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The skeletons of free distributive lattices

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Abstract

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The skeletons of free distributive lattices are studied by methods of formal concept analysis; in particular, a specific closure system of sublattices is elaborated to clarify the structure of the skeletons. Up to five generators, the skeletons are completely described.

1. Introduction

The knowledge about the structure of free distributive lattices is very limited; in the finite case, even the number of elements is not known for more than seven generators [2]. In [7] an approach is described in which the structural analysis of finite free distributive lattices is based on the notion of the skeleton of a finite lattice. The main concern of this paper is to clarify the structure of the skeletons of free distributive lattices via distinguished sublattices. The skeletons are not only valuable for analysing the structure of free distributive lattices, they are also interesting by themselves because the rth skeleton of the free bounded distributive lattice with n generators is isomorphic to the lattice of all maximal convex r-subsets in the power set of an n-element set (see [7]). As in [7], this paper also uses methods of formal concept analysis which have been developed in [4–6, 8]. In Section 2, free completely distributive lattices and their skeletons are represented as concept lattices. These representations are used in Section 3 to study a closure system of distinguished sublattices of the skeletons. In Section 4 the results of Section 2 and 3 are applied to describe the skeletons of the free bounded distributive lattice generated by five elements; as a particular corollary we obtain that the number of maximal antichains in the power set of a 5-element set is 376.

2. Free completely distributive lattices and their skeletons

The *free completely distributive lattice* FCD(S) *generated by the set* S is, up to isomorphism, characterized by the property that every map α from S into a completely distributive complete lattice L can be extended to a complete homomorphism α from FCD(S) into L. In [1] one can find a representation of FCD(S) for an arbitrary set S. Here, this representation is modified to obtain a representation of FCD(S) and its skeletons as concept lattices.

First we have to introduce the notion of a skeleton for complete lattices. For this we recall that a *complete tolerance relation* Θ of a complete lattice L is a binary relation on L which is reflexive, symmetric and compatible with the operations $\bigwedge$ and $\bigvee$, i.e., $x_t\Theta y_t$ for $t \in T$ imply $(\bigwedge_{t\in T} x_t)\Theta(\bigwedge_{t\in T} y_t)$ and $(\bigvee_{t\in T} x_t)\Theta(\bigvee_{t\in T} y_t)$.

The *blocks* of Θ are the maximal intervals B of L satisfying $x\Theta y$ for all $x, y \in B$. The set L/Θ of all blocks of Θ becomes a complete lattice by defining

$$B_1 \leqslant B_2 :\Leftrightarrow \bigwedge B_1 \leqslant \bigwedge B_2 \; (\Leftrightarrow \bigvee B_1 \leqslant \bigvee B_2) \qquad \text{(see [5])}.$$

A complete tolerance relation is called *glued* if for every two of its blocks $B_1 < B_2$ there are blocks B_3 and B_4 with $B_1 \leqslant B_3 \leqslant B_4 \leqslant B_2$ and $B_3 \cap B_4 \neq \emptyset$. The smallest glued complete tolerance relation of L is, if it exists, denoted by $\Sigma\,(L)$ and the complete lattice $S(L) := L/\Sigma\,(L)$ is called the *skeleton* of L [5]. This construction may be iterated as follows: $S_0(L) := L$ and $S_r(L) := S(S_{r-1}(L))$ for $r = 1, 2, 3, \ldots$; $S_r(L)$ is called the rth skeleton of L.

Next we define specific relations on the power set $\mathfrak{P}(S)$ of a set S. For $X, Y \subseteq S$, let $X\Delta Y :\Leftrightarrow X \cap Y \neq \emptyset$ and, for $r = 1, 2, 3, \ldots$, let $X\Sigma_r^S Y :\Leftrightarrow |S\setminus(X \cup Y)| \leqslant r-1$; let $\Sigma_0^S := \Delta$. With these definitions we can state the representation theorem.

Theorem 1. FCD(S) $\cong \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ *and, moreover,* $S_r(\text{FCD}(S)) \cong \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_r^S)$.

Proof. The extents and intents of the context $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ are order filters on S and, for each order filter $\mathscr{F}$ on S, the pair $(\mathscr{F}, \mathscr{F}^{\#})$ is a concept of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ where $\mathscr{F}^{\#} := \{Y \subseteq S \mid S\setminus Y \notin \mathscr{F}\}$ (cf. [6]); this follows from $\exists X \in \mathscr{F}: X \cap Y = \emptyset \Leftrightarrow \exists X \in \mathscr{F}: X \subseteq S\setminus Y \Leftrightarrow S\setminus Y \in \mathscr{F}$. Hence $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ consists of all pairs $(\mathscr{F}, \mathscr{F}^{\#})$ for which $\mathscr{F}$ is an order filter on S (notice that $\mathscr{F}^{\#\#} = \mathscr{F}$). $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ is completely distributive because

$$\bigwedge_{t\in T} (\mathscr{F}_t, \mathscr{F}_t^{\#}) = \Big(\bigcap_{t\in T} \mathscr{F}_t, \bigcup_{t\in T} \mathscr{F}_t^{\#}\Big)$$

and

$$\bigvee_{t\in T} (\mathscr{F}_t, \mathscr{F}_t^{\#}) = \Big(\bigcup_{t\in T} \mathscr{F}_t, \bigcap_{t\in T} \mathscr{F}_t^{\#}\Big) \qquad \text{(cf. [4])}.$$

