistic Tables, which will be used for the proofs of Section 5.
- In Section 5, we determine the computational complexity of the algorithmic problems presented in Section 3.

Our main result is the PSPACE-completeness of the strong recurrence problem on non-homogeneous Markov chains, which implies other decidability results. This contrasts to most analogous algorithmic problems defined so far on Probabilistic Automata on finite and infinite words, which happened to be undecidable (see [2]).

**Related work.** The asymptotic properties of sums of random variables on non-homogeneous Markov chains, which is related to our work by the notion of support, have been studied in probability theory since the seminal work of Markov, [14], continued by Dobrushin in [9]. Other researchers have been interested in the structure of the tail $\sigma$-field of a non-homogeneous Markov chain, in particular Cohn, in [6,7]. In [15], the authors consider related questions in the context of the simulated annealing algorithm. All these papers study the evolution of an infinite product of Markov matrices. However, none of them focuses on the special case where the matrices belong to a finite family, which is crucial for the algorithmic aspects. This model is introduced as the model of Finite Probabilistic Tables (FPT), in [17]. A. Paz considers notions of weak ergodicity on FPT, but does not consider notions of recurrence. In [2,1], the authors consider Probabilistic Automata on infinite words. This model is equivalent to the model of FPT, and the acceptance notion for a Probabilistic Büchi Automaton is equivalent to our notion of weak recurrence. Baier et al., in [1], prove that the associated weak recurrence problem is undecidable. In [5], the authors present a notion of accepting run on automata which bounds the distance between two accepting states. This can be related to our notion of support of a run. However, we only ask for asymptotic boundedness, whereas the authors of [5] assume the existence of a fixed bound. See the discussion of Section 2.2 on the subject.

Most of the results of the present paper have been presented in [21].

## 2. Preliminaries

A matrix $A = \{a_{ij}\}_{i,j \in [1,p]}$ of size $p \in \mathbb{N}$ is a **Markov matrix** if the $a_{ij}$ are non-negative and the lines of the matrix sum to one: for all $i \in [1, p], \sum_{j=1}^{p} a_{ij} = 1$.

A **finite non-homogeneous Markov chain** is a process $(X_n)_{n \in \mathbb{N}}$, where the $X_n$ takes values in a finite state space $S = \{s_1, \ldots, s_p\}$ of size $|S| = p$, and which satisfies the Markov property: for all $n \in \mathbb{N}$ and all $t_0, \ldots, t_{n+1}$ in $S$, we have

$$P[X_{n+1} = t_{n+1}|X_n = t_n \wedge X_{n-1} = t_{n-1} \wedge \cdots \wedge X_0 = t_0] = P[X_{n+1} = t_{n+1}|X_n = t_n].$$

This implies that for all $n \in \mathbb{N}$, there exists a Markov matrix $A_n = \{a_{ij}\}_{i,j \in [1,p]}$ of size $p$ such that for all $i, j$ in $[1, p]$ we have

$$P[X_{n+1} = s_j|X_n = s_i] = a_{ij}.$$

In particular, we do not assume that $A_n$ is independent of $n$, which is the case when we consider homogeneous Markov chains.

### 2.1. Finite Probabilistic Tables and Probabilistic Automata

If $S$ is a finite set, we write $\Delta(S)$ for the set of probability distributions on $S$.

**Definition 1 (Finite Probabilistic Tables [17]).** A **Finite Probabilistic Table** (FPT), is a tuple $T = (S, \Sigma, \{M^a, a \in \Sigma\}, \alpha)$ where

- $S$ is a finite set (representing the states).
- $\Sigma$ is a finite set (representing the alphabet).
For all $a \in \Sigma$, $M^a$ is a Markov matrix of order $|\Sigma|$ ($M^a$ represents the transition probabilities from state to state related to the symbol $a$).

$\alpha \in \Delta(S)$ is the initial distribution on states.

**Notations.** We write $M^a = (m^a_{s,t})_{s,t \in \{1, \ldots, |\Sigma|\}}$. The component $m^a_{s,t}$ corresponds to the probability of going from state $s$ to state $t$ when the transition matrix $M^a$ is chosen. If $w = a_1 \cdots a_n \in \Sigma^n$, we write $M^w$ for the product $M^{a_1} \cdots M^{a_n}$, whose components are the $m^w_{s,t}$. Often, we will use the notation $\delta$ for the transition function: if $w \in \Sigma^*$ and $s, t \in S$, $\delta(s, w)(t)$ is the probability to arrive in $t$ if we start on $s$ and read $w$. In other words, $\delta(s, w)(t) = m^w_{s,t}$. We generalize the notation and write $\delta(s, w)$ for the set of states $t \in S$ such that $\delta(s, w)(t) > 0$. Finally, if $A \subseteq S$, $\delta(A, w)$ is the set of states $t \in S$ such that there exists $s \in A$ with $\delta(s, w)(t) > 0$. Also, if $\alpha \in \Delta(S)$, $\delta(\alpha, w)$ is the set of states $t \in S$ such that there exists $s \in S$ s.t. $\alpha(s) > 0$ and $\delta(s, w)(t) > 0$.

We will often define an FPT as a tuple $\mathcal{T} = (S, \Sigma, \delta, \alpha)$, since we can recover easily the $M^a, a \in \Sigma$ from the transition $\delta$.

**Runs on an FPT.** Let $\mathcal{T} = (S, \Sigma, \delta, \alpha)$ be an FPT. A run on $\mathcal{T}$, or a run on $S$ and $\Sigma$, is an alternating sequence $s_0a_1s_1a_2 \ldots$, finite or infinite, of states in $S$ and letters in $\Sigma$. The trace of a run $r$, written $Tr(r)$, is the sequence of its letters in $\Sigma$, and $\text{Inf}(r)$ is the set of states which appear infinitely often in $r$. Given a finite run $r = s_0a_1s_1a_2 \ldots a_n$, we denote by $|r| = n$ the length of $r$ and by $r|_k = s_0a_1s_1 \ldots a_k$ its prefix of length $k$. Similarly for a finite word $w \in \Sigma^*$, $|w|$ is the length of $w$ and $w|_k$ denotes its prefix of length $k$. We write $\Omega$ for the set of infinite runs on $\mathcal{T}$.

The $\sigma$-field of the set of runs. If $n \in \mathbb{N}$, $X_n$ is the random variable on $\Omega$ which associates with a run $r$ its $n$th state. The set of cones of the form $C_w = \{r \in \Omega | Tr(r|_n) = w\}$, for $w \in \Sigma^n$, induces a $\sigma$-field $\mathcal{F}$ on $\Omega$ which is the smallest $\sigma$-field with respect to which all the $X_n, n \geq 0$, are measurable. The initial distribution $\alpha$ on $S$, and an infinite word $w = a_1a_2 \ldots \in \Sigma^\omega$, uniquely determine a probability measure $\mathbb{P}_w$ on $\mathcal{F}$ such that $X_n, n \geq 0$ is a non-homogeneous Markov chain on $(\Omega, \mathcal{F}, \mathbb{P}_w)$, with $\mathbb{P}_w(X_0 = s) = \alpha(s)$, and

$$\mathbb{P}_w(X_{n+1} = t | X_n = s) = \delta(s, a_{n+1})(t)$$

for all $n \in \mathbb{N}$ and $s, t \in S$ (see [11,13,22,8]). We may forget the $\alpha$ in the notation when it is clear from the context.

**Definition 2** (Reachability). A state $s \in S$ is said to be accessible in $\mathcal{T}$ if there exists $n \in \mathbb{N}$ and a word $\rho \in \Sigma^n$ such that $\delta(\alpha, \rho)(s) > 0$.

By simple reachability considerations, we can compute the set $\text{Acc}(\mathcal{T})$ of the accessible states in $\mathcal{T}$ in time polynomial in the size of the FPT.

### 2.2. The notion of Support

The following notion of support is at the core of our approach. Intuitively, a factor (or subword) is in the support of an infinite sequence if it appears with positive asymptotic proportion as a factor of the sequence.

**Definition 3** (Support of an Infinite Sequence). Let $\Sigma$ be a finite alphabet, and $w = a_0, a_1, \ldots \in \Sigma^\omega$. Let $\rho = b_0b_1 \ldots b_l \in \Sigma^\omega$. We call the proportion of $\rho$ in $w$ the sup limit of the proportion of time $\rho$ appears in $a_1, \ldots, a_n$ as a factor:

$$\text{prop}(\rho, w) = \lim_{n \to \infty} \sup_{n \geq l} \frac{|\{i \in [1; n] | s.t. a_i = b_0 \land \cdots \land a_{i+l-1} = b_l\}|}{n}.$$ 

The support of the sequence $w$, written $\text{Supp}(w)$, is the set of words $\rho \in \Sigma^*$ such that $\text{prop}(\rho, w) > 0$.

A run $r$ on an FPT $\mathcal{T} = (S, \Sigma, \delta, \alpha)$, i.e., an infinite sequence of states in $S$ and letters in $\Sigma$, can be seen as an infinite sequence of letters in the alphabet $S \cup \Sigma$. Given $s \in S$, $\text{prop}(s, r)$ is in fact the asymptotic proportion of time with which a state $s$ appears in the run $r$. As a consequence, the support $\text{Supp}(r)$ of a run can contain states in $S$, and finite sequences of the type $t_1b_1t_2b_2 \ldots b_Lt_L$, where the $t_j$ are in $S$ and the $b_j$ in $\Sigma$.

