Spanning trees in graphs of minimum degree 4 or 5

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Abstract


For a connected simple graph G let \( L(G) \) denote the maximum number of leaves in any spanning tree of G. Linial conjectured that if G has N vertices and minimum degree \( k \), then
\[
L(G) \geq \left( \frac{k-2}{k+1} \right) N + c_k,
\]
where \( c_k \) depends on \( k \). We prove that if \( k = 4 \), \( L(G) \geq \frac{3}{5} N + \frac{2}{5} \) if \( k = 5 \), \( L(G) \geq \frac{1}{3} N + 2 \). We give examples showing that these bounds are sharp.

1. Introduction

Is there a spanning tree of a connected simple graph G with many leaves? To find a spanning tree with the maximum number of leaves is an NP-complete problem, even when restricted to cubic (3-regular) graphs [4]. So people want to know for a given graph G with N vertices and minimum degree \( k \), how many leaves at least are there for some spanning tree of G?

Throughout this paper G always denotes a connected simple graph. Let \( L(G) \) denote the maximum number of leaves in any spanning tree of G. In 1981, Storer [6] announced that \( L(G) \geq \frac{1}{3} N + 2 \) for any 3-regular graph G with N vertices. The most interesting problem in this area is a conjecture due to Linial [5, cf. [1]], which generalizes Storer’s result.

Conjecture. Let the minimum degree of G be \( k \). Then
\[
L(G) \geq \frac{k-2}{k+1} N + c_k,
\]
where \( c_k \) depends on \( k \).

This bound is attained with \( c_k = 2 \) by the following family of \( k \)-regular graphs: Construct a ‘necklace’ with any number of beads, where each bead is \( K_{k+1} - e \) (Fig. 1).
Kleitman and West [3] introduced a new method, the 'dead leaves' approach, with which they gave a proof of Linial's Conjecture for \( k = 3 \) with a best possible \( c_k = 2 \). The special case where \( G \) is cubic, i.e., Storer's Theorem, had not been proven rigorously before.

Through a complicated proof using dead leaves, Griggs, Kleitman, and Shastri [1] proved that \( L(G) \geq \frac{1}{3}(N + 4) \) if a cubic graph \( G \) with \( N \) vertices has no subgraph isomorphic to \( K_4 - e \). This bound is also tight, being attained by many graphs.

In Section 2 we prove Linial's Conjecture for \( k = 4 \) with the best possible value of \( c_4 = \frac{3}{2} \). We use the dead leaves approach. Kleitman and West [2] have independently developed a somewhat different proof for this case \( k = 4 \). While they originally obtained a proof that \( L(G) \geq \frac{3}{2}N + c \), we discovered the sharp result presented here.

Building on our work to settle \( k = 4 \), we prove our main result, which is Linial's Conjecture for \( k = 5 \), in Section 3. The best possible value for \( c_5 \) is 2.

A weaker general result than Linial's Conjecture would be to show that for every \( \varepsilon > 0 \), \( L(G) \geq (1 - \varepsilon)N \) for all graphs with sufficiently large minimum degree. This has just been proved by Kleitman and West [2].

It is worth pointing out that the proofs given in Sections 2 and 3, in fact, provide a polynomial algorithm to find a spanning tree which attains the lower bounds on \( L(G) \).

We conclude the paper by presenting in Section 4 a new family of graphs attaining Linial's bound.

2. The lower bound for \( k = 4 \)

Suppose \( T \) is a partial tree of \( G \). If \( v \) is a vertex of \( G \), let \( N_T(v) \) denote the set of neighbors of \( v \) inside \( T \) and \( N_T(v) \) the set of neighbors of \( v \) outside \( T \). Let \( N(T) \) denote the set of neighbors of \( T \), i.e., \( N(T) = \bigcup_{v \in T} N_T(v) \).

A leaf \( r \) of \( T \) is dead if \( |N_T(r)| = 0 \), otherwise it is alive. We call \( r \) \( k \)-split if \( |N_T(r)| = k \). We shall form a cost function involving the number of leaves, dead leaves, and vertices of \( T \), and we shall always seek to enlarge \( T \) while not decreasing the cost function. To consider dead leaves is a crucial idea, because we cannot gain enough new leaves in many cases, but we do gain some dead leaves to improve the value of the cost function.

**Theorem 1.** If \( G \) is a connected simple graph with \( N \) vertices and minimum degree 4, then \( L(G) \geq \frac{3}{2}N + \frac{3}{2} \).
**Proof.** First notice that

\[ L(G) \geq \frac{3}{2}N + \frac{5}{2} \text{ if and only if } 5L(G) \geq 2N + 8 \]

and

\[ \text{if and only if } 5L(G) > 2N + 7. \]  

(1)

Define a cost function

\[ \Delta(L, D, N) = \frac{3}{2}L + \frac{3}{2}D - 2N. \]

Then (1) holds if and only if there exists some spanning tree \( T \) for \( G \) such that

\[ \Delta(L, D, N) > 7, \]

where \( D \) is the number of dead leaves of \( T \), since every leaf in \( T \) is dead.

Our proof follows such procedures: First we find a partial tree with \( N_0 \) vertices, \( L_0 \) leaves and \( D_0 \) dead leaves such that

\[ \Delta(L_0, D_0, N_0) > 7. \]

Then we expand it to a spanning tree of \( G \) by a series of steps, where for each step we add some number of vertices \( n \), such that there is a net gain of \( l \) leaves and \( d \) dead leaves, satisfying the cost function \( \Delta(l, d, n) \geq 0 \). Finally the initial tree becomes a spanning tree \( T \) with all leaves dead, and clearly if \( L \) is the total number of leaves in \( T \), then \( \frac{12}{3}L + \frac{3}{2}L = 5L > 2N + 7 \), and we are done.