For $p \in S$, let $\mathscr{C}_p^S := \{X \subseteq S \mid p \in X\}$. $\mathscr{C}_p^S$ is an order filter on S with $(\mathscr{C}_p^S)^{\#} = \mathscr{C}_p^S$. We obtain that the pairs $(\mathscr{C}_p^S, \mathscr{C}_p^S)$ with $p \in S$ are concepts which generate $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ as complete lattice because $\mathscr{F} = \bigcup_{X \in \mathscr{F}} \bigcap_{p \in X} \mathscr{C}_p^S$ for every order filter $\mathscr{F}$ on S. Now, let α be a map from $\{(\mathscr{C}_p^S, \mathscr{C}_p^S) \mid p \in S\}$ into a completely distributive complete lattice L. For $(\mathscr{F}, \mathscr{F}^{\#}) \in \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$, we define

$$\hat{\alpha}(\mathscr{F}, \mathscr{F}^{\#}) := \bigvee_{X \in \mathscr{F}} \bigwedge_{p \in X} \alpha(\mathscr{C}_p^S, \mathscr{C}_p^S).$$

Then

$$\hat{\alpha} \bigvee_{t \in T} (\mathscr{F}_t, \mathscr{F}_t^{\#}) = \hat{\alpha}\Big(\bigcup_{t \in T} \mathscr{F}_t, \bigcap_{t \in T} \mathscr{F}_t^{\#}\Big) = \bigvee_{t \in T} \bigvee_{X \in \mathscr{F}_t} \bigwedge_{p \in X} \alpha(\mathscr{C}_p^S, \mathscr{C}_p^S) = \bigvee_{t \in T} \hat{\alpha}(\mathscr{F}_t, \mathscr{F}_t^{\#}).$$

Dually, we obtain that $\hat{\alpha}$ is also $\bigwedge$-preserving by using

$$\hat{\alpha}(\mathscr{F}, \mathscr{F}^{\#}) = \bigvee_{X \in \mathscr{F}} \bigwedge_{p \in X} \alpha(\mathscr{C}_p^S, \mathscr{C}_p^S) = \bigwedge_{\sigma \in \prod \mathscr{F}} \bigvee_{X \in \mathscr{F}} \alpha(\mathscr{C}_{\sigma X}^S, \mathscr{C}_{\sigma X}^S) = \bigwedge_{Y \in \mathscr{F}^{\#}} \bigvee_{p \in Y} \alpha(\mathscr{C}_p^S, \mathscr{C}_p^S).$$

Now, we can conclude that the inverse of the bijection ι_S which assigns to each $p \in S$ the concept $(\mathscr{C}_p^S, \mathscr{C}_p^S)$, extends to a complete homomorphism which is the inverse of $\hat{\iota}_S : \mathrm{FCD}(S) \to \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$; hence $\hat{\iota}_S$ is an isomorphism. This completes the proof of the first assertion.

Next we prove that $\Delta \cup \sum_r^S$ is a *block relation* of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \sum_{r-1}^S)$, i.e., $\Delta \cup \sum_{r-1}^S \subseteq \Delta \cup \sum_r^S$ and, for each $X \subseteq S$, $X^{\Delta \cup \Sigma_r^S}$ $(= \{Y \subseteq S \mid X(\Delta \cup \sum_r^S)Y\})$ is an extent and an intent of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \sum_{r-1}^S)$, respectively (cf. [5]). Obviously $\Delta \cup \sum_{r-1}^S \subseteq \Delta \cup \sum_r^S$. For $p \in S \setminus X$ we have $X^{\Delta \cup \Sigma_r^S} \subseteq (X \cup \{p\})^{\Delta \cup \Sigma_{r-1}^S}$. Let $Y \subseteq S$ with $Y \notin X^{\Delta \cup \Sigma_r^S}$, i.e., $X \cap Y = \emptyset$ and $|S \setminus (X \cup Y)| \geq r$. We choose $p \in S \setminus (X \cup Y)$. Then $(X \cup \{p\}) \cap Y = \emptyset$ and $|S \setminus (X \cup \{p\} \cup Y) \geq r - 1$; hence $Y \notin (X \cup \{p\})^{\Delta \cup \Sigma_{r-1}^S}$. This proves that

$$X^{\Delta \cup \Sigma_r^S} = \bigcap_{p \in S \setminus X} (X \cup \{p\})^{\Delta \cup \Sigma_{r-1}^S}$$

for $X \subset S$; furthermore, $S^{\Delta \cup \Sigma_r^S} = \mathfrak{P}(S)$. Therefore $\Delta \cup \sum_r^S$ is shown to be a block relation of $(\mathfrak{P}(S), \mathfrak{P}(S), \sum_{r-1}^S)$ for $r = 1, 2, 3, \ldots$.

Now, by Theorem 8 in [5], the block relation $\Delta \cup \sum_r^S$ yields a complete tolerance relation Θ of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \sum_{r-1}^S)$ via the following definition:

$$(\mathfrak{C}_1, \mathfrak{D}_1)\Theta(\mathfrak{C}_2, \mathfrak{D}_2) :\Leftrightarrow \mathfrak{C}_1 \times \mathfrak{D}_2 \cup \mathfrak{C}_2 \times \mathfrak{D}_1 \subseteq \Delta \cup \textstyle\sum_r^S;$$

furthermore,

$$\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \textstyle\sum_{r-1}^S)/\Theta \cong \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \sum_r^S).$$

If we show that Θ is the smallest glued complete tolerance relation of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \sum_{r-1}^S)$, the second assertion of the theorem is also proved. Let $X, Y \subseteq S$ with $X \cap Y = \emptyset$ and $|S \setminus (X \cup Y)| = r - 1$.