**Remark 1.** Taking a limit $\inf$ instead of a limit $\sup$ in the definition of the support, we could express the fact that the proportion with which a given factor appears stays bounded away from zero as the length increases. It is not difficult to see that the same algorithms can be used for both notions, for the natural problems we will be interested in.

The notion of support of a run can be related to classical notions of acceptance for infinite words on automata. The classical Büchi condition says that an infinite execution on a system is valid if it visits infinitely often a particular (called accepting) configuration. For instance, we can say that an infinite execution on an automatic elevator system is valid if it visits infinitely often the configuration where the elevator waits for people on the ground floor. However, in that case, an execution where the delay it takes for the elevator to come back to the ground floor goes to infinity is still defined as valid. This may not be coherent with our intuition, as we may want that the time people are waiting for the elevator is bounded. Asking for a uniform bound is too stringent in practice, as it may happen that an elevator get used for a long time on the upper stairs.

Our notion of support is a notion of "asymptotic boundedness" of the return time to the ground floor.
2.3. Undecidable Problems concerning Probabilistic Tables

Finite Probabilistic Tables are closely related to the Finite Probabilistic Automata, introduced by Rabin in [18]. Relatively few algorithmic results have been proved on Finite Probabilistic Automata, and most of them are undecidability results. The seminal result is the undecidability of the Emptiness Problem, which can be defined in our context as follows.

**Problem 1 (Emptiness Problem for Finite Probabilistic Automata).**
*Input:* A Finite Probabilistic Table $T = (S, \Sigma, \delta, \alpha)$, $\lambda \in ]0[, \text{ and } X \subseteq S$.
*Question:* Is there $w \in \Sigma^*$ such that $\delta(\alpha, w)(X) > \lambda$?

**Theorem 1 ([17]).** The Emptiness Problem for Finite Probabilistic Automata is undecidable.

The following problem, considered in [1], is equivalent to the Emptiness Problem for Probabilistic Büchi Automata, a class of probabilistic automata on infinite words.

**Problem 2 (The Emptiness Problem for Probabilistic Büchi Automata).**
*Input:* A Finite Probabilistic Table $T = (S, \Sigma, \delta, \alpha)$, and $X \subseteq S$.
*Question:* Is there $w \in \Sigma^\omega$ such that $\mathbb{P}_w[\{r \in \Omega \mid X \cap \text{Inf}(r) \neq \emptyset\}] > 0$?

We will use in our reductions the following theorem of [1]. The theorem is proved by a reduction to Problem 1 of the Emptiness for Probabilistic Automata.

**Theorem 2 ([1]).** The Emptiness Problem for Probabilistic Büchi Automata is undecidable.

Finally, we will also use the undecidability of problems related to Isolated cut-points. Given $\lambda \in [0; 1]$, the $\lambda$-isolated cut-point problem is the following.

**Problem 3 ($\lambda$-isolated Cut-point Problem).**
*Input:* A Finite Probabilistic Table $T = (S, \Sigma, \delta, \alpha)$, and $X \subseteq S$.
*Question:* Is there $\epsilon > 0$ such that

$$\forall w \in \Sigma^* \mid \delta(\alpha, w)(X) - \lambda \mid \geq \epsilon.$$

A real number $\lambda$ in $[0; 1]$ which satisfies the condition of the problem is called an isolated cut-point. The definition of isolated cut-points has been motivated in [18] by the fact that for any isolated $\lambda$, the language

$$\mathcal{L} = \{w \in \Sigma \text{ s.t. } \delta(\alpha, w)(X) \geq \lambda\}$$

is rational. The following theorem has been proven by Bertoni in [3] for the case $\lambda \in ]0[, \text{ and for the extremal cases } \lambda = 0 \text{ and } \lambda = 1$ in [10].

**Theorem 3.** For any $\lambda \in [0; 1]$, the $\lambda$-isolated cut-point problem is undecidable.

3. Problems on finite non-homogeneous Markov chains

In this section, we present the problems we will consider, in the general framework of finite non-homogeneous Markov chains. In the past, researchers working in this domain seem to have been mostly interested in considerations on the ergodic properties of such chains [16,20]. In general they did not take into account the fact that the number of transition functions of the process may be finite, which is crucial when considering algorithmic aspects. We start with some remarks on homogeneous Markov chains, and next we study several problems of interest concerning non-homogeneous Markov chains.

3.1. Recurrence and transience on homogeneous Markov chains

We fix $X_i$, $i \geq 0$ a homogeneous Markov chain on a finite state space $S$. If $\alpha \in \Delta(S)$, $\mathbb{P}^\alpha$ is the probability distribution on the set of runs on the chain with initial distribution $\alpha$. Recall, [11], that a state $s \in S$ is called recurrent if $\mathbb{P}^\alpha(\{r \mid s \in \text{Inf}(r)\}) > 0$. Otherwise it is called transient. Note that we sometimes identify $s$ with the Dirac distribution $\mu_\delta \in \Delta(S)$ with $\mu_\delta(s) = 1$.

**Theorem 4 (Recurrence and the Ergodic Theorem, [11]).** Given a homogeneous Markov chain with finite state space $S$ and $s \in S$, $s$ is recurrent iff $\mathbb{P}^\alpha(\{r \mid s \in \text{Inf}(r)\}) = 1$, iff $\mathbb{P}^\alpha(\{r \mid s \in \text{Supp}(r)\}) > 0$, iff $\mathbb{P}^\alpha(\{r \mid s \in \text{Supp}(r)\}) = 1$.

Thus, in the homogeneous case, a state $s$ is recurrent if almost all the runs on the chain visit infinitely often $s$, or equivalently if almost all the runs spend a non-negligible amount of time on $s$. We will see in the next subsection that this equivalence does not hold in the context of non-homogeneous Markov chains.
3.2. Recurrence and transience for non-homogeneous Markov chains

Given a homogeneous Markov chain on $S$ and $s \in S$, Theorem 4 shows that $\mathbb{P}^x(\{r|s \in \text{Inf}(r)\}) = 0$ iff $\mathbb{P}^x(\{r|s \in \text{Supp}(r)\}) > 0$. This is not the case in the context of non-homogeneous Markov chains, since there exist sequences in which a given letter appears infinitely often, but is not in the support. This motivates the following two notions of recurrence. Given a non-homogeneous Markov chain with initial distribution $\alpha$, we write $\mathbb{P}^\alpha$ for the probability distribution on the set of runs of the chain.

**Definition 4** (Strong Recurrence, Weak Recurrence). Let $X_n, n \in \mathbb{N}$ be a non-homogeneous Markov chain on a finite state space $S$, and $s \in S$. We say that $s$ is weakly recurrent if

$$\mathbb{P}^\alpha(\{r|s \in \text{Inf}(r)\}) = 0.$$  

We say that $s$ is strongly recurrent if

$$\mathbb{P}^\alpha(\{r|s \in \text{Supp}(r)\}) > 0.$$  

Otherwise, $s$ is said to be weakly transient (resp. strongly transient).

3.3. Problems on non-homogeneous Markov chains

Given an FPT $\mathcal{T} = (S, \Sigma, \delta, \alpha)$, we consider several natural problems related to transience and recurrence and study their algorithmic complexity. In the following we list our problems and give their complexity, and postpone the proofs to the next sections.

3.3.1. The weak and strong recurrence problems

The first question is whether we can find $w \in \Sigma^\omega$ such that a given state $s \in S$ is weakly, or strongly, recurrent for the associated non-homogeneous Markov chain on $\mathcal{T}$.

**Problem 4** (Weak Recurrence (resp. Strong Recurrence)).

**Input:** An FPT $\mathcal{T} = (S, \Sigma, \delta, \alpha), F \subseteq S$.

**Question:** Is there $w \in \Sigma^\omega$ such that

$$\mathbb{P}^\alpha[w|F \cap \text{Inf}(r) \neq \emptyset] > 0$$ (resp. $\mathbb{P}^\alpha[w|F \cap \text{Supp}(r) \neq \emptyset] > 0$).

The weak recurrence problem is undecidable, and this follows from undecidability results on probabilistic automata by Theorem 2 of [1]. In contrast, we will see in Theorem 6 that the strong recurrence problem is PSPACE-complete. We will study several problems related to recurrence and transience on non-homogeneous Markov chains, comparing the complexities when we consider the classical “infinitely often” condition, and when we consider the new “support” condition. We underline the fact that Problem 4 is the only one where the complexities of the problems are different for the “infinitely often” and the “support” condition.

A natural extension to the Büchi acceptance condition which asks a run to visit infinitely often a final state, is a condition which specifies precisely the set of states visited infinitely often by a run, as is done in the case of Müller condition for automata. We prove that we cannot generalize our approach to several states, as Theorem 7 proves that the following problem is undecidable:

**Problem 5** (Two States Strong Recurrence).

**Input:** An FPT $\mathcal{T} = (S, \Sigma, \delta, \alpha), s, t \in S$.

**Question:** Is there $w \in \Sigma^\omega$ s.t. $\mathbb{P}^\alpha[w|s \in \text{Supp}(r) \text{ and } t \in \text{Supp}(r)] > 0$?

