Poisson structures on tangent bundles

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Abstract

The paper starts with an interpretation of the complete lift of a Poisson structure from a manifold $M$ to its tangent bundle $TM$ by means of the Schouten–Nijenhuis bracket of covariant symmetric tensor fields defined by the cotangent Lie algebroid of $M$. Then, we discuss Poisson structures of $TM$ which have a graded restriction to the fiberwise polynomial algebra; they must be $\pi$-related ($\pi : TM \rightarrow M$) with a Poisson structure on $M$. Furthermore, we define transversal Poisson structures of a foliation, and discuss bivector fields of $TM$ which produce graded brackets on the fiberwise polynomial algebra, and are transversal Poisson structures of the foliation by fibers. Finally, such bivector fields are produced by a process of horizontal lifting of Poisson structures from $M$ to $TM$ via connections.

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1. The complete lift of a Poisson structure

Let $M$ be an $n$-dimensional differentiable manifold with local coordinates $(x^i) (i = 1, \ldots, n)$, $\pi : TM \rightarrow M$ its tangent bundle and $(y^i)$ the vector coordinates with respect to the basis $\{\partial / \partial x^i\}$. (We assume that everything is $C^\infty$ in this paper.)

Let us consider a Poisson structure on the manifold $M$, given by the Poisson bivector

$$w = \frac{1}{2} w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \quad \quad \quad (1.1)$$

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(throughout the paper, we use Einstein’s summation convention). The complete lift of $w$ in the sense of [16] is given by

$$w^C = w^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2} \gamma^k \frac{\partial w^{ij}}{\partial x^k} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j},$$

(1.2)

and it follows easily that $w^C$ is a Poisson bivector field on $TM$ since the Poisson condition, namely, that the Schouten-Nijenhuis bracket $[w^C, w^C] = 0$ [12], is satisfied.

The Poisson bivector $w^C$ has already been studied by several authors [2,4,5,13,14], and it can also be derived from the bracket of the 1-forms of $M$ with respect to the Poisson structure $w$ (e.g., [12])

$$\{\alpha, \beta\} = \mathcal{L}_\alpha \beta - \mathcal{L}_\beta \alpha - d(w(\alpha, \beta)) \quad (\alpha, \beta \in \Omega^1(M)).$$

(1.3)

A Pfaff form $\alpha = \alpha \, dx^i$ on $M$ may be regarded as a fiberwise linear function $l(\alpha) := \alpha(x) y^i$ on $TM$ (denotes a definition). A Poisson structure $W$ on $TM$ is completely determined by the brackets $\{f \circ \pi, g \circ \pi\}_w, \{l(\alpha), f \circ \pi\}_w$ and $\{l(\alpha), l(\beta)\}_w$, where $f, g \in C^\infty(M)$ and $\alpha, \beta \in \Omega^1(M)$, the space of Pfaff forms on $M$, since it is completely determined by the brackets of the local coordinates $x^i$ and $y^j$.

The Poisson bivector $w^C$ is exactly the one defined by:

(i) $\{f \circ \pi, g \circ \pi\}_w = 0, \forall f, g \in C^\infty(M)$;
(ii) $[l(\alpha), f \circ \pi]_w = (\alpha^\sharp f) \circ \pi, \forall f \in C^\infty(M), \forall \alpha \in \Omega^1(M)$, where $\sharp_w : T^*M \rightarrow TM$ is defined by $\beta(\alpha^\sharp) = w(\alpha, \beta), \forall \beta \in \Omega^1(M)$;
(iii) $[l(\alpha), l(\beta)]_w = l([\alpha, \beta]), \forall \alpha, \beta \in \Omega^1(M)$.

$w^C$ is a Poisson structure because the bracket (1.3) satisfies the Jacobi identity.

The Poisson structure $w^C$ also has the interesting property

$$w^C = -\mathcal{L}_E w^C, \quad E = y^i \frac{\partial}{\partial y^i},$$

($E$ is the Euler vector field), which means that $(TM, w^C)$ is a homogeneous Poisson manifold [13].