Then $Y^{\Delta \cup \Sigma_{r-1}^S}$ is maximal in

$$\{Z^{\Delta \cup \Sigma_{r-1}^S} \mid Z \in \mathfrak{P}(S) \setminus X^{\Delta \cup \Sigma_{r-1}^S}\}.$$

Therefore $\gamma X \vee \mu Y$ covers μY. Since $X(\Delta \cup \Sigma_r^S)Y$, it follows that X lies in $Y^{\Delta \cup \Sigma_r^S}$ which is already proven to be an extent of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_{r-1}^S)$. As $Y^{\Delta \cup \Sigma_{r-1}^S} \subseteq Y^{\Delta \cup \Sigma_r^S}$, the extent $\mathfrak{C}$ of $\gamma X \vee \mu Y$ is contained in $Y^{\Delta \cup \Sigma_r^S}$. Let $Z \in \mathfrak{C} \setminus Y^{\Delta \cup \Sigma_{r-1}^S}$. Then $Z^{\Delta \cup \Sigma_r^S}$ has to contain the intent $\mathfrak{D}$ of μY because μY is covered by $\gamma X \vee \mu Y$. Therefore $\mathfrak{C} \times \mathfrak{D} \subseteq \Delta \cup \Sigma_r^S$; hence $(\gamma X \vee \mu Y)\Theta\mu Y$. Now, if $(\mathfrak{C}_1, \mathfrak{D}_1) < (\mathfrak{C}_2, \mathfrak{D}_2)$ in $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_{r-1}^S)$ then there are $X \in \mathfrak{C}_2$ and $Y \in \mathfrak{D}_1$ with $X \cap Y = \emptyset$ and $|S \setminus (X \cup Y)| = r - 1$. Since $(\gamma X \vee \mu Y)\Theta\mu Y$ we have

$$(((\mathfrak{C}_1, \mathfrak{D}_1) \vee \gamma X) \wedge \mu Y)\Theta((\mathfrak{C}_2, \mathfrak{D}_2) \wedge (\gamma X \vee \mu Y)).$$

Together with

$$(\mathfrak{C}_1, \mathfrak{D}_1) \leq ((\mathfrak{C}_1, \mathfrak{D}_1) \vee \gamma X) \wedge \mu Y < (\mathfrak{C}_2, \mathfrak{D}_2) \wedge (\gamma X \vee \mu Y) \leq (\mathfrak{C}_2, \mathfrak{D}_2),$$

this proves that Θ is glued. An arbitrary glued complete tolerance relation Ψ of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_{r-1}^S)$ contains, by definition, all covering pairs, in particular $(\gamma X \vee \mu Y, \mu Y)$ with $X \cap Y = \emptyset$ and $|S \setminus (X \cup Y)| = r - 1$. These pairs generate Θ by Theorem 8 in [5] because

$$(\Delta \cup \Sigma_r^S)/(\Sigma_{r-1}^S) = \{(X, Y) \in \mathfrak{P}(S)^2 \mid X \cap Y \neq \emptyset \text{ and } |S \setminus (X \cup Y)| = r - 1\}.$$

Thus, $\Theta \subseteq \Psi$ and therefore Θ is the smallest glued tolerance relation of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_{r-1}^S)$. This completes the proof. $\square$

The connection to the approach in [7] can be described using the following relation on $\mathfrak{P}(S)$: $X \supseteq_r Y :\Leftrightarrow X \supseteq Y$ and $|X \setminus Y| \geq r$. Then $X \not\supseteq_r Y$, i.e., $X \not\supseteq Y$ or $|X \setminus Y| \leq r - 1$, is equivalent to $(S \setminus X) \cap Y \neq \emptyset$ or $|S \setminus ((S \setminus X) \cup Y)| \leq r - 1$. Therefore $X \mapsto S \setminus X$ and $Y \mapsto Y$ describes a context isomorphism from $(\mathfrak{P}(S), \mathfrak{P}(S), \not\supseteq_r)$ onto $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_r^S)$; hence, by Theorem 1, we have $\mathrm{FCD}(S) \cong \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \not\supseteq)$ and, moreover, $S_r(\mathrm{FCD}(S)) \cong \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \not\supseteq_r)$.

In the case of a finite set S, it is more appropriate to use instead of Σ_r^S the relation $\bar{\Sigma}_r^S$ which is defined as follows: $X\bar{\Sigma}_r^S Y :\Leftrightarrow |X| + |Y| \geq |S| + 1 - r$. Obviously, $\Delta \cup \Sigma_r^S = \Delta \cup \bar{\Sigma}_r^S$; but $\bar{\Sigma}_r^S$ yields a simpler structure than Σ_r^S. For describing this, we define $\mathfrak{P}_{\geq k}(S) := \{X \subseteq S \mid |X| \geq k\}$ and $C_n := (\{0, 1, \ldots, n - 1\}, \leq)$. The context $(\mathfrak{P}(S), \mathfrak{P}(S), \bar{\Sigma}_r^S)$ with $r \leq s := |S|$ has exactly the concepts $(\mathfrak{P}_{\geq s+1-r-k}(S), \mathfrak{P}_{\geq k}(S))$ with $k = 0, \ldots, s + 1 - r$; hence $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \bar{\Sigma}_r^S) \cong C_{s+2-r}$. The pairs $(\mathfrak{P}_{\geq s+1-j}(S), \mathfrak{P}_{\geq j}(S))$ with $j = 1, \ldots, s$ form a distinguished chain in $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ which is under the isomorphism $\hat{\iota}_s$ the image of the chain $d_1 < \cdots < d_s$ in $\mathrm{FCD}(S)$ where

$$d_j := \bigwedge \{\bigvee X \mid X \subseteq S \text{ with } |X| = j\} \qquad (= \bigvee \{\bigwedge Y \mid Y \subseteq S \text{ with } |Y| = s + 1 - j\}).$$

For a finite set S, the free completely distributive lattice $\mathrm{FCD}(S)$ is also the *free bounded distributive lattice generated by* S and denoted by $\mathrm{FBD}(S)$; in particular $\mathrm{FBD}(n) := \mathrm{FBD}(\{e_1, \ldots, e_n\})$.