3.3.2. The weak and strong transience problems

The condition

$$\mathbb{P}^\alpha[w|F \cap \text{Inf}(r) \neq \emptyset] > 0,$$

as well as the condition $\mathbb{P}^\alpha[w|F \cap \text{Supp}(r) \neq \emptyset] > 0$, can be seen as a Büchi condition. One can be interested in the co-Büchi condition: a run is accepted if no state in $F$ is visited infinitely often. This gives the weak and strong transience problems.

**Problem 6** (Weak Transience (resp. Strong Transience)).

**Input:** An FPT $\mathcal{T} = (S, \Sigma, \delta, \alpha), F \subseteq S$.

**Question:** Is there $w \in \Sigma^\omega$ such that

$$\mathbb{P}^\alpha[w|F \cap \text{Inf}(r) = \emptyset] > 0$$ (resp. $\mathbb{P}^\alpha[w|F \cap \text{Supp}(r) = \emptyset] > 0$).

In Theorem 8 we prove that the weak transience and strong transience problems are both PSPACE-complete. The complexities of these problems were already partially known: in [1] the authors prove that the weak transience problem is in EXPTIME, and the PSPACE-completeness of the weak transience problem has been independently proved in [4]. In Section 5.3, we shows that the “infinitely often” and the “support” conditions can be seen as equivalent in this context: Given $\mathcal{T} = (S, \Sigma, \delta, \alpha)$ an FPT and $F \subseteq S$, $\exists w \in \Sigma^\omega$ s.t. $\mathbb{P}^\alpha[w|F \cap \text{Inf}(r) = \emptyset] > 0$ if and only if $\exists w \in \Sigma^\omega$ s.t. $\mathbb{P}^\alpha[w|F \cap \text{Supp}(r) = \emptyset] > 0$. 

1158

3.3.3. The universal weak and strong recurrence problems

Consider now the universal analog of the weak recurrence problem (resp. of the strong recurrence problem): do we have that for all \( w \in \Sigma^\omega \), \( P_w^\omega(\{r | s \in \text{Inf}(r)\}) > 0 \) (resp. \( P_w^\omega(\{r | s \in \text{Supp}(r)\}) > 0 \)). By contraposition, these problems can be reformulated as follows.

**Problem 7** (Universal Weak Recurrence (resp. Universal Strong Recurrence)).

**Input:** An FPT \( \mathcal{T} = (S, \Sigma, \delta, \alpha), F \subseteq S \).

**Question:** Is there \( w \in \Sigma^\omega \) such that

\[
P_w^\omega(\{r | s \in \text{Inf}(r)\}) = 1 \quad (\text{resp. } P_w^\omega(\{r | s \in \text{Supp}(r)\}) = 1.)
\]

By the results of [1], the universal weak recurrence problem is undecidable. In Theorem 10, we will prove that the universal strong recurrence problem is also undecidable.

3.3.4. The universal weak and strong transience problems

We consider the universal versions of the weak and strong transience problems.

**Problem 8** (Universal Weak Transience (resp. Universal Strong Transience)).

**Input:** An FPT \( \mathcal{T} = (S, \Sigma, \delta, \alpha), F \subseteq S \).

**Question:** Is there \( w \in \Sigma^\omega \) such that

\[
P_w^\omega(\{r | s \in \text{Inf}(r)\}) = 1 \quad (\text{resp. } P_w^\omega(\{r | s \in \text{Supp}(r)\}) = 1).\]

We prove in Theorem 11 that the universal weak and strong transience problems are PSPACE-complete. As for the weak and strong transience problems, we prove in Section 5.5 that the “infinitely often” and the “support” condition are equivalent in this context: Given \( \mathcal{T} = (S, \Sigma, \delta, \alpha) \) an FPT and \( F \subseteq S \), \( \exists w \in \Sigma^\omega \) s.t. \( P_w^\omega(\{r | s \in \text{Inf}(r)\}) = 1 \) if and only if \( \exists w \in \Sigma^\omega \) s.t. \( P_w^\omega(\{r | s \in \text{Supp}(r)\}) = 1 \).

4. Loops and Filters

In this section, we present the notions of **Loops** and **Filters**, that we will use in the proofs of the next section. The notion of probabilistic loop corresponds to the set of homogeneous Markov chains that one can define on an FPT. A filter on a finite alphabet is a finite sequence of letters in which we allow empty holes, i.e. undefined letters. Thus, a filter can be seen as a mask, and the proportion of a filter on a sequence measures the number of places at which the mask can be put on the sequence such that the defined letters of the mask coincide with the associated letters of the sequence.

For the following of the section, we fix an FPT \( \mathcal{T} = (S, \Sigma, \{M^a, a \in \Sigma\}, \alpha) \).

4.1. Probabilistic loops

Our decision procedures will often rely on the notion of probabilistic loop.

**Definition 5** (Probabilistic Loop). A probabilistic loop in \( \mathcal{T} \) is a couple \((C, \rho)\), where \( C \subseteq S \) and \( \rho \in \Sigma^* \) are such that \( \delta(C, \rho) \subseteq C \).

If \( F \subseteq S \), a probabilistic loop around \( F \) in \( \mathcal{T} \) is a probabilistic loop \((C, \rho)\) in \( \mathcal{T} \) such that for all \( s \in C \), there exists \( \rho_s \) a prefix of \( \rho \) such that \( \delta(s, \rho_s) \cap F \neq \emptyset \).

A probabilistic loop \((C, \rho)\) in \( \mathcal{T} \) induces a homogeneous Markov chain \( X_n, n \in \mathbb{N} \) with state space \( C \) and transition probabilities given, for all \( s, t \in C \), by \( P[X_{n+1} = t | X_n = s] = \delta(s, \rho)(t) \). Let \( A \) be the set of states in \( C \) which are recurrent for this chain. The **Support** of the loop \((C, \rho)\) is the set of states \( t \in S \) such that there exists \( s \in A \) and \( \rho' \) a prefix of \( \rho \) with \( \delta(s, \rho')(t) > 0 \).

**Example 1.**

- \( (s_2, s_4), bb \) is a probabilistic loop around \( s_2 \) and \( s_4 \).
- There exists no probabilistic loop around \( s_1 \).
4.2. Filters

**Definition 6 (Filters).** Let $S$ be a finite state space, and $\Sigma$ be a finite alphabet. A filter on $S$ and $\Sigma$ is a finite sequence of couples on $S \cup \{\}$ and $\Sigma \cup \{\cdot\}$, where the symbol $\cdot$ is a special symbol denoting an “indefinite place”.

A filter can be seen as a word on the alphabet $S \cup \Sigma \cup \{\cdot\}$. We define the binary relation $\equiv$ on $S \cup \Sigma \cup \{\cdot\}$: given $u, v \in S \cup \Sigma \cup \{\cdot\}$, let $u \equiv v$ if $u = v$ or $u = \cdot$ or $v = \cdot$. Two filters $x = u_0u_1\ldots u_k$ and $y = v_0v_1\ldots v_l$ will be said to coincide, written $x \equiv y$, if $k = l$ and for all $i \in [0; k]$ we have $u_i \equiv v_i$.

We introduce the notion of filter to be able to consider the asymptotic proportions of sequences of letters which appear in infinite runs. Typically, let $r = s_0, a_1, s_1, a_2, \ldots$ be an infinite run on $S$ and $\Sigma$, and let $\rho = b_0b_1\ldots b_l \in \Sigma^*$. We want to consider the asymptotic proportion of $\rho$ in $r$, without considering the states which appear in $r$. For this we consider $\rho$ as the filter $\tilde{\rho} = b_0 \cdot b_1 \cdot \ldots \cdot b_l$, and we compute the asymptotic proportion of the filter $\tilde{\rho}$ in $r$ as follows.

**Definition 7 (Proportion of a Filter in a Run).** Let $r = s_0, a_1, s_1, a_2, \ldots$ be an infinite run on $S$ and $\Sigma$. Let $\rho = b_0b_1\ldots b_l \in (S \cup \Sigma \cup \{\cdot\})^*$ be a filter. We call the proportion of $\rho$ in $r$ the sup limit of the proportion of time $\rho$ appears in $s_0, a_1, s_1, \ldots, a_n, s_n$ as a factor, allowing differences only on places with a $\cdot$:

$$\text{prop}(\rho, r) = \limsup_{n \to \infty} \frac{|\{i \in [1; n - 1] : a_i \equiv b_0 \land \ldots \land a_{i+1} \equiv b_l\}|}{n}.$$

In the following, when $\rho \in \Sigma^*$ and $r$ is a run, we will write $\text{prop}(\rho, r)$ indifferently for $\text{prop}(\tilde{\rho}, r)$, where $\tilde{\rho}$ is the filter naturally associated with $\rho$. If $u$ and $v$ are two filters on $S$ and $\Sigma$, then $uv$ is the natural concatenated filter. For instance, if $w = a_0\cdot a_1 \in \Sigma^*$, $s \in S$ and $r$ is a run, then we write $\text{prop}(s, w, s)$ for the asymptotic proportion of the filter $s, a_1, \cdot, a_2, \ldots, a_n, s$ in $r$.

The following is a purely combinatorial lemma.

**Lemma 1.** Let $S$ be a finite state space and $\Sigma$ be a finite alphabet. Let $r$ be a run on $S$ and $\Sigma$, and let $u$ be a filter on $S$ and $\Sigma$. Suppose $\text{prop}(u, r) > 1/N$, where $N \in \mathbb{N}$ and $N > |u|$. Then there exists $v \in \Sigma^*$ such that $\text{prop}(uvu, r) > 0$. Moreover, we can choose $u$ such that $|v| \leq 2 \cdot N \cdot |u|$.