**Initial procedure:** Pick one vertex \( v \), and add all edges incident on \( v \) along with their endpoints. Such a star is required since \( L_0 = \deg(v) \geq 4 \) implies we have \( L_0 \) leavcs and \( L_0 + 1 \) vertices so that \( \Delta(L_0, 0, L_0 + 1) > 7. \)

**Expansion procedure:** Let \( T \) be the current tree. Before doing the next step, we repeatedly add the vertices, each of which is adjacent to some internal vertex of \( T \), to \( T \). Then only leaves of \( T \) may have neighbors outside \( T \). We do this without mentioning it again.

Next we list a collection of acceptable operations, at least one of which is available for the next step, until \( T \) becomes a spanning tree of \( G \).

(O1) **There is a leaf \( r \) of \( T \) with \( |N_T(r)| = k \geq 2. \)**

Expanding \( T \) at \( r \) to all \( N_T(r) \) gives \( \Delta(k - 1, 0, k) > 0 \). If we assume (O1) fails, then each live leaf of \( T \) has exactly one neighbor outside \( T \). Now we look at the neighbors of \( T \).

(O2) **There is a vertex \( x \in N(T) \) with \( |N_T(x)| > 4. \)**

Adding \( x \) to \( T \) kills at least \( k = 3 \) leaves and \( \Delta(0, k, 1) \geq 0. \)

Assuming (O1) and (O2) both fail, we have \( |N_T(v)| \leq 3 \) for each \( v \in N(T). \)

Now we consider a vertex \( v \) of \( T \) with \( |N_T(v)| = 1, 2, 3 \) separately.

(O3) **There exists \( v \in N(T) \) and \( |N_T(v)| = 1. \)**

Since \( \deg(v) \geq 4 \), \( v \) splits into (at least) 3 vertices outside \( T \). Expanding these 4 vertices gives \( \Delta(2, 0, 4) > 0 \) (Fig. 2).

If we assume (O1)–(O3) all fail, then \( 2 \leq |N_T(v)| \leq 3 \) for each \( v \in N(T) \).

(O4) **There is \( y_1 \in N(T) \) and \( |N_T(y_1)| = 2. \)**

Assume \( \deg(y_1) = 4 \) and \( y_1 \) splits into \( x_1 \) and \( y_2 \) outside \( T \) (if \( |N_T(y_1)| > 2, \)
expanding at $y_1$ as in Fig. 2 again, we are done by $\Delta(2, 1, 4) > 0$). We may also assume that $x_i \sim y_2$, $\deg(x_i) = \deg(y_2) = 4$ and none of them is adjacent to $T$, since otherwise we are done by $\Delta = \frac{1}{3}$ (Fig. 3) or $\Delta = \frac{2}{3}$ (Fig. 4). Let $x_2, y_3$ be the neighbors of $y_2$ besides $x_1, y_1$. For the same reason, we may assume none of them is adjacent to $T$. If $\{x_1, x_2, y_2, y_3\}$ form a $K_4$, we expand $T$ as in Fig. 5, so $y_2$ is dead, and $\Delta(2, 2, 5) = 0$. If $\{x_1, x_2, y_2, y_3\}$ do not form a $K_4$ (recall $x_1 \sim y_2$), then one of $x_2, y_3$ must split, say $y_3$ splits into $x_3, y_4$ (as before assume $x_3, y_4 \not\sim T$). Notice that $y_3$ should be adjacent to $x_1$ or $x_2$ (or both), otherwise we are done easily.

Set $B = T \cup \{x_i, y_j: 1 \leq i \leq 3, 1 \leq j \leq 4\}$. Referring to Fig. 6, so far the cost function $\Delta = -\frac{3}{2}$, so we need just one dead leaf or a 2-split to balance the deficit (each 2-split increases $\Delta$ by $\frac{1}{2}$). Clearly if one of $\{x_1, x_2, x_3, y_4\}$ is dead, we are done by $\Delta(3, 2, 7) > 0$; if one of $\{x_1, x_2, x_3, y_4\}$ splits into two vertices outside $B$, we are done by $\Delta(4, 1, 9) = 0$. In fact, once we get a ‘4-2-split’ structure, i.e., expand $T$ from $y_1$ by a full binary tree with four internal vertices (Fig. 7), and we win. So each one of the $\{x_1, x_2, x_3, y_4\}$ has exactly one neighbor outside $B$. Let $a \sim x_1, b \sim x_2, c \sim x_3, d \sim y_4$, where $a, b, c, d \not\in B$. If $a = b = c = d$ and $\deg(a) = 4$, expand $T$ to $B \cup \{a\}$, then $\Delta(3, 5, 8) = \frac{3}{2} > 0$ (Fig. 8). If $\deg(a) \geq 5$, we are done by $\Delta(4, 1, 8) > 0$. Fig. 9 shows the case $x_1 \sim x_3$.

Now we go back and look at $y_3$.

(1) $y_3 \sim x_1$.

Assume $x_2 \not\sim x_1$, otherwise refer to Fig. 4. Then $x_2$ must be adjacent to two of $\{x_3, y_3, y_4\}$. If $x_2 \sim y_3$, we are done by Fig. 10, and $\Delta(4, 2, 8) > 0$; otherwise we are done by Fig. 11, killing $t$ and $y_3$.