As an immediate consequence of Theorem 1 we obtain tensorial decompositions of FCD(S). Let us recall that *the direct product* of contexts $\mathbb{K}_t := (G_t, M_t, I_t)$ $(t \in T)$ is defined by

$$\prod_{t\in T} \mathbb{K}_t := \left(\prod_{t\in T} G_t, \prod_{t\in T} M_t, \nabla\right)$$

where

$$(g_t)_{t\in T} \nabla (m_t)_{t\in T} :\Leftrightarrow \exists t \in T: g_t I_t m_t;$$

the definition of the tensor product of complete lattices yields that $\mathfrak{B}(\prod_{t\in T} \mathbb{K}_t) \cong \bigotimes_{t\in T} \mathfrak{B}(\mathbb{K}_t)$ (see [6, 9]). If a set S is the disjoint union of sets S_t $(t \in T)$, we can easily deduce the context isomorphy $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta) \cong \prod_{t\in T} (\mathfrak{P}(S_t), \mathfrak{P}(S_t), \Delta)$. Thus, we have derived the following theorem (cf. [3]).

Theorem 2. *If* $S = \dot{\bigcup}_{t\in T} S_t$, *then* $\mathrm{FCD}(S) \cong \bigotimes_{t\in T} \mathrm{FCD}(S_t)$; *in particular* $\mathrm{FCD}(S) \cong \bigotimes_{s\in S} \mathrm{FCD}(\{s\})$ *where* $\mathrm{FCD}(\{s\}) \cong C_3$.

Corollary 3. $\mathrm{FBD}(m+n) \cong \mathrm{FBD}(m) \otimes \mathrm{FBD}(n)$.

3. Sublattices of the skeletons

In this section we study sublattices of the skeletons of free distributive lattices with the aim to understand the skeletons as a union of distinguished sublattices. In [8] it is shown how to recognize complete sublattices of a concept lattice via its underlying context; for that a *closed relation* of a context (G, M, I) is defined to be a subset J of $G \times M$ such that every concept of (G, M, J) is already a concept of (G, M, I). A complete sublattice $\mathfrak{S}$ of $\mathfrak{B}(G, M, I)$ yields the closed relation $C(\mathfrak{S}) := \bigcup (A \times B \mid (A, B) \in \mathfrak{S})$ of (G, M, I) by the following proposition from [8].

Proposition 4. *C is a bijection from the set of all complete sublattices of $\mathfrak{B}(G, M, I)$ onto the set of all closed relations of (G, M, I); in particular, $C^{-1}(J) = \mathfrak{B}(G, M, J)$ for each closed relation J of (G, M, I).*

For the application of Proposition 4 we generalise the relation Δ on the power set $\mathfrak{P}(S)$ using an arbitrary subset T of S:

$$X \Delta_T Y :\Leftrightarrow X \cap T \cap Y \neq \emptyset$$

obviously, $\Delta = \Delta_S$. For $\mathfrak{G}, \mathfrak{M} \subseteq \mathfrak{P}(S)$ and $T \subseteq S$ such that $X \in \mathfrak{G}$ implies $X \cap T \in \mathfrak{G}$ and $Y \in \mathfrak{M}$ implies $T \cap Y \in \mathfrak{M}$, it can be shown by Proposition 4 that $\mathfrak{B}(\mathfrak{G}, \mathfrak{M}, \Delta_T)$ is a complete sublattice of $\mathfrak{B}(\mathfrak{G}, \mathfrak{M}, \Delta)$. The main argument in the proof of this assertion also occurs in the following proposition which directly contributes to the theme of this paper.

Proposition 5. *For a finite set S and for a subset T of S, $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ is a 0–1-sublattice of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$.*

Proof. Let $(\mathfrak{C}, \mathfrak{D}) \in \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$. Obviously, $\mathfrak{D} \subseteq \mathfrak{C}^{\Delta \cup \bar{\Sigma}_r^S}$. Let $Z \in \mathfrak{C}^{\Delta \cup \bar{\Sigma}_r^S}$. Suppose $Z \notin \mathfrak{D}$. Then there exists an $X \in \mathfrak{C}$ with $X \cap T \cap Z = \emptyset$ and $|X| + |Z| \leq |S| - r$. Let $U \subseteq S$ such that $X \cap T \subseteq U$, $U \cap Z = \emptyset$, and $|U| = |X|$. To show that U must be an element of $\mathfrak{C}$, we choose an arbitrary element Y of $\mathfrak{D}$.

Case 1: $|Y| \geq |S| + 1 - r - |X|$.

Then $|U| + |Y| \geq |X| + |S| + 1 - r - |X| = |S| + 1 - r$; hence $U\, \bar{\Sigma}_r^S\, Y$.

Case 2: $|Y| \leq |S| - r - |X|$, *i.e.*, $|X| + |Y| \leq s - r$.

As $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$, it follows that $X \Delta_T Y$ and so $U \Delta_T Y$ because $X \cap T \subseteq U$. Case 1 and 2 yield $U \in \mathfrak{C}$. But this contradicts $Z \in \mathfrak{C}^{\Delta \cup \bar{\Sigma}_r^S}$, $U \cap Z = \emptyset$, and $|U| + |Z| \leq |S| - r$. Therefore $\mathfrak{D} = \mathfrak{C}^{\Delta \cup \bar{\Sigma}_r^S}$ and symmetrically $\mathfrak{C} = \mathfrak{D}^{\Delta \cup \bar{\Sigma}_r^S}$; hence $(\mathfrak{C}, \mathfrak{D}) \in \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$. This together with Proposition 4 proves that $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ is a complete sublattice of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$ which for finite lattices is the same as a 0–1-sublattice. □

In a context the intersection of closed relations need not to be a closed relation again so that it is not clear how to describe in general the intersection of sublattices via an underlying context. In the case of the sublattices $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$, the intersections have a natural description by closed relations which is given in the following proposition.