**Proof.** We define the following sequences.

- $(x_i)_{i \in \mathbb{N}} \subset \{0, 1\}^N$ is defined as
  $$x_i = 1 \text{ iff } (r(i), r(i + 1), \ldots, r(i + |u| - 1)) = (u(0), u(1), \ldots, u(|u| - 1)).$$
- Let $k \in \mathbb{N}$ be greater than $|u|$. Then $(y_k^i)_{i \in \mathbb{N}} \subset \{0, 1\}^N$ is defined as
  $$y_k^i = 1 \text{ iff } x_i = 1 \text{ and } x_{i+k} = 1.$$

Suppose that we can find $k \in [|u|; 2 \cdot N]$ such that $\limsup_{T \to \infty} \frac{\sum_{i=0}^{T-|u|} y_k^i}{T} > 0$. Then, for a positive asymptotic proportion of indices $i$ on the run $r$ as the length goes to infinity, we have

$$(r(i), r(i + 1), \ldots, r(i + |u| - 1)) = (u(0), u(1), \ldots, u(|u| - 1)),$$

and

$$(r(i + k), r(i + k + 1), \ldots, r(i + k + |u| - 1)) = (u(0), u(1), \ldots, u(|u| - 1)).$$

In other words, the filter $u, \cdot, \ldots, \cdot, u$, with $k - |u|$ symbols $\cdot$, appears with positive asymptotic proportion in the run $r$. There exists only a finite number of words in $\Sigma^{k-|u|}$ which can fit in the sequence $\cdot, \ldots, \cdot$. As a consequence, there exists $v \in \Sigma^*$, with length $k - |u|$, hence with length at most $2 \cdot N$, such that the filter $uvu$ belongs to the support of $r$. This gives the result. All we have to do is to prove the existence of such a $k$.

By hypothesis, $\text{prop}(u, r) > 1/N$, i.e. $\limsup_{T \to \infty} \frac{\sum_{i=0}^{T-|u|} y_k^i}{T} > 1/N$. We prove that this implies $\limsup_{T \to \infty} \frac{\sum_{i=0}^{T-|u|} y_k^i}{T} > 0$ for some $k$. For this, we use a pigeon hole method. Given $m \in \mathbb{N}$, let

$$B_m = \{i \in \{2 \cdot m \cdot N - |u|; 2 \cdot (m + 1) \cdot N - |u|\} \text{ s.t. } x_i = 1\}.$$

By hypothesis, we have $\limsup_{T \to \infty} \frac{\sum_{i=0}^{T-|u|} |B_m|}{2N \cdot T} > \frac{1}{N}$, i.e.

$$\limsup_{T \to \infty} \frac{\sum_{i=0}^{T-|u|} |B_m|}{T} > 2 \cdot |u|.$$

Given $i \in \mathbb{N}$, let $z_i = 1$ if $|B_i| \geq 2 \cdot |u|$, and $z_i = 0$ elsewhere. Then, since the $|B_i|, i \in \mathbb{N}$ are all bounded by $2 \cdot |u| \cdot N$, we have that

$$\limsup_{T \to \infty} \frac{\sum_{i=0}^{T-|u|} z_i}{T} > 0.$$

(1)
If $|B_i| \geq 2 \cdot |u|$, by a pigeon hole principle, there exists $i_1, i_2 \in B_i$ such that $i_2 - i_1 \geq |u|$. As a consequence, if $i$ is such that $z_i = 1$, then there exists $j \in B_i$ and $k \in [|u|, 2 \cdot N \cdot |u|]$ such that $y^k_j = 1$. Along with Eq. (1), this implies

$$
\lim_{T \to \infty} \sup_{i=|u|} \frac{1}{T} \sum_{i=|u|}^{2 \cdot N \cdot |u|} \sum_{j=0}^{T} y^k_j > 0.
$$

This implies that there exists $k \in [|u|, 2 \cdot N \cdot |u|]$ such that

$$
\lim_{T \to \infty} \sup_{i=|u|} \frac{1}{T} \sum_{i=|u|}^{T} y^k_j > 0
$$

hence the result. □

5. Computational complexity of the problems on NHMCs

In this section, we determine the computational complexity of the set of problems defined in Section 3.

5.1. The weak and strong recurrence problems, Problem 4

We start by a theorem proved in [1].

**Theorem 5 ([1]).** The weak recurrence problem is undecidable.

**Proof.** This follows directly from the undecidability results of the emptiness problem on Probabilistic Büchi Automata given in Theorem 2 of [1]. The result is proven by a reduction to the emptiness problem on finite Probabilistic Automata. □

Next, we consider the strong recurrence problem. Let $\mathcal{T} = (S, \Sigma, \delta, \alpha), F \subseteq S$ be an instance of the strong recurrence problem. We can assume that $F = \{s\}$, with no loss of generality. We will prove in this subsection that $s$ is strongly recurrent for a non-homogeneous Markov chain on the probabilistic table iff $s$ is accessible and there exists a probabilistic loop around $s$ in $\mathcal{T}$ (Proposition 1). This will imply that the strong recurrent problem is PTIME-equivalent to the following probabilistic loop problem:

**Problem 9 (The Probabilistic Loop Problem).**

**Input:** An FPT $\mathcal{T} = (S, \Sigma, \delta, \alpha), s \in S$.

**Question:** Is there a probabilistic loop around $s$ in $\mathcal{T}$?

We will prove that the probabilistic loop problem is PSPACE-complete: the PSPACE-hardness is proved by a reduction of the Finite Intersection of Regular Language problem, which is known to be PSPACE-complete, to our problem; we prove also that the probabilistic loop problem is in NPSPACE, hence in PSPACE by a theorem of Savitch [19]. These results imply that the strong recurrence problem is PSPACE-complete (Theorem 6).

The next example shows that the equivalence between the fact that a state $s$ may be recurrent and the existence of probabilistic loop around $s$ does not hold if $\Sigma$ is infinite.