(2) $y_3 \not\sim x_1, y_3 \sim x_2$.

(a) $x_1 \sim x_2$: Since $x_1$ must be adjacent to two of $\{x_1, x_2, y_4\}$, $x_3$ should be adjacent to at least one of $x_1, x_2$. If $x_3 \sim x_1$, expanding $T$, gives $\Delta(3, 1, 6) > 0$.
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Fig. 6. Fig. 7. Fig. 8.

Fig. 9. Fig. 10. Fig. 11.

(Fig. 12). If \( x_3 \sim x_2 \), expand \( T \) as in Fig. 13, so \( t \) and \( y_2 \) are killed and \( \Delta(3, 2, 7) > 0 \).

(b) \( x_1 \not\sim x_2 \), \( x_1 \sim x_3 \): Then \( x_3 \) must be adjacent to one of \( x_2, y_4 \). If \( x_3 \sim x_2 \), we expand \( T \) as in Fig. 14, so that \( t, y_2 \) are dead, and \( \Delta(3, 2, 7) > 0 \). If \( x_3 \not\sim x_2 \), and \( x_3 \sim y_4 \), then \( x_2 \sim y_4 \). Now if \( d \neq a \), expand \( T \) by Fig. 15 while if \( a = d \) and \( a \neq b \), expand \( T \) as in Fig. 16; if \( a = d = b \) but \( a \neq c \), expand \( T \) as in Fig. 17. We have a 4-2-split for each case.

(c) \( x_1 \sim \{x_2, x_3\}, x_1 \sim y_4 \): Then \( x_3 \) must be adjacent to \( x_2 \) and \( y_4 \). Now if \( d \neq a \), expand \( T \) by Fig. 18; if \( a = d \), \( a \neq b \) expand \( T \) by Fig. 19; if \( a = b = d \) but \( a \neq c \), expand \( T \) by Fig. 20. Again we have a split for each case.

It remains to consider the case that (O1)–(O4) all fail. Then each \( v \in N(T) \) has \( |N_T(v)| = 3 \).

(O5) There exists \( y_1 \in N(T) \) with \( |N_T(y_1)| = 3 \).

If \( |N_T(y_1)| > 2 \), refer to Fig. 2. Hence we may assume \( N_T(y_1) = \{x_1, y_2\} \). Assume \( x_1 \not\sim T \not\sim y_2 \) (otherwise done by killing many leaves). One of \( x_1, y_2 \) must split, say \( y_2 \) splits into \( x_2, y_3 \), and expanding gives \( \Delta(2, 2, 5) = 0 \) (Fig. 21). Finally assume \( N_T(y_1) = \{y_3\} \). If \( y_2 \sim T \), we get \( \Delta(0, 6, 2) = 0 \) (Fig. 22). If \( y_2 \not\sim T \), \( y_2 \) should split into at least 3 vertices outside \( T \), and expanding \( T \) gives \( \Delta(2, 2, 5) = 0 \) (Fig. 23).

Clearly (O1) (O5) cover all cases, and we are done. \( \square \)
Notice that the lower bound of $L(G)$ given in Theorem 1 is sharp. For example, the graph $G(4, 6)$ in Fig. 24, which is 4-regular with 6 vertices, is such an example. Another graph $G(4, 8)$ almost matches this lower bound (Fig. 25). It is not clear whether there are some other graphs matching this lower bound, but we know that such graphs should be 4-regular, and each edge is involved in a triangle.

3. The lower bound for $k = 5$

Now let us consider graphs $G$ with minimum degree 5.

**Theorem 2.** If $G$ is a connected simple graph with $N$ vertices and minimum degree at least 5, then $L(G) \geq \frac{1}{2}N + 2$. 
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Proof. First notice that if $N$ is even, then $L(G) \geq \frac{1}{2}N + 2$ if and only if $L(G) > \frac{1}{2}N + 1$, i.e., $2L(G) > N + 2$. Define the cost function $\Delta(L, D, N) = \frac{3}{2}L + \frac{1}{4}D - N$. It is enough to show that

$$\Delta(L, D, N) > 2. \quad (2)$$

If $N$ is an odd number, then $L(G) \geq \frac{1}{2}N + 2$ if and only if $L(G) > \frac{1}{2}N + \frac{3}{2}$, i.e., $2L(G) > N + 3$, so it is enough to show that

$$\Delta(L, D, N) > 3. \quad (3)$$

As before, we find a partial tree which satisfies (2) or (3) according to whether $N$ is even or odd, and then expand it by a finite sequence of steps, such that each step preserves $\Delta \geq 0$. The proof depends on a series of lemmas.

Initial procedure: Pick $v \in V(G)$ with maximum degree in $G$, adding all edges incident on $v$ with the end points. If $N$ is even, $d(v) \geq 5$, so this star has $n_0 \geq 6$ vertices and $n_0 - 1$ leaves and $7(n_0 - 1)/4 - n_0 > 2$, while if $N$ is odd, then $d(v) \geq 6$, so this star has $n_0 \geq 7$ vertices and $7(n_0 - 1)/4 - n_0 > 3$.

Expansion procedure: We list a collection of acceptable operations, such that if $T$ is not yet a spanning tree, then certainly at least one of the operations is available for the next step.