Proposition 6. *Let T and U be subsets of a finite set S. Then $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S) \cap \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_U \cup \bar{\Sigma}_r^S) = \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_{T \cap U} \cup \bar{\Sigma}_r^S)$.*

Proof. The concept lattice of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_{T \cap U} \cup \bar{\Sigma}_r^S)$ is contained in each concept lattice of the contexts $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ and $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_U \cup \bar{\Sigma}_r^S)$ and so also in their intersection. Now, let $(\mathfrak{C}, \mathfrak{D}) \in \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S) \cap \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_U \cup \bar{\Sigma}_r^S)$. Suppose that there are $X \in \mathfrak{C}$ and $Z \in \mathfrak{D}$ with $X \cap T \cap U \cap Z = \emptyset$ and $|X| + |Z| \leq |S| - r$. Let us choose $V \subseteq S$ with $X \cap T \subseteq V$, $V \cap U \cap Z = \emptyset$, and $|V| = |X|$. For $Y \in \mathfrak{D}$ we have $X \cap T \cap Y \neq \emptyset$ or $|X| + |Y| \geq |S| + 1 - r$ and so $V \cap T \cap Y \neq \emptyset$ or $|V| + |Y| \geq |S| + 1 - r$; hence $V \in \mathfrak{C}$. But, because of $Z \in \mathfrak{D}$, $V \cap U \cap Z = \emptyset$, and $|V| + |Z| \leq |S| - r$, this contradicts $(\mathfrak{C}, \mathfrak{D}) \in \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_U \cup \bar{\Sigma}_r^S)$. Therefore $(\mathfrak{C}, \mathfrak{D})$ is a concept of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_{T \cap U} \cup \bar{\Sigma}_r^S)$ which proves the assertion of the proposition. □

By Proposition 6, the distinguished sublattices $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ form a closure system on the underlying set of the lattice $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$ which, up to isomorphism, is the rth skeleton of the free distributive lattice generated by S. The structure of these sublattices is clarified by the next proposition (for the definition of the tensorial operation $\circledvee$ see [6]).

Proposition 7. *For finite sets $T \subset S$ with $s := |S|$, $t := |T|$, and $s - t \geq r \geq 1$, $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ is isomorphic to the sublattice of* $\mathrm{FBD}(t) \otimes C_{s-t+2-r}$ *generated by $e_1 \circledvee 0, \ldots, e_t \circledvee 0$ and the chain $\{\bigwedge_{j=0}^{k} d_j \circledvee (k-j) \mid k = 1, \ldots, s - t - r\}$.*

Proof. Obviously, the context $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ is isomorphic to the context $\mathbb{K}_I := (\mathfrak{P}(T) \times \mathfrak{P}(S \setminus T), \mathfrak{P}(T) \times \mathfrak{P}(S \setminus T), I)$ with $(X_1, X_2) I (Y_1, Y_2) :\Leftrightarrow X_1 \Delta Y_1$ or $|X_1| + |Y_1| + |X_2| + |Y_2| \geq s + 1 - r$. The question arises whether I is a closed relation of the direct product context $(\mathfrak{P}(T), \mathfrak{P}(T), \Delta) \times (\mathfrak{P}(S \setminus T), \mathfrak{P}(S \setminus T), \bar{\Sigma}_r^{S \setminus T})$. Since this need not to be true, we extend I to a closed relation J of the direct product context by defining $(X_1, X_2) J (Y_1, Y_2) :\Leftrightarrow (X_1, X_2) I (Y_1, Y_2)$ or $|X_2| \geq s - t + 1 - r$ or $|Y_2| \geq s - t + 1 - r$. Before proving the closeness of J, we show that $\mathfrak{B}(\mathbb{K}_I) \cong \mathfrak{B}(\mathbb{K}_J)$ for $\mathbb{K}_J := (\mathfrak{P}(T) \times \mathfrak{P}(S \setminus T), \mathfrak{P}(T) \times \mathfrak{P}(S \setminus T), J)$. Let $(X_1, X_2) \in \mathfrak{P}(T) \times \mathfrak{P}(S \setminus T)$ with $|X_2| \geq s - t + 1 - r$. Then $(X_1, X_2)^J = \mathfrak{P}(T) \times \mathfrak{P}(S \setminus T)$ so that (X_1, X_2) can be deleted from $\mathbb{K}_J$ without changing the structure of the concept lattice. If we can prove this also for $\mathbb{K}_I$, we obtain the desired isomorphy. We may assume that $|X_1| \leq s - r - |X_2|$ because otherwise $(X_1, X_2)^I = \mathfrak{P}(T) \times \mathfrak{P}(S \setminus T)$. Let $\mathfrak{Z}$ be the set of all (Z_1, Z_2) with $X_1 \subseteq Z_1 \subseteq T$, $Z_2 \subseteq X_2$, $|Z_1| = |X_1| + |X_2| - s + t + r$, and $|Z_2| = s - t - r$. Because of

$$|X_1| + |X_2| - s + t + r \leq s - r - |X_2| + |X_2| - s + t + r = t,$$

$\mathfrak{Z}$ is not empty. If $(X_1, X_2) I (Y_1, Y_2)$ then, for $(Z_1, Z_2) \in \mathfrak{Z}$, $X_1 \Delta Y_1$ would imply $Z_1 \Delta Y_1$ and $|X_1| + |Y_1| + |X_2| + |Y_2| \geq s + 1 - r$ would imply $|Z_1| + |Y_1| + |Z_2| + |Y_2| \geq s + 1 - r$; hence $(Z_1, Z_2) I (Y_1, Y_2)$ and so $(X_1, X_2)^I \subseteq (Z_1, Z_2)^I$. If $\neg(X_1, X_2) I (Y_1, Y_2)$ then $X_1 \cap Y_1 = \emptyset$ and $|X_1| + |Y_1| + |X_2| + |Y_2| \leq s - r$; therefore, because of

$$|Y_1| + |X_1| + |X_2| - s + t + r \leq s - r - s + t + r = t,$$

there exists a (Z_1, Z_2) in $\mathfrak{Z}$ with $Z_1 \cap Y_1 = \emptyset$ and so $\neg(Z_1, Z_2) I (Y_1, Y_2)$. This proves that

$$(X_1, X_2)^I = \bigcap_{(Z_1, Z_2) \in \mathfrak{Z}} (Z_1, Z_2)^I.$$

Thus, (X_1, X_2) can also be deleted from $\mathbb{K}_I$ and we obtain

$$\mathfrak{B}(\mathbb{K}_J) \cong \mathfrak{B}(\mathbb{K}_I) \cong \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$$

so that we can perform our structural analysis with the concept lattice of the context $\mathbb{K}_J$.