**Example 2.** Let $S = \{s, t\}$. For $\delta \in \{0; 1\}$ consider the Markov matrix $M_\delta = \begin{pmatrix} 1 - \delta & \delta \\ 0 & 1 \end{pmatrix}$. The graph of the associated Markov chain is

```
 1-\delta
 S ---\delta--- 1
    \_ \_ \_

Suppose that the chain is initiated on state $s$: $\alpha = \{s\}$. Consider now the family of matrices $M = \{M_{i/2^j}, i \in \mathbb{N}\}$. For any finite product of matrices in $M$, the associated homogeneous Markov chain $X_n, n \geq 0$ on $S$ is aperiodic and $t$ is the only state in the support of the stationary distribution. Indeed, the probability to go from state $s$ to state $t$ will always be non-zero, and the probability to go from state $t$ to state $s$ will always be zero. By Theorem 4, this implies that $s$ is transient for the (homogeneous) chain. This implies that there exists no probabilistic loop around $s$ in $\mathcal{T}$. However, if we consider the non-homogeneous Markov chain $X_n, n \geq 0$ on $S$ whose transition probabilities are given by the matrices $M_{1/2}, M_{1/2^2}, M_{1/2^3}, \ldots$, then $P^\text{out}_{1/2, 1/2^2} \left([r\mid \forall n \in \mathbb{N} X_n(r) = s]\right) > 0$, and in particular $P^\text{out}_{1/2, 1/2^2} \left([r\mid s \in \text{Supp}(r) > 0]\right) > 0$, which proves that $s$ is strongly recurrent for the (non-homogeneous) chain.

The following two lemma will be applied recursively to build probabilistic loops in the proof of Proposition 1. As before, given $\rho = b_0, b_1, \ldots, b_i \in \Sigma^*$, let $\tilde{\rho} = b_0, \cdot, b_1, \cdot, \ldots, b_i$ be the filter on $S$ and $\Sigma$ associated with $\rho$. More generally, if $\rho \in (S \cup \Sigma)^*$, $\tilde{\rho}$ is the filter naturally associated with $\rho$: this is done by adding some ‘’ symbols in $\rho$ in order to get an alternating sequence of states or ‘’ symbols, and letters in $\Sigma$ or ‘’ symbols. We will write $\text{prop}(\rho, r)$ for $\text{prop}(\tilde{\rho}, r)$.

**Lemma 2.** Let $\rho \in \Sigma^*$. Suppose $P^\text{out}_w([r\mid \text{prop}(\rho, r), r > 0]) > 0$, and let $t \in \delta(s, \rho)$. Then, there exists $\rho' \in \Sigma^*$ such that

$s \in \delta(t, \rho'), \text{ and } P^\text{out}_w([r \mid \text{prop}(s, \rho'), r > 0]) > 0$. 

Proof. Since $P^\omega_u((r | \text{prop}(s, r)) > 0)) > 0$, there exists $N \in \mathbb{N}$ such that $P^\omega_u((r | \text{prop}(s, r) > 1/N)) > 0$. If not, $P^\omega_u((r | \text{prop}(s, r)) \geq 0)$ would be a countable union of sets of probability zero, and hence would have probability zero. Let $\Delta = [r | \text{prop}(s, r)) > 1/N$, and let $\gamma = \delta(s, r)(t)$. By hypothesis, $\gamma > 0$. Using a strong law of large numbers, we have that

$$P_u^\omega((r \in \Delta | \text{prop}(s, r, t), r) > \gamma/N)) \geq P_u^\omega(\Delta).$$

By Lemma 1, we know that for all $r$ such that $\text{prop}(s, r, t), r > \gamma/N$, there exists $\rho_i \in \Sigma^*$ of length not greater than $2 \cdot N/\gamma \cdot (2 + |\rho|)$ such that $\text{prop}(s, r, t, \rho_i, s, r, t, r) > 0$. Thus,

$$P_u^\omega((r \in \Delta | \text{prop}(s, r, t, \rho_i, s, r, t), r) > 0)) \geq P_u^\omega(\Delta).$$

$\Sigma$ is finite, hence there is a finite number of words of length lower than $2 \cdot N \cdot (2 + |\rho|) / \gamma$. Since $P_u^\omega(\Delta) > 0$, we can find a $\rho' \in \Sigma^*$ of length lower than $2 \cdot N \cdot (2 + |\rho|) / \gamma$ such that

$$P_u^\omega([r \text{prop}(s, r, t, \rho'), s, r, t), r) > 0).$$

This proves the result, since $P_u^\omega([r \text{prop}(s, r, t, \rho'), s, r, t), r) > 0) > 0$ implies that $s \in \delta(t, \rho')$. □

Lemma 3. Let $\rho \in \Sigma^*$. Suppose

$$P_u^\omega([r \text{prop}(s, r, t), r) > 0)) > 0.$$ 

Let $\{t_1, \ldots, t_l\} = \delta(s, r)$. Then there exists $\rho' \in \Sigma^*$ such that

- For all $t \in \{t_1, \ldots, t_l\}$, there exists a prefix $\rho'_i$ of $\rho'$ such that $s \in \delta(t, \rho'_i)$.
- $P_u^\omega([r \text{prop}(s, r, \rho'), r) > 0)) > 0$.

Proof. We build $\rho'$ iteratively for all $t \in \{t_1, \ldots, t_l\}$ using Lemma 2.

- For $t_1$: let $\rho_1$ be such that
  - $s \in \delta(t_1, \rho_1)$.
  - $P_u^\omega([r \text{prop}(s, \rho_1), r) > 0)) > 0)$.

- Suppose that we have constructed $\rho_1, \ldots, \rho_i \in \Sigma^*$, for $i \in [1; l - 1]$, such that
  - For all $j \in [1; i]$, $s \in \delta(t_j, \rho_1 \rho_2 \ldots \rho_i)$.
  - $P_u^\omega([r \text{prop}(s, \rho_1 \ldots \rho_i), r) > 0)) > 0$.

Let $t_{i+1} \in \delta(s, \rho_1 \ldots \rho_i)$. Using Lemma 2, let $\rho_{i+1}$ be such that

- $s \in \delta(t_{i+1}, \rho_{i+1})$.
- $P_u^\omega([r \text{prop}(s, \rho_1 \ldots \rho_i \rho_{i+1}), r) > 0)) > 0$.

- The construction ends when all the $\rho_i, i \in [1; l]$ have been constructed. By construction $\rho' = \rho_1 \ldots \rho_l$ satisfies the requirements of the lemma. □

The following proposition is at the core of our approach. Since we can decide in PTIME if a state $s$ of a given FPT is accessible from the initial distribution, the proposition implies that the strong recurrence problem is PTIME equivalent to the probabilistic loop problem.

Proposition 1. Let $\mathcal{T} = (\Sigma, \delta, \alpha, F = \{s\}$, with $s \in S$, be an instance of the strong recurrence problem. Then the following are equivalent.

- There exists $w \in \Sigma^\omega$ such that $s$ is strongly recurrent for the associated non-homogeneous Markov chain on $\mathcal{T}$.
- $s$ is accessible, and there exists a probabilistic loop around $s$ in $\mathcal{T}$.

Moreover, in the positive case, the letters of the trace of the loop can all be taken in the support of $w$.

Proof. Notice that one implication is easy: if there exists $\rho_0 \in \Sigma^n$ such that $\delta(\alpha, \rho_0)(s) > 0$ and if there exists a probabilistic loop $(C, \rho)$ around $s$, then $P^\omega_{\rho_0 \rho'}([r \in \text{Supp}(r)]) > 0$ and $s$ is strongly recurrent for the chain associated with $w = \rho_0 \cdot \rho^\omega$.

For the other implication, we proceed as follows. We build a sequence $\rho_1, \rho_2, \ldots$ of non-empty finite words of $\Sigma$ and a sequence $E_0, E_1, E_2, \ldots$ of subsets of $S$ as follows.

- Let $E_0 = \{s\}$.
- $\rho_1$ is such that $P_u^\omega([r \text{prop}(s, r_1), r) > 0)) > 0$. (We know that we can take for instance one of the one-letter words of $\Sigma$ for $\rho_1$.) Let $E_1 = \delta(s, \rho_1)$.

Suppose that we have built $\rho_1, \ldots, \rho_{i-1}$ and $E_1, \ldots, E_i$, $i \in \mathbb{N}$, such that for all $j \in [1; i]$:

- For all $t \in E_{j-1}$, there exists a prefix $\rho'_j$ of $\rho_j$ such that $s \in \delta(t, \rho'_j)$.
- $P_u^\omega([r \text{prop}(s, \rho_1 \ldots \rho_j), r) > 0)) > 0$.

(remark that this condition is satisfied for $j = 1$, taking the empty word for the prefix of $\rho_1$).

Then $\rho_{i+1}$ and $E_{i+1}$ are constructed as follows.
Consider the following regular automata

\[ A_1 \quad \text{and} \quad A_2. \]

Fig. 1. Automata \( A_1 \) and \( A_2. \)

- Using Lemma 3, let \( \rho_{t+1} \) be such that
  - For all \( t \in \delta(s, \rho_1, \ldots, \rho_t) \), there exists a prefix \( \rho' \) of \( \rho_{t+1} \) such that \( s \in \delta(t, \rho') \).
  - \( \mathbb{P}_w^\rho((r|\text{prop}(s, \rho_1, \ldots, \rho_{t+1}), r) > 0)) > 0. \]

- \( E_{t+1} = \delta(s, \rho_1, \ldots, \rho_{t+1}). \)

  - The construction ends when there exists \( l, m \in [1; l] \) such that \( E_l = E_m. \) It happens in at most \( 2^{|l|} \) steps. Then \( E_l, \rho_{t+1}\rho_{t+2} \ldots \rho_{m-1} \) is a probabilistic loop around \( s. \) The fact that

\[ \mathbb{P}_w^\rho(( r|\text{prop}(s, \rho_1, \ldots, \rho_{m-1}), r) > 0)) > 0 \]

implies that all the letters of the word \( \rho_{t+1}\rho_{t+2} \ldots \rho_{m-1} \) are in the support of \( w. \) □

We prove now that the probabilistic loop problem, Problem 9, is PSPACE-complete. First, we reduce the problem of Finite Intersection of Regular Languages, which is known to be PSPACE-complete [12], to our problem. The size of the input of the Finite Intersection of Regular Languages Problem is the sum of the number of states of the automata.