We remind that a Lie algebroid over a manifold $M$ is a triple $(A, [\, , \, ]_A, \sigma)$, where $\rho : A \rightarrow M$ is a vector bundle, $[\, , \, ]_A$ is an $\mathbb{R}$-Lie algebra structure on the space $\Gamma A$ of the global cross sections of $A$ and $\sigma : A \rightarrow TM$ is a morphism of vector bundles, called anchor, such that

(i) $\sigma([s_1, s_2]_A) = [\sigma(s_1), \sigma(s_2)],$ 
(ii) $[s_1, f s_2]_A = f [s_1, s_2]_A + ((\sigma(s_1)) f) s_2$

for every $s_1, s_2 \in \Gamma A, f \in C^\infty(M)$, and where $[\, , \, ]$ is the Lie bracket of vector fields on $M$.

In what follows, we give one more interpretation of the Poisson structure $w^C$ by means of a Schouten-Nijenhuis bracket on a Lie algebroid $A$.

There exists a well known operation, called the Schouten-Nijenhuis bracket, on cross sections of $\mathcal{V}(A) := \bigoplus_k \Gamma \wedge^k A$ (e.g., see [7]). A less popular operation, the Schouten-Nijenhuis bracket of symmetric tensors also exists, and was studied in an algebraic context and for $TM$ [1]. Here we present this second operation on the algebra of cross sections $S(A) = \bigoplus_{k \geq 0} S_k(A)$, where $S_k(A) = \Gamma \odot^k A$, $A$ is a Lie algebroid, $\odot$ denotes the symmetric tensor product, and $\Gamma$ denotes spaces of global cross sections of bundles. Then, we show that this operation leads to another definition of the complete lift $w^C$.

For any Lie algebroid $A$, one has the Lie derivative [1] which is defined by putting $L^A_{s'} f = \mathcal{L}_{\sigma(s')} f$ for functions $f \in C^\infty(M)$, and $L^A_{s'} s' = [s, s']_A$ for cross sections $s' \in \Gamma A$, and by extending it to arbitrary
cross sections of $\Gamma((\otimes^k A) \otimes (\otimes^l A^*))$ as in the case of the classical Lie derivative. In particular, we have the restriction $L_A^k : S_k(A) \to S_k(A)$.

**Proposition 1.1.** There exists a well defined unique extension of local type of the Lie derivative $L_A$ to an $R$-bilinear operation

$$\langle \cdot, \cdot \rangle : S_p(A) \times S_q(A) \to S_{p+q-1}(A) \quad (p, q \geq 1),$$

such that

$$\langle s_1 \circ \cdots \circ s_p, t_1 \circ \cdots \circ t_q \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} [s_i, t_j]_A \circ s_1 \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_p \circ t_1 \circ \cdots \circ \hat{t}_j \circ \cdots \circ t_q,$$

(1.4)

where the hat denotes the absence of the corresponding factor.

**Proof.** “Local type” means that $\forall x_0 \in M$ and $\forall G \in S_p(A), H \in S_q(A), \langle G, H \rangle(x_0)$ depends only on the restriction of the cross sections $G, H$ to a neighborhood of $x_0$.

Since, around $x_0 \in M$ $G, H$ are decomposable into finite sums of products of the form appearing in (1.4), the required extension has an obvious definition, and we only must show that the result does not depend on the decomposition.

If $H = t_1 \circ \cdots \circ t_q$, formula (1.4) becomes

$$\langle s_1 \circ \cdots \circ s_p, H \rangle = \sum_{i=1}^{p} (L_A^s_i H) \circ s_1 \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_p.$$

(1.5)

Similarly, if $G = s_1 \circ \cdots \circ s_p$ then

$$\langle G, t_1 \circ \cdots \circ t_q \rangle = -\sum_{j=1}^{q} (L_A^t_j G) \circ t_1 \circ \cdots \circ \hat{t}_j \circ \cdots \circ t_q.$$

(1.6)

Formulas (1.5), (1.6) show the required independence of the bracket $\langle G, H \rangle$ of the decomposition of $G, H$. $\square$

Note that we may also consider (1.5), (1.6) as the definition of the bracket $\langle G, H \rangle$.