We define a saw path $SP$ to be a path (no repeated vertices, as usual) of $G$ such that the vertices of $SP$ are alternatively inside $T$ and outside $T$. A saw cycle $SC$ is a saw path such that the first vertex is outside $T$, and adjacent to the last vertex inside $T$. The length of the $SC$ is defined as the number of vertices outside $T$ in $SC$ (Fig. 26).

(O1) If one leaf $r$ is $k$-split with $k \geq 3$, we expand $r$ to all of its neighbors, and $\Delta(k-1, 0, k) > 0$.

Now if we assume (O1) fails, then $|N_T(r)| \leq 2$ for every leaf $r$ of $T$.

(O2) There is a leaf $r_1$ with $N_T(r_1) = \{a_1, a_2\}$.

If one of $\{a_1, a_2\}$, say $a_2$, is not adjacent to $T$ by at least one other edge, then $a_2$ has at least 4 neighbors outside $T$, and expanding at $r_1$ to all neighbors of $a_2$, we have $\Delta(3, 0, 5) > 0$. Assume $a_2 \sim r_2 \in T$. If $N_T(r_2) = \{a_2\}$, we are done by expanding $r_1$ to $a_1, a_2$, killing $r_2$, and $\Delta(1, 1, 2) = 0$. So we may assume $N_T(r_2) = \{a_2, a_3\}$, where $a_3 \neq a_1$, or otherwise we have a saw cycle with length 2, and we are done by Lemma 1 below. Repeatedly searching, if some $r_i \sim a_j$ for some $j < i$, there is a saw cycle, and we are done by Lemma 1. Otherwise at the very end of the finite saw path, either it ends inside $T$, so we are done by expanding the last 2-split killing one old leaf, or it ends outside $T$, so we are done by the argument for $a_2$.

Fig. 26.
If we assume (O1), (O2) both fail, then $|N_T(r)| \leq 1$ for every leaf $r$ of $T$. Now we look at the neighbors of $T$.

(O3) There is $v \in N(T)$ with $|N_T(v)| = 1$.

Since $|N_T(v)| = k \geq 4$, expand $T$ to $v$, then to all $N_T(v)$, and $\Delta(k - 1, 0, k + 1) > 0$.

Suppose (O3) also fails, then $|N_T(v)| \geq 2$ for each $v \in N(T)$.

(O4) There exists $v \in N(T)$ with $|N_T(v)| = 2$.

We are done by Lemma 2.

If (O1)–(O4) all fail, clearly $|N_T(v)| \geq 3$ for each $v \in N(T)$.

(O5) There exists $v \in N(T)$ with $|N_T(v)| = 3$.

We are done by Lemma 3.

If we assume (O1)–(O5) all fail, then $|N_T(v)| \geq 4$ for each $v \in N(T)$.

(O6) There exists $v \in N(T)$ with $|N_T(v)| = 4$.

Then $v \sim x \not\in T$. If $x \sim T$ (at least 4 times) expand $T$ to $v$ and $x$, killing 8 leaves, so $\Delta(0, 8, 2) = 0$. If $x \not\sim T$, $x$ should split into 4 vertices outside $T$, expanding these vertices gives $\Delta(3, 3, 6) = 0$. (It is trivial if $\deg(v) > 5$ or $\deg(x) > 5$.)

Finally we assume (O1)–(O6) all fail.

(O7) There exists a $v \in N(T)$ with $N_T(v) \geq 5$.

Expand $T$ by $v$, killing at least 4 old leaves, and we are done by $\Delta(0, 4, 1) = 0$.

(O1)–(O7) cover all cases which may appear when expanding $T$ to a spanning tree of $G$. The summation of the costs of all steps including the initial procedure gives $L(G) \geq \frac{1}{2}N + 2$. We have completed the proof, subject to proving the lemmas that follow.

**Lemma 1.** Suppose $G$ is a simple connected graph with minimum degree at least 5. Let $T$ be a tree in $G$ that does not span it. Assume $|N_T(r)| \leq 2$ for every leaf $r$ of $T$. If there is a saw cycle $SC$, then we may expand $T$ preserving $\Delta \geq 0$.

**Proof.** Let SC be a shortest saw cycle with length $k$. Clearly $k \geq 2$. Label all vertices of SC outside $T$ in order by $a_1, a_2, \ldots, a_k$, and set $A = \{a_1, a_2, \ldots, a_k\}$. Correspondingly label vertices of SC inside $T$ by $r_1, r_2, \ldots, r_k$, i.e., $a_1 \sim r_1 \sim a_2 \sim r_2 \sim \cdots$, and so on.

**Case 1:** $k = 2m$, integer $m \geq 1$.

Expand every other 2-split, say expand $r_i$ to $\{a_1, a_2\}$, $r_i$ to $\{a_3, a_4\}, \ldots, r_{2m-1}$ to $\{a_{2m-1}, a_{2m}\}$, killing $r_2, r_4, \ldots, r_{2m}$, so $\Delta(m, m, 2m) = 0$ (Fig. 27).

**Case 2:** $k = 2m + 1$, where $m \geq 1$.

(1) Assume there is an edge from $A$ to $T$ besides SC. We assume $a_{2m+1} \sim \cdots
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Fig. 28.