**Problem 10 (Finite Intersection of Regular Languages).** Input: \( A_1, \ldots, A_l \) a family of regular deterministic automata (on finite words) on the same finite alphabet \( \Sigma. \)

Question: Do we have \( L(A_1) \cap \cdots \cap L(A_l) = \emptyset? \)

**Proposition 2.** The probabilistic loop problem is PSPACE-hard.

**Proof.** Let \( A_1, \ldots, A_l \) be a family of regular automata on the same finite alphabet \( \Sigma, \) with respective state space \( S \) and transition functions \( \delta_i: \delta_i(s, a)(t) = 1 \) if there exists a transition from \( s \) to \( t \) with label \( a \in \Sigma \) in \( A_i. \) We build an FPT \( T = (S, \Sigma, \delta, \alpha) \) and identify a state \( s \in S \) such that there exists a probabilistic loop around \( s \) in \( T \) iff \( L(A_1) \cap \cdots \cap L(A_l) \neq \emptyset. \)

Let \( x \) be a new letter, not in \( \Sigma, \) and let \( \Sigma' = \Sigma \cup \{x\}. \)

- \( s \) is the union of the state spaces of the \( A_i, \) plus two extra states \( s, \perp. \) That is \( S = \bigcup_{i=1}^l S_i' \cup \{s, \perp\}, \) where the \( S_i' \) are disjoint copies of the \( S_i. \)

- The state \( \perp \) is a sink: for all \( a \in \Sigma', \delta(\perp, a)(\perp) = 1. \)

- If \( u' \) is the copy of a non-accepting state \( u \) of \( A_i, \) we allow in \( T \) the same transitions from \( u' \) as in \( A_i \) for \( u: \) if \( a \in \Sigma, \delta(u', a)(v') = 1 \) iff \( v' \) is the copy of a state \( v \) in \( S \) such that \( \delta_i(u, a)(v) = 1. \) Moreover, we add a transition from \( u \) with label \( x: \delta(u, x)(\perp) = 1. \)

- If \( u' \) is the copy of an accepting state \( u \) of \( A_i, i \in [1; l], \) the transitions from \( u' \) in \( T \) are the same as in \( A_i, \) plus an extra transition \( \delta(u', x)(s) = 1. \)

- From state \( s \) in \( T, \) with uniform probability on \( i \in [1; l], \) when reading \( x, \) the system goes to one of the copies of an initial state of the \( A_i.'s. \)

- For the transitions which have not been precised, for instance if \( a \in \Sigma \) is read in state \( s, \) the system goes with probability one to the sink \( \perp. \)

- The initial distribution is the Dirac distribution on \( s. \)

Given \( \rho \in L(A_1) \cap \cdots \cap L(A_l), (C = \{s\}, x \cdot \rho \cdot x) \) is clearly a probabilistic loop around \( s \) in \( T. \)

Conversely, suppose that there exists a probabilistic loop \( (C, \rho) \) around \( s \) in \( T. \) Let \( t \in C, \) and let \( \rho_0 \) be a prefix of \( \rho, \) such that \( s \in \delta(t, \rho_0). \) We distinguish two cases.

- Suppose first that \( \rho_0 = \rho. \) Then, since \( (C, \rho) \) is a probabilistic loop, \( s \in C. \) By the structure of the automaton, since the only transition from \( s \) which does not go to the sink has label \( x, \) this implies that \( C = \{s\}, \) and that \( \rho = x \cdot \rho' \cdot x \) where \( \rho' \in L(A_1) \cap \cdots \cap L(A_l). \)

- If \( \rho_0 \neq \rho, \) let \( \rho_1 \) be such that \( \rho = \rho_0 \cdot \rho_1. \) Since \( x \) is the only letter allowed on state \( s \) which does not lead to the sink, we must then have \( \delta(C, \rho_0) = \{s\}, \) \( \rho_0 \) must end with \( x, \) and \( \rho_1 \) must start with \( x. \) Let \( \rho_0 = \rho_2 \cdot x, \) and \( \rho_1 = x \cdot \rho_3, \) so that \( \rho = \rho_3 \cdot x \cdot \rho_4. \) Then, \( (\{s\}, x \cdot \rho_2 \cdot \rho_3 \cdot x) \) is also a probabilistic loop around \( s \) in \( T, \) and this implies that \( \rho_4 \cdot \rho_3 \in L(A_1) \cap \cdots \cap L(A_l). \) If it were not the case, then with positive probability, when reading \( \rho_4 \cdot \rho_3, \) we would go to the sink \( \perp, \) and thus we would not have a probabilistic loop.

In any case, there exists a probabilistic loop around \( s \) in \( T \) of type \( (C = \{s\}, \rho') \) with \( \rho' \in L(A_1) \cap \cdots \cap L(A_l). \)

Finally, we have proved that there exists a probabilistic loop around \( s \) in \( T \) iff \( L(A_1) \cap \cdots \cap L(A_l) \neq \emptyset. \) □

We give an example of the last reduction.

**Example 3.** Consider the following regular automata \( A_1 \) and \( A_2, \) and the associated FTP \( T \) (see Figs. 1 and 2).

For instance, \((\{s\}, xbaax)\) is a probabilistic loop around \( s. \)
The strong recurrence problem is PSPACE-complete.

Proposition 3. The probabilistic loop problem is in PSPACE.

Proof. We prove that our problem is in NPSPACE, which will give our result, using a theorem of Savitch [19]. The point is that the probabilistic loop condition can be seen as a graph theoretic notion, with no consideration for the exact values of the probabilities. Given a state \( t \in S \), we want a nondeterministic Turing machine which finds a probabilistic loop around \( t \) in \( T \) if it exists. That is, our machine is looking for a couple \((C, w)\) with \( A \subseteq S \) and \( w \in \Sigma^* \) such that

1. \( \delta(C, w) \subseteq A \)
2. for all \( s \in A \), there exists \( w' \) a prefix of \( w \) such that \( t \in \delta(s, w') \).

For this, we consider a nondeterministic Turing machine which guesses \( A \) and one letter of \( w = a_1 a_2 \ldots \in \Sigma^* \) at each step. Formally, a state of the machine is a tuple \( \{ (U_s, s \in S), \{ X_s, s \in A \} \} \) where the \( U_s \), \( s \in S \) associate a boolean \( U_s \) to each state, and where the \( X_s \) are sets of states. The update of the machine is as follows: for the initial state of the machine, all the \( U_s \) are false. At each step \( k \), the machine keeps in memory the value \( X_s = \delta(s, a_1 \ldots a_k) \) for all \( s \in A \), and it updates the value of \( U_s \): if \( t \in \delta(s, a_1 \ldots a_k) \) then \( U_s \) becomes true.

Finally, the machine accepts when it has found \( w \) which satisfies conditions 1 and 2. Since \( \delta(s, a_1 \ldots a_k) \) can be computed using only the values of \( \delta(s, a_1 \ldots a_k) \) for all \( s \in A \), we are indeed in NPSPACE. Remark that in general the length of \( w \) can be exponential in the size of the systems. \( \square \)

As a consequence of Proposition 1 and the previous discussion, we get our theorem.

Theorem 6. The strong recurrence problem is PSPACE-complete.

5.2. The two states strong recurrence problem

We reduce the emptiness problem of a PBA\( ^{\leq 0} \), Problem 2, which is known to be undecidable, by Theorem 2 of [1], to Problem 5:

Given an FPT \( T = (S, \Sigma, \delta, \alpha) \) and \( s, t \in S \), is there an \( w \in \Sigma^\omega \) such that \( P^w_\alpha[\{ r \in \text{Supp}(r) \text{ and } t \in \text{Supp}(r) \}] > 0 \)?

The fact that the two states strong recurrence problem is undecidable shows that we cannot generalize our decidability results on the strong recurrence to several states.

Let \( T = (S, \Sigma, \delta, \alpha) \) be an FPT, and \( F \subseteq S \). We build an associated “layered” FPT \( T' = (S', \Sigma, \delta', \alpha') \), with \( S' \) partitioned into two subsets \( L \) and \( H \), such that if \( P^w_\alpha \) and \( P^w_\alpha' \) are the respective probability distributions on \( T \) and \( T' \), for all \( w \in \Sigma^\omega \) we have

\[
P^w_\alpha[\{ r \in \text{Inf}(r) \cap F \neq \emptyset \}] > 0 \text{ if } P^w_\alpha'[\{ r \in \text{Supp}(r) \cap L \neq \emptyset \text{ and } \text{Supp}(r) \cap H \neq \emptyset \}] > 0.
\]

Thus, if we could decide, given \( s, t \in S' \), Problem 5 on instance \( T' \), \( s, t \), then we could decide the emptiness problem for the Probabilistic Büchi Automaton \( A = (T, F) \). The construction is as follows.

Construction:

- Let \( L = \{ u_1, \ldots, u_l \} \), and \( H = \{ v_1, \ldots, v_l \} \) be two sets of new states, copies of \( S \). Let \( S' = L \cup H \).
- Let \( a \in \Sigma \). Suppose \( i \in [1; l] \) is such that \( s_i \in S \setminus F \). Then the associated transitions of \( T' \) stay on the same level: for all \( j \in [1; l], \delta'(u_i, a)(u_j) = \delta(s_i, a)(s_j), \delta'(u_i, a)(v_j) = 0, \delta'(v_i, a)(v_j) = \delta(s_i, a)(s_j), \) and \( \delta'(v_i, a)(u_j) = 0 \).
- If \( i \in [1; l] \) is such that \( s_i \in F \), then we can move between the levels \( L \) and \( H \) with positive probability, if \( \delta(s_i, a)(s_j) > 0 \): \( \delta'(u_i, a)(u_j) = \delta'(u_i, a)(v_j) = \delta'(v_i, a)(v_j) = \delta'(v_i, a)(u_j) = \delta(s_i, a)(s_j)/2 \).