Let $(X_1, X_2) J (Y_1, Y_2)$ in $\mathbb{K}_J$. If $X_1 \cap Y_1 = \emptyset$ and $|X_2|, |Y_2| \leq s - t - r$ then $|X_1| + |Y_1| + |X_2| + |Y_2| \geq s + 1 - r$ and, because of $|X_1| + |Y_1| \leq t$, $|X_2| + |Y_2| \geq s - t + 1 - r$; hence $(X_1, X_2) \nabla (Y_1, Y_2)$ in the direct product context, i.e., $J \subseteq \nabla$.

Now, let $(\mathfrak{C}, \mathfrak{D}) \in \mathfrak{B}(\mathbb{K}_J)$ and let $(Y_1, Y_2) \in \mathfrak{C}^\nabla$. We want to show that $(Y_1, Y_2) \in \mathfrak{C}^J = \mathfrak{D}$. This is true if $|Y_2| \geq s-t+1-r$; so we assume that $|Y_2| \leq s-t-r$. Let $(X_1, X_2) \in \mathfrak{C}$ with $X_1 \cap Y_1 = \emptyset$. Suppose $t - |Y_1| \geq |X_1| + |X_2|$. Then there exists a $Z_1 \subseteq T$ with $X_1 \subseteq Z_1$, $Z_1 \cap Y_1 = \emptyset$ and $|Z_1| = |X_1| + |X_2|$. It follows that $(Z_1, \emptyset) \in \mathfrak{D}^J = \mathfrak{C}$ what contradicts $\neg(Z_1, \emptyset)\nabla(Y_1, Y_2)$. Thus, we have $t - |Y_1| < |X_1| + |X_2|$. This guarantees the existence of a $Z_2 \subseteq X_2$ with $|T \setminus Y_1| + |Z_2| = |X_1| + |X_2|$. Again we conclude that $(T\setminus Y_1, Z_2) \in \mathfrak{D}^J = \mathfrak{C}$. $(T \setminus Y_1, Z_2)\nabla(Y_1, Y_2)$ implies $|Z_2| + |Y_2| \geq s-t+1-r$ and so we obtain

$$\begin{aligned}|X_1| + |Y_1| + |X_2| + |Y_2| &= |T\setminus Y_1| + |Y_1| + |Z_2| + |Y_2| \\ &\geq t - |Y_1| + |Y_1| + s - t + 1 - r = s + 1 - r.\end{aligned}$$

This proves $(Y_1, Y_2) \in \mathfrak{C}^J = \mathfrak{D}$; hence $\mathfrak{C}^\nabla = \mathfrak{D}$ and symmetrically $\mathfrak{D}^\nabla = \mathfrak{C}$. Therefore $(\mathfrak{C}, \mathfrak{D})$ is a concept of $(\mathfrak{P}(T) \times \mathfrak{P}(S\setminus T), \mathfrak{P}(T)\times\mathfrak{P}(S\setminus T), \nabla)$ so that J is a closed relation of this context. Since $\mathrm{FBD}(T) \cong \mathfrak{B}(\mathfrak{P}(T), \mathfrak{P}(T), \Delta)$ and $C_{s-t+2-r} \cong \mathfrak{B}(\mathfrak{P}(S\setminus T), \mathfrak{P}(S\setminus T), \bar{\Sigma}_r^{S\setminus T})$, the concept lattice of $\mathbb{K}_J$ is isomorphic to a sublattice of $\mathrm{FBD}(t) \otimes C_{s-t+2-r}$ by Proposition 4 and Theorem 1 in [6]. By Section 2, we have $\hat{\iota}_T p = (\mathfrak{C}_p^T, \mathfrak{C}_p^T)$ for all $p \in T$ and $\hat{\iota}_T d_j = (\mathfrak{P}_{\geq t-j}(T), \mathfrak{P}_{\geq j}(T))$ for $j = 0, 1, \ldots, t$; furthermore

$$(\mathfrak{C}_p^T, \mathfrak{C}_p^T) \circledvee (\emptyset, \mathfrak{P}(S\setminus T)) = (\mathfrak{C}_p^T \times \mathfrak{P}(S\setminus T), \mathfrak{C}_p^T + \mathfrak{P}(S\setminus T)),$$

and

$$\begin{aligned}&\bigwedge_{j=0}^{k} (\mathfrak{P}_{\geq t-j}(T), \mathfrak{P}_{\geq j}(T)) \circledvee (\mathfrak{P}_{\geq s-t+1-r-k+j}(S\setminus T), \mathfrak{P}_{\geq k-j}(S\setminus T)) \\ &= \bigwedge_{j=0}^{k} ((\mathfrak{P}_{\geq j}(T) \times \mathfrak{P}_{\geq k-j}(S\setminus T))^\nabla, \mathfrak{P}_{\geq j}(T) \times \mathfrak{P}_{\geq k-j}(S\setminus T)) \\ &= \left(\bigcup_{j=0}^{k} \mathfrak{P}_{\geq t-j}(T) \times \mathfrak{P}_{\geq s-t+1-r-k+j}(S\setminus T), \bigcup_{j=0}^{k} \mathfrak{P}_{\geq j}(T) \times \mathfrak{P}_{\geq k-j}(S\setminus T)\right).\end{aligned}$$