If \( |N_T(r_0)| = 1 \), we simply expand \( r_{2m+1} \) to \( a_i \) and \( a_{2m+1} \), killing \( r_0 \), so \( \Delta(1, 1, 2) = 0 \). Assume \( |N_T(r_0)| = 2 \), say \( r_0 \sim a_0 \notin T, \ a_0 \neq a_{2m+1} \). If \( a_0 \notin A \) we have a shorter saw cycle, a contradiction. Therefore, \( a_0 \notin A \), and we expand \( T \) as in Case 1, and \( r_0 \) to \( \{a_{2m+1}, a_0\} \), killing \( r_2, \ldots, r_{2m+1} \), giving us \( \Delta(m + 1, m + 1, 2m + 2) = 0 \) (Fig. 28).

(2) Assume there is no edge from \( A \) to the complement of \( A \cup T \). While expanding every other 2-split as in Case 1, we expand \( r_{2m+1} \) to \( a_{2m+1} \), killing \( 3m + 1 \) leaves (all \( a_i \)'s and \( r_2, r_4, \ldots, r_{2m} \)), so that \( \Delta(m, 3m + 1, 2m + 1) \geq 0 \) for \( m > 1 \).

Here we must point out that under the assumption of (2), if \( m = 1 \), then each of \( \{a_1, a_2, a_3\} \) must have one more edge incident on \( T \) besides \( SC \), so refer to Case (1).

(3) Neither (1) nor (2) happens, say \( a_i \sim x \notin A \cup T \).

Claim. \( \{a_1, a_2, \ldots, a_{2m+1}, a_1\} \) form a cycle, otherwise we may expand \( T \) preserving \( \Delta \geq 0 \).

Proof. To prove the claim, observe that if for some \( i, \ a_i \neq a_{i+1} \) (mod \( 2m + 1 \)), then since there are no edges from \( a_i \) to \( T \) besides \( SC \), \( a_i \) should split into 3 vertices outside \( T \) other than \( a_{i+1} \). Then we expand \( r_i \) to \( \{a_i, a_{i+1}\} \), and \( a_i \) to the 3 vertices, giving \( \Delta(3, 0, 5) > 0 \).

We have \( a_i \sim a_{2m+1} \) by the claim. Expand \( T \) by \( m \) 2-splits as in Case 1, and expand \( a_1 \) to \( \{x, a_{2m+1}\} \), killing \( r_2, r_4, \ldots, r_{2m} \), and \( r_{2m+1} \), so that \( \Delta(m + 1, m + 1, 2m + 2) = 0 \) (Fig. 29).

This completes Lemma 1.

Lemma 2. Suppose \( G \) is a simple connected graph with minimum degree at least 5. Let \( T \) be a tree in \( G \) that does not span it. Assume \( |N_T(r)| \leq 1 \) for any leaf \( r \) of \( T \).
and $|N_T(v)| > 2$ for each $v \in N(T)$. If there exists $v \in N(T)$ with $|N_T(v)| = 2$, then we may expand $T$ preserving $\Delta \geq 0$.

**Proof.** If $\deg(v) > 5$, then $|N_T(v)| \geq 4$, and expanding $T$ to all $N_T(v)$, we have $\Delta(3, 1, 5) > 0$. Hence assume $\deg(v) = 5$. Let $v$ split into $\{x, y, z\}$. Assume none of $\{x, y, z\}$ is adjacent to $T$, since otherwise we are done by $\Delta(2, 3, 4) > 0$. None of $\{x, y, z\}$ has degree $>5$, since otherwise it should split into 3 new vertices other than its brothers ('brothers' means that they grow from the same vertex in $T$), and we have $\Delta(4, 1, 7) > 0$ (Fig. 30).

Furthermore, $\{x, y, z\}$ should form a triangle (Fig. 31), since otherwise one of $\{x, y, z\}$ must split into 3 new vertices outside $T$, and we are done as above.

Let $x$ split into $\{a, b\}$, $y$ into $\{c, d\}$, $z$ into $\{e, f\}$.

**Case 1:** One of $\{a, b, c, d, e, f\}$ is adjacent to $T$.

This gives $\Delta(3, 3, 6) = 0$. Fig. 32 shows the case $a \sim T$.

**Case 2:** One of $\{a, \ldots, f\}$ is adjacent to only one of $\{x, y, z\}$.

Then it must split into 3 new vertices other than $\{x, y, z\}$ and its brother, so $\Delta(5, 1, 9) = 0$ (Fig. 33).

If neither Case 1 nor Case 2 happens, then we need to consider the following two more cases.

**Case 3:** $\{a, b\} = \{c, d\} = \{e, f\}$.

Expand $x$ to $a$, $b$, killing $y$, $z$, $\Delta(3, 3, 6) = 0$ (Fig. 34).

**Case 4:** Assume $a = f$, $b = d$, $c = e$ (Fig. 35).

We may assume $a \sim b \sim c$, because if there is only one edge among $a$, $b$ and $c$, then one of $\{a, b, c\}$ should split into 3 new vertices, and we can refer to Case 2. For the same reason we may assume $\deg(a) = \deg(b) = \deg(c) = 5$.

Assume $a \sim c$. Then each of $\{a, b, c\}$ has exactly one edge to a vertex besides $\{a, b, c, x, y, z\}$, say $a \sim g$. If $g \sim b$ or $c$, say $g \sim b$, then expand $x$ to $\{a, b\}$, $a$ to $\{g, c\}$, which kills $y$, $z$, $s$, and $b$, so we have $\Delta(4, 4, 8) = 0$ (Fig. 35). Otherwise, suppose $b \not\sim g \not\sim c$. If $g \sim T$, we are done by expanding $x$ to $\{a, b\}$, $a$ to $\{c, g\}$,
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which kills many leaves and $\Delta(4, 5, 8) > 0$. If $g \not= T$, $g$ splits into 4 new vertices, so expanding all such neighbors gives $\Delta(7, 3, 12) > 0$ (Fig. 36).