- If \( i \in [1; l], \alpha'(v_i) = 0, \alpha'(u_i) = \alpha(s_i) \).

Fig. 2. The FPT \( T \).
Given $w \in \Sigma^\omega$, we write $\mathbb{P}_w^\alpha$ (resp. $\mathbb{P}_w^\alpha'$) for the probability distribution on the set of runs induced by $\alpha$ and $w$ on $\mathcal{T}$ (resp. by $\alpha'$ and $w$ on $\mathcal{T}'$).

**Proposition 4.** For $w \in \Sigma^\omega$,
\[
\mathbb{P}_w^\alpha([r \mid \text{Inf}(r) \cap F \neq \emptyset]) > 0 \text{ iff } \mathbb{P}_w^\alpha'([r \mid \text{Supp}(r) \cap L \neq \emptyset \text{ and } \text{Supp}(r) \cap H \neq \emptyset]) > 0.
\]

**Proof.** First, a notation: given a run $r \in (2)^\omega$, $\phi(r)$ is the proportion of states which are in $H$ in the portion of the run $r$ between time $i$ and time $j$.

The implication $\Leftarrow$ is simple, we prove the implication $\Rightarrow$. Let
\[
\gamma = \mathbb{P}_w^\alpha([r \mid \text{Inf}(r) \cap F \neq \emptyset]),
\]
and suppose $\gamma > 0$. We write $\Omega$ for the set of runs on $\mathcal{T}$, and $\Omega'$ for the set of runs on $\mathcal{T}'$. Let $\epsilon > 0$, and $\delta > 0$ be such that $\prod_{i \geq 1}(1 - \delta/2^i) \geq 1 - \epsilon$. Let $F' \subseteq \Omega'$ be the set of copies of states in $F$, and let
\[
Z = \{r' \in \Omega' \mid \text{Inf}(r') \cap F' \neq \emptyset\}.
\]

By “set of copies of $F$”, we mean the set of states in $L$ and $H$ which are copies of states in $F$. Then, by the construction, we can see that
\[
\gamma = \mathbb{P}_w^\alpha([r \mid \text{Inf}(r) \cap F \neq \emptyset]) = \mathbb{P}_w^\alpha[Z].
\]

We can write
\[
Z = Z \cap \{r' \in \Omega' \mid \text{Supp}(r') \subseteq H\} \cup Z \cap \{r' \in \Omega' \mid \text{Supp}(r') \subseteq L\}
\]
\[
\cup \{r' \in \Omega' \mid \text{Supp}(r') \cap L \neq \emptyset \text{ and } \text{Supp}(r') \cap H \neq \emptyset\}.
\]

Let $A = Z \cap \{r' \in \Omega' \mid \text{Supp}(r') \subseteq H\}$ and $B = Z \cap \{r' \in \Omega' \mid \text{Supp}(r') \subseteq L\}$. We just have to prove that $A$ and $B$ have probability zero to get the result, since in that case we have
\[
\gamma = \mathbb{P}_w^\alpha[Z] = \mathbb{P}_w^\alpha'[\{r' \in \Omega' \mid \text{Supp}(r') \cap L \neq \emptyset \text{ and } \text{Supp}(r') \cap H \neq \emptyset\}].
\]

We prove that $\mathbb{P}_w^\alpha'[A] = 0$, by symmetry we could prove that $\mathbb{P}_w^\alpha'[B] = 0$ using the same method.

First, we build inductively the following function $\phi$ on $\mathbb{N}$:
- Let $\phi(1) = 1$.
- Let $n \in \mathbb{N}$, and suppose that $\phi(n)$ is already defined. Then $\phi(n + 1)$ is defined such that $\phi(n + 1) - \phi(n) \geq \phi(n)$, and
\[
\mathbb{P}_w^\alpha'[\{r' \in \Omega' \mid \exists i \in [\phi(n); \phi(n + 1)) \text{ s.t. } r'(i) \in F'\}] \geq \gamma \cdot (1 - \delta/2^n).
\]

That is, $\phi(n + 1)$ is large enough so that the probability of a run in $\Omega'$ to meet a copy of a final state between step $\phi(n)$ and step $\phi(n + 1)$ is not too small. This is always possible, since by definition any run which belongs to $Z$ visits infinitely often a state in $F'$. Given $n \in \mathbb{N}$, let
\[
D_n = \{r' \in \Omega' \mid \text{prop}(H, r'_{\phi(n)}) \geq 1/2\}.
\]

By the construction of the function $\phi$, in particular the fact that $\phi(n + 1) - \phi(n) \geq \phi(n)$, if a run spends half of its time in $H$ between step $\phi(n)$ and step $\phi(n + 1)$, then it must have spent at least a quarter of its time in $H$ between step $0$ and step $\phi(n + 1)$. That is, for all $n$, if $r' \in D_n$ we have
\[
\text{prop}(H, r'_{\phi(n + 1)}) \geq 1/4.
\]

Given $n \in \mathbb{N}$, let
\[
E_n = \{r' \in \Omega' \mid \exists i \in [\phi(n); \phi(n + 1)) \text{ s.t. } r'(i) \in F'\}.
\]

By definition of $\phi$, for all $n \in \mathbb{N}$, we have $\mathbb{P}_w^\alpha'[E_n] \geq \gamma \cdot (1 - \delta/2^n)$. Let $E = \bigcap_{n \in \mathbb{N}} E_n$. By the choice of $\delta$, we know that $\mathbb{P}_w^\alpha'[E] \geq \gamma \cdot (1 - \epsilon)$. Now, we claim that, if $n, k \in \mathbb{N}$, then
\[
\mathbb{P}_w^\alpha'[D_n \cap D_{n+1} \cap \cdots \cap D_{n+k} \cap E] = \frac{\mathbb{P}_w^\alpha'[E]}{2^k}. \qquad (2)
\]

Using multiple conditioning in the probabilities, this comes from the fact that for all $i \in [0; k]$, we have
\[
\mathbb{P}_w^\alpha'[D_n \cap D_{n+1} \cap \cdots \cap D_{n+i-1} \cap E] = \mathbb{P}_w^\alpha'[D_n+i|E] = 1/2.
\]

Indeed, when a run visits the copy of $s$ state in $F$ between time $\phi(n)$ and time $\phi(n + 1)$, the probability that it spends more than half of the time between $\phi(n + 1)$ and $\phi(n + 2)$ on $H$ is exactly one half; after a visit to a copy of a final state, the probabilities to go to $H$ or to $L$ are both $1/2$. We can now come back to our proof that $\mathbb{P}_w^\alpha'[A] = 0$. Recall that
\[
A = Z \cap \{r' \in \Omega' \mid \text{Supp}(r') \subseteq H\}.
\]
First, remark that
\[ A \subseteq \bigcup_{n \geq 0} \cap_{n \geq N} D_n. \tag{3} \]

Indeed, if \( r' \not\in \bigcup_{n \geq 2} \cap_{n \geq N} D_n \), then there exists a subsequence \( D_{i(n)} \), \( n \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \) we have \( r' \not\in D_i(n) \). Now, \( r' \not\in D_i(n) \) implies \( \text{prop}(L, r_{\phi(n+1)}) \geq 1/2 \). Thus, by the construction of \( \phi \), \( r' \not\in \bigcup_{n \geq 0} \cap_{n \geq N} D_n \) would imply that \( \text{Supp}(r') \cap L \neq \emptyset \), and then \( r' \not\in A \).

By continuity of the measure, we have
\[ \mathbb{P}_w^A[A \cap E] = \lim_{N \to \infty} \mathbb{P}_w^A[\cap_{n \geq N} D_n \cap E]. \]

Now, using Eqs. (2) and (3), we get that \( \lim_{N \to \infty} \mathbb{P}_w^A[\cap_{n \geq N} D_n \cap E] = 0 \), hence \( \mathbb{P}_w^A[A \cap E] = 0 \). Since \( \mathbb{P}_w^A[E] \geq \gamma \cdot (1 - \epsilon) \), \( A \subseteq Z \), and \( \mathbb{P}_w^A[Z] = \gamma \), this proves \( \mathbb{P}_w^A[A] \leq \epsilon \). This is true for all \( \epsilon > 0 \), so we get that \( \mathbb{P}_w^A[A] = 0 \), hence the result. \( \Box \)

**Theorem 7. Problem 5 is undecidable.**

**Proof.** We have \( \mathbb{P}_w^A[\{r| \text{Supp}(r) \cap L \neq \emptyset \text{ and } \text{Supp}(r) \cap H \neq \emptyset \}] > 0 \) iff there exists \( u \in L \) and \( v \in H \) such that \( \mathbb{P}_w^A[\{r|u \in \text{Supp}(r) \text{ and } v \in \text{Supp}(r)\}] > 0 \). This corresponds to Problem 5. Thus, we have reduced the emptiness problem for Probabilistic Büchi Automata, which is known to be undecidable, to Problem 5. \( \Box \)

4.3. The weak and strong transience problem

We prove in this subsection that both the weak and strong transience problems are PSPACE-complete. Moreover, we prove that they are equivalent, in that they are satisfied on the same instances, and that they are equivalent to a probabilistic loop problem.

**Proposition 5.** Let \( \mathcal{T} = (\Sigma, \delta, \alpha) \) be a FPT, and \( F \subseteq S \). Then the following are equivalent.

1. \( \exists w \in \Sigma^* \text{ s.t. } \mathbb{P}_w^\alpha[\{r|F \cap \text{Inf}(r) = \emptyset\}] > 0 \).
2. \( \exists w \in \Sigma^* \text{ s.t. } \mathbb{P}_w^\alpha[\{r|F \cap \text{Supp}(r) = \emptyset\}] > 0 \).
3. There exists an accessible probabilistic loop on \( S \) whose support does not contain any state in \( F \).

**Proof.** 3 \( \Rightarrow \) 1 and 1 \( \Rightarrow \) 2 are simple. Suppose 2:

\[ \exists w \in \Sigma^* \text{ s.t. } \mathbb{P}_w^\alpha[\{r|F \cap \text{Supp}(r) = \emptyset\}] > 0. \]

Then there exists \( s_0 \in S \) such that \( \alpha(s_0) > 0 \) and \( \mathbb{P}_w^\alpha[\{r|F \cap \text{Supp}(r) = \emptyset\}] > 0 \). By contradiction, suppose there exists \( B \in \mathbb{N} \) such that \( \forall i \in \mathbb{N}, \text{ if } t_i \in \delta(s_0, w_i) \), then \( \exists k \in [1; B] \) with \( F \cap \delta(t_i, w_{i+1} \ldots w_{i+k}) \neq \emptyset \). Let
\[ \gamma = \min_{s \in S, \rho \in \Sigma^*} \delta(s, \rho)(t). \]

We have \( \gamma > 0 \). Then, for all \( i \in \mathbb{N} \) and \( s \in \delta(s_0, w_i) \),
\[ \mathbb{P} \left[ \left\{ r : \sum_{j=0}^{k} \mathbb{P}[X_j = s] \right\} \right] \geq \gamma. \]