Now, we may extend the bracket $\langle \cdot, \cdot \rangle$ to the case where the factors belong to $S_0(A) = C^\infty(M)$. Namely, we will put

$$\langle g, h \rangle = 0,$$

$$\langle s_1 \circ \cdots \circ s_p, f \rangle = -\langle f, s_1 \circ \cdots \circ s_p \rangle = \sum_{i=1}^{p} (L_A^s_i f) s_1 \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_p,$$

(1.7)

$\forall f, g, h \in C^\infty(M), s_i \in \Gamma A$. The fact that the second formula (1.7) does not depend on the decomposition $G = s_1 \circ \cdots \circ s_p$ follows by noticing that

$L_A^s f = (d_A f)(s_i)$.
where $d_A$ is the exterior differential for the Lie algebroid $A$ \cite{7}, whence

\begin{equation}
\langle G, f \rangle = i (d_A f) G,
\end{equation}

where the definition used for the operator $i$ is that of \cite{6}.

**Proposition 1.2.** The bracket $\langle , \rangle$ has the following properties

\begin{equation}
\langle H, G \rangle = - \langle G, H \rangle,
\end{equation}

\begin{equation}
\langle G, H \odot K \rangle = \langle G, H \rangle \odot K + H \odot \langle G, K \rangle,
\end{equation}

$\forall G, H, K \in S(A)$.

**Proof.** The bracket $\langle , \rangle$ is extended to $S(A)$ by $\mathbb{R}$-bilinearity. Both relations easily follow from (1.4) and (1.7). $\square$

**Proposition 1.3** (The Jacobi identity). $\forall F, G, H \in S(A)$,

\begin{equation}
\langle \langle G, H \rangle, K \rangle + \langle \langle H, K \rangle, G \rangle + \langle \langle K, G \rangle, H \rangle = 0.
\end{equation}

**Proof.** It suffices to prove (1.11) for decomposable $G, H, K$ and this follows by a technical computation based on the Jacobi identity satisfied by the bracket of the cross sections of $A$. $\square$

**Corollary 1.4.** $(S(A), \langle , \rangle)$ is a Poisson algebra \cite{1} with respect to the symmetric product $\odot$ and the Schouten–Nijenhuis bracket $\langle , \rangle$.

Let us consider the particular case of the cotangent Lie algebroid $(T^*M, \{ , \}, \#^w)$ of a Poisson manifold $(M, w)$, where the bracket is that defined by formula (1.3).

Then

\begin{equation}
S(T^*M) := \bigoplus_k S_k(T^*M)
\end{equation}

is the algebra of the covariant symmetric tensor fields on $M$, and Corollary 1.4 shows that $(S(T^*M), \langle , \rangle)$ is a Poisson algebra with respect to the symmetric product $\odot$ and the Schouten–Nijenhuis bracket $\langle , \rangle$ of the Lie algebroid $T^*M$.

Notice that, in the present case $d_{T^*M} f = - X_f$, where $X_f$ is the $w$-Hamiltonian vector field of $f$. Accordingly, (1.8) yields

\begin{equation}
\langle G, f \rangle = - i_{X_f} G \quad (f \in C^\infty(M), \ G \in S_k(T^*M)).
\end{equation}

The function space $C^\infty(TM)$ has some interesting subspaces. Namely, the spaces of fiberwise homogeneous $k$-polynomials

\begin{equation}
\mathcal{H}(T^k TM) := \{ \tilde{G} = G_{i_1 \ldots i_k} y^{i_1} \ldots y^{i_k} \mid G = G_{i_1 \ldots i_k} d x^{i_1} \odot \cdots \odot d x^{i_k} \in S_k(T^*M) \},
\end{equation}

and we have an isomorphism of algebras

\begin{equation}
\iota : (S(T^*M), \odot) \rightarrow \left( \mathcal{P}(TM) := \bigoplus_k \mathcal{H}(T^k TM), \cdot \right)
\end{equation}
mapping $G$ to $\tilde{G} := \iota(G)$ (the dot denotes usual multiplication). With this isomorphism, the bracket $\{ , \}$ of symmetric covariant tensor fields is translated into a bracket of polynomials. Moreover, since the local coordinates $x^i, y^j$ are polynomials of degree zero and one respectively, this bracket defines a Poisson structure with a Poisson bivector, say $W$, on $TM$; the bracket will be denoted by $\{,\}_W$.

**Proposition 1.5.** The Poisson structure $W$ defined on the tangent bundle of a Poisson manifold $(M, w)$ by the bracket $\{,\}_W$ coincides with the Poisson structure $w^C$.