It only remains to suppose $a \not= c$. Then $a$ has two new neighbors, say $g$ and $h$. Expand $z$ to $\{a, c\}$, $a$ to $\{b, g, h\}$, killing $\{s, y, x\}$, and we are done by $\Delta(5, 3, 9) > 0$ (Fig. 37).

Lemma 3. Suppose $G$ is a simple connected graph with minimum degree at least 5. Let $T$ be a tree in $G$, that does not span it. Assume $|N_T(r)| \leq 1$ for every leaf $r$ of $T$, and $|N_T(v)| = 3$ for every $v \in N(T)$. If there exists $v \in N(T)$ with $|N_T(v)| = 3$ then we may expand $T$ preserving $\Delta \geq 0$.

Proof. Notice the following points first:

1. $\deg(v) = 5$. Otherwise $|N_T(v)| \geq 3$, and expanding $v$ to all $N_T(v)$ gives $\Delta(2, 2, 4) = 0$.

2. Let $N_T(v) = \{x, y\}$, and $x \not= T \not= y$. Otherwise, we are done by killing many leaves (Fig. 38).

3. $x \sim y$. Otherwise, $x$ has 4 neighbors other than $y$, and expanding $T$ as in Fig. 39 gives $\Delta(4, 2, 7) > 0$. For the same reason we may assume $\deg(x) = \deg(y) = 5$.

4. Let $x$ split into $\{a, b, c\}$, $y$ split into $\{d, e, f\}$, where none of $\{a, b, c, d, e, f\}$ is adjacent to $T$, since otherwise we have $\Delta(3, 5, 6) > 0$ (Fig. 40).

Next we need to take care of the following cases.

Case 1: $|\{a, b, c\} \cap \{d, e, f\}| = 0$.

Expand $x$ to $\{a, b, c\}$, $y$ to $\{d, e, f\}$, and we have $\Delta(5, 2, 9) > 0$.

Case 2: $|\{a, b, c\} \cap \{d, e, f\}| = 1$, where, say, $c = f$.

Notice that $\{a, b, c\}$ should form a triangle. for otherwise one of $\{a, b\}$, say $a$, has 3 neighbors other than its brothers, so we are done by $\Delta(5, 2, 9) > 0$ (Fig. 41). Similarly $\{c, d, e\}$ form a triangle also. But this is impossible if $\deg(c) = 5$.
deg(c) > 5. \( \{a, b, c\} \) and \( \{c, d, e\} \) do form two triangles, and we are done by Lemma 4 below.

Case 3: \(|\{a, b, c\} \cap \{d, e, f\}| = 2\), and we assume \( b = e, c = f \).

It is clear by the above argument that \( a, d \sim b, c \).

Subcase 3.1: \( a \sim d, b \sim c \).

We may assume \( \deg(a) = \deg(b) = \deg(c) = \deg(d) = 5 \), or otherwise one of them must split to 3 neighbors other than its brothers, and we can expand \( T \) as in Fig. 41 again. So each one of \( \{a, b, c, d\} \) has exactly one more new neighbor. Assume \( a \sim g \). If \( g \sim T \), we are done by Fig. 42 and \( \Delta(4, 6, 8) > 0 \). If \( a \) is expanded to \( \{d, g\} \), killing one of \( \{b, c, d\} \), we have \( \Delta(4, 4, 8) = 0 \) (Fig. 43 but with \( b \neq c \), and \( g \neq b, c \) or \( d \)). Otherwise \( g \) splits into 4 vertices other than \( \{b, c, d\} \), so \( \Delta(7, 3, 12) > 0 \) (Fig. 44).

Subcase 3.2: \( a \sim d, b \sim c \).

As before we are done unless \( \deg(a) = \deg(d) = 5 \), and \( \deg(b), \deg(c) \leq 6 \). Let \( g \) be another neighbor of \( a \). If one of \( \{b, c\} \) is of degree 5, we expand \( T \) as in Fig. 43, killing \( s, t, y, \) and \( b \) (or \( c \)), so \( \Delta(4, 4, 8) = 0 \). Assume \( \deg(b) = \deg(c) = 6 \). Each one of \( \{a, b, c, d\} \) has exactly one new neighbor. Then we may refer to the last part of Subcase 3.1 (Fig. 43 or 44).

Subcase 3.3: \( d \sim a \) and \( b \sim c \).

As above we assume \( \deg(a) = \deg(d) = 5 \), \( \deg(b), \deg(c) \leq 6 \). Then \( a \) has two other neighbors \( g \) and \( h \), \( d \) has \( i \) and \( j \). Assume \( T \not\sim g, h, i, j \) (else we are done by killing many leaves).

If \( \{g, h, i, j\} \) are all distinct, there are at most two of them which may be adjacent to \( b \) or \( c \), so one of \( \{g, h, i, j\} \) must split into 3 vertices other than its brothers, say \( h \), so \( \Delta(6, 7, 11) = 0 \) (Fig. 45).