Since the closed relation J is the union of the product sets

$$\begin{aligned}&(\mathfrak{C}_p^T \times \mathfrak{P}(S\setminus T)) \times (\mathfrak{C}_p^T \times \mathfrak{P}(S\setminus T)) \quad \text{for } p \in T, \\ &(\mathfrak{P}(T) \times \mathfrak{P}_{\geq s-t+1-r}(S\setminus T)) \times (\mathfrak{P}(T)\times\mathfrak{P}(S\setminus T), \\ &(\mathfrak{P}(T) \times \mathfrak{P}(S\setminus T)) \times (\mathfrak{P}(T) \times \mathfrak{P}_{\geq s-t+1-r}(S\setminus T)),\end{aligned}$$

and

$$\left(\bigcup_{j=0}^{k} \mathfrak{P}_{\geq t-j}(T) \times \mathfrak{P}_{\geq s-t+1-r-k+j}(S\setminus T)\right) \times \left(\bigcup_{j=0}^{k} \mathfrak{P}_{t \geq j}(T) \times \mathfrak{P}_{\geq k-j}(S\setminus T)\right)$$

$$\text{for } k = 1, \ldots, s-t-r,$$

again by Proposition 4, $\mathfrak{B}(\mathbb{K}_J)$ is generated by the concepts $(\mathfrak{C}_p^T, \mathfrak{C}_p^T) \circledvee (\emptyset, \mathfrak{P}(S\setminus T))$ with $p \in T$ and

$$\bigwedge_{j=0}^{k} (\mathfrak{P}_{\geq t-j}(T), \mathfrak{P}_{\geq j}(T)) \circledvee (\mathfrak{P}_{\geq s-t+1-r-k+j}(S\setminus T), \mathfrak{P}_{\geq k-j}(S\setminus T)).$$

Now the isomorphisms above yield the assertion of the proposition. □

Corollary 8. *For* $s - t = r$, $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S) \cong \mathrm{FBD}(t)$.

4. Skeletons as unions of free distributive sublattices

In this final section we discuss which skeletons of the free bounded distributive lattices $\mathrm{FBD}(n)$ can be determined by the distinguished sublattices described in Section 3. For a finite set S we define a *bicover of degree r with bound k* to be a pair $(\mathfrak{X}, \mathfrak{Y})$ with $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{P}(S)$ such that, for each $R \subseteq S$ with $|R| = r$, there are $X_R \in \mathfrak{X}$ and $Y_R \in \mathfrak{Y}$ with $X_R \cap Y_R \subseteq R$ and $|X_R| + |Y_R| \leq k$ and, for $X \in \mathfrak{X}$ and $Y \in \mathfrak{Y}$, $X \cap Y \neq \emptyset$ or $|X| + |Y| > k$. Let $\mathrm{bic}_r(S)$ $(=: \mathrm{bic}_r(|S|))$ be the smallest number k for which S admits a bicover of degree r with bound k.

Proposition 9. *For a non-empty finite set S and for* $r = 1, 2, 3, \ldots, |S|$, $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$ *is the union of all* $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ *with* $|T| = |S| - r$ *if and only if* $|S| - r < \mathrm{bic}_r(S)$.

Proof. Let $|S| - r \geq \mathrm{bic}_r(S)$. Then S admits a bicover $(\mathfrak{X}, \mathfrak{Y})$ of degree r with bound $|S| - r$ because a bicover of degree r with bound k can always be extended to a bicover of degree r with bound $k + 1$. Since $X(\Delta \cup \bar{\Sigma}_r^S)Y$ for all $X \in \mathfrak{X}$ and $Y \in \mathfrak{Y}$, there is a concept $(\mathfrak{C}, \mathfrak{D})$ of $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$ with $\mathfrak{X} \subseteq \mathfrak{C}$ and $\mathfrak{Y} \in \mathfrak{D}$. There does not exist a $T := S \setminus R$ with $|R| = r$ and $(\mathfrak{C}, \mathfrak{D}) \in \mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \in \bar{\Sigma}_r^S)$ because, for $X_R \in \mathfrak{X} \subseteq \mathfrak{C}$ and $Y_R \in \mathfrak{Y} \subseteq \mathfrak{D}$ with $X_R \cap Y_R \subseteq R$,

$$X_R \cap T \cap Y_R = \emptyset \quad \text{and} \quad |X_R| + |Y_R| \leq |S| - r.$$

Thus, $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r^S)$ is not the union of all $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ with $|T| = |S| - r$. Let us, conversely, start with this statement as assumption. Then there is a concept $(\mathfrak{C}, \mathfrak{D})$ of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_r)$ which is not a concept of $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_r^S)$ if $|T| = |S| - r$. Hence, for each $R \subseteq S$ with $|R| = r$, there exist $X_R \in \mathfrak{C}$ and $Y_R \in \mathfrak{D}$ with $X_R \cap (S \setminus R) \cap Y_R = \emptyset$ and $|X_R| + |Y_R| \leq |S| - r$. Thus, $(\mathfrak{C}, \mathfrak{D})$ is a bicover of degree r with bound $|S| - r$ and so $|S| - r \geq \mathrm{bic}_r(S)$. □

From Theorem 1 we can deduce that each generator e_i of $\mathrm{FBD}(n)$ is contained in a unique block $[e_i]_1$ of $\Sigma\,(\mathrm{FBD}(n))$ and, moreover, $[e_i]_r$ is contained in a unique block $[e_i]_{r+1}$ of $\Sigma\,(S_r(\mathrm{FBD}(n))$ for $r = 1, 2, 3, \ldots$. By Corollary 8, each $(n - r)$-element subset of $\{[e_1]_r, \ldots, [e_n]_r\}$ generate in $S_r(\mathrm{FBD}(n))$ a sublattice isomorphic to $\mathrm{FBD}(n - r)$. When these sublattices cover $S_r(\mathrm{FBD}(n))$, this describes the following corollary of Proposition 9.