**Proof.** The brackets $\{x^i, x^j\}_W$, $\{x^i, y^j\}_W$ and $\{y^i, y^j\}_W$ computed with $\{ , \}$, are the same as those produced by (1.2).

**Corollary 1.6.** If $G$ and $H$ are symmetric covariant tensor fields on $M$, then

$$\tilde{\langle G, H \rangle} = \{\tilde{G}, \tilde{H}\}_{w^C},$$

and (1.15) is a corresponding isomorphism of Poisson algebras.

In the remaining part of this section we will compute the modular class of the Poisson structure $w^C$.

If $\mu$ is a volume form on an orientable manifold $M$, the divergence $\text{div}_\mu X$ of a vector field $X$ is defined by the condition

$$L_X\mu = (\text{div}_\mu X)\mu,$$

and one has

$$\text{div}_\mu(fX) = f \text{div}_\mu X + Xf, \quad f \in C^\infty(M).$$

Accordingly, if $(M, w)$ is a Poisson manifold endowed with a volume form $\mu$, the operator

$$\Delta_\mu: f \in C^\infty(M) \mapsto \text{div}_\mu X_f \in C^\infty(M)$$

is a derivation on $C^\infty(M)$, so it is a vector field on the manifold $M$, called the modular vector field of $(M, w, \mu)$ (see [8,15]).

Denote by $\mathcal{V}^i(M)$ the space of $i$-vector fields of a manifold $M$, i.e., skew symmetric contravariant tensor fields of type $(i, 0)$ on $M$, and $\mathcal{V}(M) = (\bigoplus_{i=1}^n \mathcal{V}^i(M), \wedge)$ the contravariant Grassmann algebra of $M$. On a Poisson manifold $(M, w)$, the Lichnerowicz–Poisson coboundary operator is

$$\sigma := -[w, , ]: \mathcal{V}^k(M) \to \mathcal{V}^{k+1}(M),$$

where $[ , ]$ is the Schouten–Nijenhuis bracket, and one has the Lichnerowicz–Poisson (LP) cohomology spaces (e.g., [12])

$$H^k_{LP}(M, w) = \frac{\text{Ker}(\sigma: \mathcal{V}^k(M) \to \mathcal{V}^{k+1}(M))}{\text{Im}(\sigma: \mathcal{V}^{k-1}(M) \to \mathcal{V}^k(M))}.$$ 

For a modular vector field one has $\sigma \Delta_\mu = 0$ [8], and $\Delta_\mu$ is a 1-cocycle. Therefore it defines a 1-dimensional LP-class $\Delta = [\Delta_\mu] \in H^1_{LP}(M, w)$. It is easy to see that this class does not depend on $\mu$; it is called the modular class of the Poisson manifold $(M, w)$ [8,15].

We want to discuss the relation between the modular classes of $(M, w)$ and of $(TM, w^C)$.
Let $g$ be a Riemannian metric on the oriented manifold $M$. Then
\[
dV_g = \sqrt{\det g} \, dx^1 \wedge \cdots \wedge dx^n
\]
is a volume form on $M$, and it follows easily that
\[
\Phi = (\det g) \, dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n
\]
is a volume form on $TM$ (the volume form of the Sasaki metric associated to $g$ [3]).

**Proposition 1.7.** The modular vector field of $(TM, w^C, \Phi)$ is given by
\[
\Delta_{\Phi}^{TM} = 2(\Delta_{dV_g})^V,
\]
where the upper index $V$ denotes the vertical lift in the sense of [16].

**Proof.** With (1.1), a Hamiltonian vector field on $(M, w)$ has the form
\[
X^w_j = \{ f, \cdot \}_w = \frac{\partial f}{\partial x^i} w^i_j \frac{\partial}{\partial x^j} \quad (f \in C^\infty(M)),
\]
and the definition of the modular vector field leads to
\[
\Delta_{dV_g} = \sum_{k=1}^n \left( \frac{\partial w^i_k}{\partial x^k} + w^i_k \frac{\partial \ln \sqrt{\det g}}{\partial x^k} \right) \frac{\partial}{\partial x^i}.
\]