If \( \{|g, h\} \cap \{i, j\}| = 1 \), assume \( h = j \). We should have \( g \sim h \) and \( h \sim i \), or otherwise one of \( \{g, i\} \) must split into at least 3 new vertices (one such instance is
shown in Fig. 46), and we get $\Delta(6, 2, 11) = 0$. Also notice that $g$ (as well as $i$) is adjacent to one of $\{b, c\}$ (otherwise we are done by $g$ (or $i$) splitting into 3 new neighbors). Then expand $a$ to $\{g, h\}$, $h$ to $\{d, i\}$, killing $\{b, c, d, y, s, t\}$, so $\Delta(5, 6, 10)) > 0$ (Fig. 47).

Now we assume $\{g, h\} = \{i, j\}$.

If one of $\{g, h\}$ has two neighbors other than $\{a, b, c, d, g, h\}$, say $g$ has neighbors $k$ and $l$, we expand $a$ to $\{g, h\}$, $g$ to $\{d, k, l\}$, and $\Delta(6, 4, 11) > 0$ (Fig. 48). Otherwise $g \sim b$ (or $c$), $h \sim c$ (or $b$), $g$ has one new neighbor $k$. Expanding $T$ gives $\Delta(5, 6, 10) > 0$ (Fig. 49).

Subcase 3.4: $d + a$, $b - c$.

We may follow the proof of Subcase 3.3, while assuming $\deg(b) = \deg(c) = 5$, to expand $T$ preserving $\Delta \geq 0$.

Case 4: $\{a, b, c\} \cap \{d, e, f\} = 3$.

Expand $x$ to $\{a, b, c\}$, killing $s, t, y$, so that $\Delta(3, 3, 6) = 0$ (Fig. 50).

This completes the proof of Lemma 3. \(\square\)

**Lemma 4.** Suppose $G$ is a simple connected graph with minimum degree at least 5. Let $T$ be a tree in $G$, that does not span it. Suppose $|N_T(r)| \leq 1$ for every leaf $r$ of T. Assume $|N_T(v)| \geq 3$ for every $v \in N(T)$, and we have the structure as in Fig. 51, where $\deg(v) = \deg(x) = \deg(y) = 5$. Then we may suitably expand $T$ from it preserving $\Delta \geq 0$.

**Proof.** We notice that expanding out to $a$, $b$, $c$, $d$, $e$ gives

$$\Delta(4, 3, 8) = -\frac{1}{4} < 0,$$

so we need just one dead leaf to finish.

We may assume that $\deg(a) = \deg(b) = \deg(d) = \deg(e) = 5$, and $\deg(c) = 6$,
and none of them is adjacent to $T$. In fact, if one of $\{a, b, d, e\}$ is of degree $>5$ or if $\deg(c) > 6$, it should split into at least 3 vertices other than its brothers, so that $\Delta(5, 2, 9) > 0$; if one of $\{a, b, c, d, e\} \sim T$, we are done by killing many leaves. Now we discuss the following cases.

Case 1: $a \sim d, e$.

Expand $a$ to $\{e, d\}$, killing 4 leaves, so $\Delta(4, 4, 8) = 0$ (Fig. 52).

Because $a$, $b$, $d$, and $e$ are symmetric in Fig. 51, we assume Case 1 does not hold for any one of $a$, $b$, $d$, or $e$, i.e., $a$ is adjacent to at most one of $\{d, e\}$ (so is $b$), $d$ is adjacent to at most one of $\{a, b\}$ (so is $e$).

Case 2: There are two edges between $a$ or $b$ and $d$ or $e$, say $a \sim e$ and $b \sim d$.

Each one of $\{a, b, d, e\}$ has a new neighbor. Let $a \sim g$. Assume $g \not\sim T$, or otherwise expand $a$ to $\{g, e\}$ (omitting $d$), then $\Delta(4, 5, 8) > 0$. If $g \not\sim e$ (Fig. 53 shows the case $g \sim d$), or $g \sim e$ and $d \not\sim g \sim b$ (Fig. 54), then $g$ has 3 new neighbors other than $a$, $b$, or $c$, so if we expand $g$ to those 3 new neighbors, we have $\Delta(6, 2, 11) = 0$. If $g \sim e$ and $g$ is adjacent to one of $\{d, e\}$ (notice Fig. 51 is symmetric), expand $T$ as in Fig. 55. If $g \sim \{a, b, d, e\}$, $g$ has another neighbor $h$, and expanding $g$ to $\{d, h\}$ we have $\Delta(5, 7, 10) > 0$.

Case 3: There is just one edge between $a$ or $b$ and $d$ or $e$.

Assume $a \sim e$, $d \not\sim b$. Then $b$ has two new neighbors $g$ and $h$, and $g$, $h \not\sim T$ for otherwise we are done by killing many leaves. But one of $\{g, h\}$ must have 3 neighbors other than $a$, $b$, $c$, and its brother, so expanding $T$ gives $\Delta(6, 2, 11) = 0$ (Fig. 56).

Case 4: $d, e \not\sim a, b$.

Each one of $\{a, b, d, e\}$ has 2 new neighbors. Assume $a$ splits to $\{g, h\}$, $b$ to $\{k, l\}$ (none of $\{g, h, k, l\}$ is adjacent to $T$, otherwise we are done by killing many leaves). In fact, if $\{g, h\} \neq \{k, l\}$, one of them should split into 3 new neighbors other than $a$, $b$, $c$, and its brothers, so we are done by $\Delta(6, 2, 11) = 0$. 
(Fig. 57 or Fig. 58). So now we assume \( \{g, h\} = \{k, l\} \). By Fig. 56 we are done unless \( g \sim h \) and \( \deg(g) = \deg(h) = 5 \). Similarly if \( m \) and \( p \) are the neighbors of \( d \), they should be the neighbors of \( e \) also, and \( m \sim p \), \( \deg(m) = \deg(p) = 5 \).