Since \( \gamma > 0 \), by standard probability theory results, this implies that with probability one, a state in \( F \) belongs to the support of a run on the process induced by \( w \) and the initial distribution \( \{s_0\} \). This is in contradiction with the choice of \( w \) and \( s_0 \).

Then, there exists an increasing sequence \( \{i_k\}_{i \in \mathbb{N}} \in \mathbb{N}^\mathbb{N} \) such that for all \( l \in \mathbb{N} \) there exists \( t_i \in \delta(s_0, w_{i_l}) \) with, for all \( k \in [0; l] \), \( F \cap \delta(t_i, w_{i_l} \ldots w_{i_l+k}) = \emptyset \). Now, let \( l \geq 2^{|B|} \). Then there exists \( k_1, k_2 \in [0; l] \) with \( \delta(t_i, w_{i_l}) \ldots w_{i_l+k_1}) = \delta(t_i, w_{i_l+k_1} \ldots w_{i_l+k_2}) \). \( C \) is accessible, since \( t_i \) is accessible by hypothesis. Moreover, \( (C, w_{i_l+k_1} \ldots w_{i_l+k_2} \ldots w_{i_l+k_2}) \) is a probabilistic loop, which does not contain any state of \( F \) in its support, hence 3. This proves the result. \( \Box \)

**Theorem 8.** The weak transience and strong transience problems (Problem 6) are PSPACE-complete.

**Proof.** The proof of the fact that these problems are in PSPACE is the same as for the strong recurrence problem: a nondeterministic Turing machine can guess \( \rho_0 \) and \( \rho \) and verify in PSPACE the requirements. Concerning the PSPACE-hardness, we point out that the exact same reduction as for the strong recurrence problem is also a reduction for the Intersection of Regular Languages problem to our problem. \( \Box \)
5.4. The universal weak and strong recurrence problems

We start by a theorem proved in [1].

**Theorem 9 ([1]).** The universal weak recurrence problem is undecidable.

**Proof.** This follows from the undecidability result of [1] that the weak recurrence problem is undecidable, and from the fact that the languages of Probabilistic Büchi Automata are closed under complementation. □

We prove that the universal strong recurrence problem is also undecidable, by a reduction to the 1-isolated cut-point problem, known to be undecidable by Theorem 3.

**Theorem 10.** The universal strong recurrence problem is undecidable.

**Proof.** Given an FPT \( T = (S, \Sigma, \delta, \alpha) \) and \( H \subseteq S \), the universal strong recurrence problem asks where exists \( w \in \Sigma^\omega \) such that

\[
P^T_w([r | H \cap \text{Supp}(r) = \emptyset]) = 1.
\]

Let \( T = (S, \Sigma, \delta, s_0, F) \subseteq S \) be an instance of the 1-isolated cut-point problem. We suppose that the initial distribution is the Dirac distribution on state \( s_0 \), which does not change the undecidability of the problem. The negation of the 1-isolated cut-point problem asks whether for all \( \epsilon > 0 \), there exists \( w_\epsilon \in \Sigma^* \) such that \( \delta(\alpha, w_\epsilon)(F) \geq 1 - \epsilon \). From \( T \) and \( F \), we build an instance \( T' = (S', \Sigma', \delta', \alpha') \), \( H \subseteq S \) of the universal strong recurrence problem such that

1. is an isolated cut-point for \( T = (S, \Sigma, \delta, \alpha), F \subseteq S \)

\[
\text{the universal strong recurrence problem is not satisfied on instance } T' = (S', \Sigma', \delta', \alpha'), H \subseteq S.
\]

We define \( H \) and the components of \( T' \) as follows.

- \( S' = S \).
- \( H = S - F \).
- \( \Sigma' = \Sigma \cup \{x, y\} \) where \( x \) and \( y \) are two new distinct symbols.
- \( \alpha' = s_0 \), i.e. the Dirac distribution on \( s_0 \).
- For \( \delta' \), we keep all the transitions of \( T \) between states of \( S = S' \), with labels in \( \Sigma \). For all \( s \in S \), given label \( x \), we add a transition which goes from \( s \) to \( s_0 \) with probability one. For all \( s \in S \), given label \( y \), we add a transition which goes from \( s \) to \( s \) with probability one. In other words, for all \( s \in S \) we define \( \delta'(s, x) = 1, \delta'(s, y)(s) = 1 \).

1. Suppose first that \( 1 \) is not an isolated cut-point for \( T = (S, \Sigma, \delta, \alpha), F \subseteq S \). Then, for all \( \epsilon > 0 \), there exists \( w_\epsilon \in \Sigma^* \) such that \( \delta(\alpha, w_\epsilon)(F) \geq 1 - \epsilon \). Given \( i \in \mathbb{N} \), let \( w_i \in \Sigma^* \) such that

\[
\delta(s_0, w_i)(F) \geq 1 - \frac{1}{2^i}.
\]

Consider the following \( w \in \Sigma^\omega \):

\[
w = w_1 \cdot y^{2^1} \cdot x \cdot w_2 \cdot y^{2^2} \cdot x \cdot w_3 \cdot y^{2^3} \cdot x \cdot \ldots
\]

Then, when we consider the word \( w \) on the FPT \( T' \), we have

\[
P^T_{w'}([r | H \cap \text{Supp}(r) = \emptyset]) = 1.
\]

That is, the universal strong recurrence problem is satisfied on \( T' = (S', \Sigma', \delta', \alpha'), H \subseteq S \).

2. Conversely, suppose that the universal strong recurrence problem is satisfied on \( T' = (S', \Sigma', \delta', \alpha'), H \subseteq S \). Let \( w \in \Sigma^\omega \) be such that

\[
P^T_w([r | H \cap \text{Supp}(r) = \emptyset]) = 1.
\]

This implies directly that for all \( \epsilon > 0 \), there exists \( w_\epsilon \in \Sigma^\omega \) such that \( \delta'(\alpha', w_\epsilon)(H) \leq \epsilon \), i.e. \( \delta'(\alpha', w_\epsilon)(F) \geq 1 - \epsilon \). By the construction of \( T' \), this implies that for all \( \epsilon > 0 \), there exists \( \overline{w_\epsilon} \in \Sigma^* \) such that \( \delta(\alpha, \overline{w_\epsilon})(F) \geq 1 - \epsilon \). \( \overline{w_\epsilon} \in \Sigma^* \) is obtained by projecting \( w_\epsilon \) on \( \Sigma \), i.e. by deleting the occurrences of \( x \) and \( y \) in \( w_\epsilon \). As a conclusion, \( 1 \) is not an isolated cut-point. □
5.5. The universal weak and strong transience problems

As in Section 5.3 concerning the weak and strong transience problems, we prove that both the universal weak and universal strong transience problems are PSPACE-complete, and equivalent.

Proposition 6. Let $T = (S, \Sigma, \delta, \alpha)$ be an FPT, and $F \subseteq S$. Then the following are equivalent.

1. $\exists w \in \Sigma^\omega$ s.t. $P_w^\alpha[\{r | F \cap \text{Inf}(r) \neq \emptyset\}] = 1$.
2. $\exists w \in \Sigma^\omega$ s.t. $P_w^\alpha[\{r | F \cap \text{Supp}(r) \neq \emptyset\}] = 1$.
3. There exists $\rho_0$ and $\rho$ in $\Sigma^*$ such that $(\delta(\alpha, \rho_0), \rho)$ is a probabilistic loop around $F$.

Proof. $3 \Rightarrow 2$ and $2 \Rightarrow 1$ are simple. Suppose $1$: $\exists w \in \Sigma^\omega$ s.t. $P_w^\alpha[\{r | F \cap \text{Inf}(r) \neq \emptyset\}] > 0$. Write $w = a_1a_2 \ldots$. For $i \in \mathbb{N}$, let $H_i = \delta(\alpha, w_i) = \bigcup_{\delta(\alpha, w_{i-l}) > 0} \delta(s, w_{i-l})$.

Since $S$ is finite, there exists $H \subseteq S$ such that infinitely often, $H_i = H$. Let $i_0 \in \mathbb{N}$ such that $H_{i_0} = H$. Let $t \in H$. Then $P_w^\alpha[\{r | F \cap \text{Inf}(r) \neq \emptyset\}] = 1$. F must be reachable from $t$ after a finite number of steps. That is, there exists $I_l \in \mathbb{N}$ such that $\delta(t, a_{i_0+1}a_{i_0+2} \ldots a_{i_0+l})$ is a probabilistic loop around $F$. Then $\rho_0 = w_{i_0}$ and $\rho = a_{i_0+1}, \ldots, a_{i_0+l}$ satisfy the conditions of 3.

Theorem 11. The universal weak and strong transience problems (Problem 8) are PSPACE-complete.

Proof. As for the strong recurrence problem, we can build a nondeterministic Turing machine which finds a relevant probabilistic loop in PSPACE if it exists. For the PSPACE-hardness, we can also reduce the Finite Intersection of Regular Languages problem to these problems.

6. Conclusion

In this paper we defined new notions of recurrence for non-homogeneous Markov chains, using a generalization of the natural notions which are equivalent in the context of homogeneous Markov chains. We worked in the model of Finite Probabilistic Tables, and studied the computational complexity of nine algorithmic problems directly related to the notions of recurrence and transience on this model. A further work could concern a quantitative notion of support: a state is in the $\lambda$-support of the run if the asymptotic proportion of the state in the run is at least $\lambda$.

Acknowledgements

This work would not have been possible without fruitful discussions with M. Grösser and C. Baier, at IT Dresden.

References