Corollary 10. *$S_r(\mathrm{FBD}(n)) = \bigcup\{\langle[e_{i_1}]_r, \ldots, [e_{i_{n-r}}]_r\rangle \mid 1 \leqslant i_1 < \cdots < i_{n-r} \leqslant n\}$ if and only if $n - r < \mathrm{bic}_r(n)$; additionally, we have $\langle[e_{i_1}]_r, \ldots, [e_{i_{n-r}}]_r\rangle \cong \mathrm{FBD}(n-r)$.*

Since $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ and $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ form a bicover of degree 1 with bound 5 for the set $\{1, \ldots, 6\}$, the skeleton $S_1(\mathrm{FBD}(6))$ is not the union of six copies of FBD(5) by Corollary 10. But, for $n \leqslant 5$, the skeletons of FBD(n) can be constructed as unions of free distributive sublattices because $n - r < \mathrm{bic}_r(n)$ for $1 \leqslant r \leqslant n \leqslant 5$ which can be easily verified. How the results of Section 2 and 3 can be applied to describe these skeletons, this we demonstrate only in the case $n = 5$ (line diagrams of the skeletons $S_r(\mathrm{FBD}(4))$ can be found in [7]).

Let $S := \{1, 2, 3, 4, 5\}$; then $S_1(\mathrm{FBD}(5)) \cong \underline{\mathfrak{B}}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_1^S)$ by Theorem 1. $\underline{\mathfrak{B}}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_1^S)$ is, by Proposition 9, the union of the five sublattices $\underline{\mathfrak{B}}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_1^S)$ with $|T| = 4$. By Corollary 8, these sublattices are isomorphic copies of FBD(4), a line diagram of which can be seen in [7]. By Proposition 5 and 6, $T \mapsto \underline{\mathfrak{B}}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T, \bar{\Sigma}_1^S)$ describes a $\cap$-preserving map

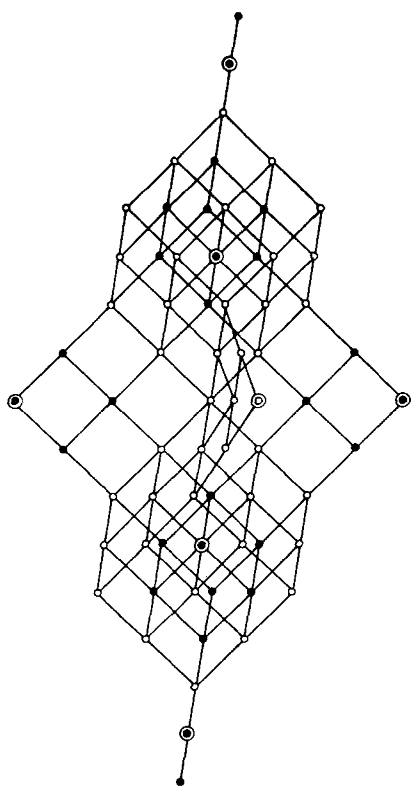

Fig. 1. $\langle[e_1]_1, [e_2]_1, [e_3]_1, [e_4]_1\rangle \cap \langle[e_2]_1, [e_3]_1, [e_4]_1, [e_5]_1\rangle$ in $S_1(\mathrm{FBD}(5))$.

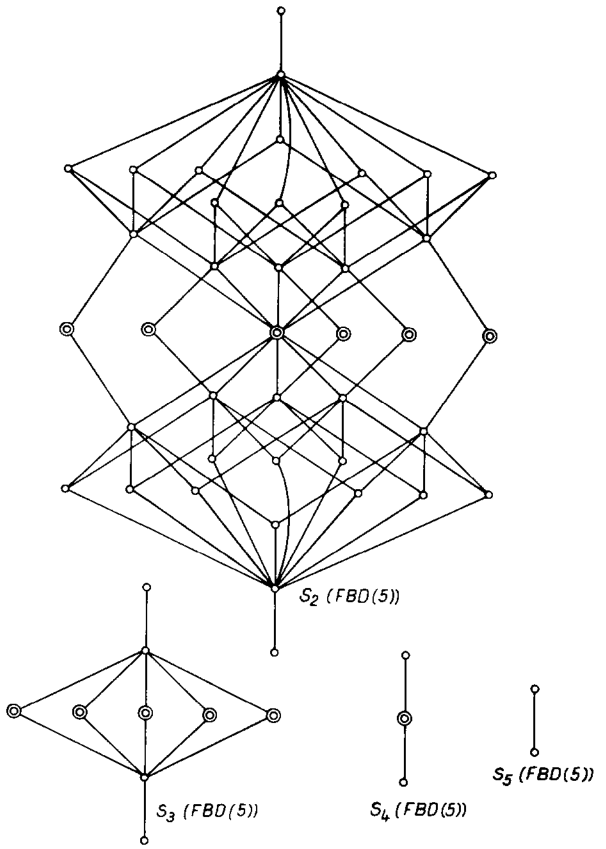

Fig. 2.

from $\mathfrak{P}(S)$ onto a closure system on $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \bar{\Sigma}_1^S)$ consisting of 0–1-sublattices. These sublattices can be determined via Proposition 7; for $|T| = 3$, $\mathfrak{B}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta_T \cup \bar{\Sigma}_1^S)$ is shown in Fig. 1 by a line diagram in which the black circles indicate a sublattice with $|T| = 2$ (the cases $|T| = 1$ and $|T| = 0$ can also be read from the diagram). Thus, it becomes clear how to glue five copies of FDB(4) together to obtain the skeleton S_1(FBD(5)). By the inclusion-exclusion-formula, we get $|S_1(\text{FBD}(5))| = 5\cdot 168 - 10\cdot 69 + 10\cdot 28 - 5\cdot 12 + 6 = 376$ which is also the number of maximal antichains in $\mathfrak{P}(\{1, 2, 3, 4, 5\})$ (cf. [7]). The skeletons S_r(FBD(5)) for $2 \leqslant r \leqslant 5$ are shown in Fig. 2.

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