We assume \( \{g, h\} \cap \{m, p\} = \emptyset \) because if \( \{g, h\} = \{m, p\} \), expanding \( a \) to \( \{g, h\} \), \( h \) to \( \{d, e\} \) kills many leaves (Fig. 59), while if \( |\{g, h\} \cap \{m, p\}| = 1 \), say \( h = p \), then \( \deg(h) = 6 \), a contradiction.

According to the above analysis, \( \{a, b, g, h\} \) form a \( K_4 \). If we expand \( a \) to \( g \) and \( h \), killing \( b \), we still need one more dead leaf to keep \( A \geq 0 \). We define a \( K_2 \)-chain to be structure formed by using \( K_3 \)'s as beads to form a chain, where each pair of adjacent \( K_3 \)'s forms a complete bipartite graph \( K_{2,2} \) besides the edges in the \( K_3 \)'s (Fig. 60).

Repeatedly applying the argument above we find that the problem occurs when there is a \( K_2 \)-chain starting at \( a \) and \( b \), and another \( K_2 \)-chain at \( d \) and \( e \) (Fig. 61). But the length of each \( K_2 \)-chain must be finite, so certainly one of the following cases should happen.

Subcase 4.1: A \( K_2 \)-chain stops outside \( T \).

The structure has to be changed at the very end of it. But one of \( \{\alpha, \beta, \rho, \tau\} \) splits into 3 new vertices besides its brother and parent (or is adjacent to \( T \)). So we win by gaining at least \( \frac{1}{4} \) in \( \Delta \) (Fig. 62).

Subcase 4.2: A \( K_2 \)-chain comes back to \( T \).
Clearly expand until the last 2-split of the $K_2$-chain, where it kills at least one extra old leaf, and we are done.

Subcase 4.3: The two $K_2$-chains meet.

We may expand the 2-splits along the upper $K_2$-chain around to $d$ and $e$, killing $d$, $e$, and $y$, and we win (Fig. 63).

This completes Lemma 4. □

These four lemmas complete the proof of Theorem 2. □

4. A new family of graphs attaining Linial’s bound

We have already seen that by taking a necklace of any number $A$ of beads, where each bead is $K_{k+1} - e$, a $k$-regular graph $G$ is obtained with

$$N = (k + 1)A \quad \text{and} \quad L(G) = \frac{k - 2}{k + 1} N + 2.$$ 

These graphs are extremal for Storer’s result ($k = 3$) and Theorem 2 ($k = 5$).

For $k = 4$ and $N = 5A$, the bound of Theorem 1 is not an integer, but in this case the implied bound is $\lfloor \frac{2}{3} N + \frac{2}{3} \rfloor = 2A + 2 = \frac{2}{3} N + 2$, and the family of necklaces attains this bound. We also saw that for $k = 4$ and general $N$, $\frac{2}{3}$ is best possible for $c_k$.

As a by-product of the proof of Theorem 2, we noticed an interesting new family of examples attaining Linial’s bound. For the case $k = 5$ of Theorem 2, let $G$ be a $K_2$-chain of $B \geq 3$ $K_2$’s that closes on itself. Then $G$ is a 5-regular graph with $N = 2B$ and $L(G) = B + 2$. It is extremal in Theorem 2 for all even $N$ (Fig. 64).

This construction extends for arbitrary $m \geq 1$ to provide a family of graphs that are regular of degree $k = 3m - 1$: For $B \geq 3$ form a $K_m$-cycle consisting of $B$ $K_m$’s in a circle such that vertices in consecutive $K_m$’s are adjacent. Such a graph $G$ has

$$N = Bm = \frac{3}{2} B(k + 1) \quad \text{and} \quad L(G) = B(m - 1) + 2 = \frac{k - 2}{k + 1} N + 2.$$ 

This is the same value for $L(G)$ attained by the necklaces and used to support Linial’s Conjecture.

Examples in the new family exist for three times as many values of $N$ as the family of necklaces, although in the new family $k$ is restricted to 2 mod 3. What is
significant is that the $K_m$-cycles are highly connected compared to the necklaces, which are not 3-connected. The connectivity is important to consider since for $k = 3$ the bound $L(G) \geq \frac{4}{3}N + 2$ rises to $L(G) \geq \frac{1}{3}(N + 4)$ when $G$ is 3-connected, by the result of Griggs, Kleitman, and Shastri [1]. Evidently 3-connectivity does not improve the bound on $L(G)$ for larger $k$.

The construction can be adapted to provide $k$-regular graphs $G$ for arbitrary $k \geq 2$. Given $A \geq 1$ and $a, b, c \geq 1$ such that $a + b + c = k + 1$, we arrange $A$ copies of the sequence of complete graphs $K_a, K_b, K_c$ in cyclic order. We form a graph $G$ by putting edges between vertices in consecutive complete graphs. Then

$$N = (k + 1)A \quad \text{and} \quad L(G) = \frac{k - 2}{k + 1}N + 2$$

In particular, if $a = b = 1$ and $c = k - 1$, then we have the familiar example of necklaces. We produce examples with high connectivity by taking each of $a, b, c$ equal to $\lfloor \frac{1}{3}(k + 1) \rfloor$ or $\lceil \frac{1}{3}(k + 1) \rceil$.

References