Then, if $F \in C^\infty(TM)$, (1.2) gives for the Hamiltonian vector field $X^w_F$ the expression
\[
X^w_F = \frac{\partial F}{\partial y^k} w^i_k \frac{\partial}{\partial x^i} + \left( \frac{\partial F}{\partial x^k} w^i_k + \frac{\partial F}{\partial y^h} \frac{\partial w^{hi}}{\partial x^k} \right) \frac{\partial}{\partial y^i},
\]
and a straightforward computation yields the modular vector field
\[
\Delta_\Phi = \sum_{k=1}^n \left( \frac{\partial w^i_k}{\partial x^k} + w^i_k \frac{\partial \ln \sqrt{\det g}}{\partial x^k} \right) \frac{\partial}{\partial y^i}.
\]
This exactly is the required result. \qed

**Corollary 1.8.** The modular class of the Poisson manifold $(TM, w^C)$ is represented by $2\Delta_{dV_g}^V$, for every modular vector field $\Delta_{\mu}$ of the base manifold $(M, w)$.

**Proof.** Proposition 1.7 shows that the result is true for the field $\Delta_{dV_g}$. Since from (1.20), (1.22) one also gets
\[
(\sigma_{w^C})^V = \sigma_{w^C} \circ (f \circ \pi), \quad f \in C^\infty(M), \quad \pi : TM \rightarrow M,
\]
the result is true for any other modular vector field of $(M, w)$. \qed

It is also worthwhile to notice the following result.

**Proposition 1.9.** The complete lift of multivector fields induces a homomorphism of cohomology algebras
\[
[\mathcal{Q}] \in H^*_{LP}(M, w) \mapsto [\mathcal{Q}^C] \in H^*_{LP}(TM, w^C).
\]
Proof. The complete lift of multivector fields is the natural extension of the complete lift of vector fields, and is compatible with the Lie bracket [16]. Therefore, since the Schouten–Nijenhuis bracket extends the Lie bracket, e.g., [12], if \( Q_1, Q_2 \in \mathcal{V}(M) \),
\[
[Q_1, Q_2]^C = [Q_1^C, Q_2^C],
\]
where the bracket is the Schouten–Nijenhuis bracket. This implies
\[
(\sigma_w Q)^C = \sigma_w Q^C,
\]
and it follows that (1.24) is a homomorphism. ✷

2. Graded Poisson structures on tangent bundles

Recall that on \( TM \) we have the spaces of fiberwise polynomial functions \( \mathcal{H}P_k \) given by (1.14) and the polynomial algebra \( \mathcal{P}(TM) \) introduced in (1.15). Denote by
\[
\mathcal{P}_k(TM) := \bigoplus_{h=0}^{k} \mathcal{H}P_h,
\]
the space of fiberwise non homogeneous polynomials of degree \( \leq k \).

In particular, we have the space \( \mathcal{A}(TM) := \mathcal{P}_1(TM) \) of affine functions
\[
a = f \circ \pi + l(\alpha), \quad f \in C^{\infty}(M), \quad \alpha \in \Omega^1(M),
\]
where \( l(\alpha) \) was defined in Section 1, and the space \( \mathcal{P}_2(TM) \) of non-homogeneous quadratic polynomials:
\[
p = f \circ \pi + l(\alpha) + s(G)
\]
where \( G = G_{ij} dx^i \otimes dx^j \) is a symmetric covariant tensor field on \( M \) and \( s(G) := \tilde{G} \) is defined in (1.14). Here and in the whole paper, when speaking of polynomials on \( TM \), we always mean fiberwise polynomials.

Definition 2.1. A Poisson structure \( W \) on \( TM \) is called polynomially graded if \( \mathcal{P}(TM) \) is closed by Poisson brackets and \( \forall F, G \in \mathcal{P}(TM) \)
\[
F \in \mathcal{P}_h, \quad G \in \mathcal{P}_k \Rightarrow [F, G]^W \in \mathcal{P}_{h+k}.
\]

Proposition 2.2. A polynomially graded Poisson structure \( W \) on \( TM \) induces a Poisson structure \( w \) on the base manifold \( M \), such that the projection \( \pi : (TM, W) \rightarrow (M, w) \) is a Poisson mapping.

Proof. If \( f \in C^{\infty}(M) \), \( f = f \circ \pi \) is a polynomial of degree zero on \( TM \). Thus, by (2.1), \( \forall f, g \in C^{\infty}(M) \), and
\[
\{f, g\}_w := \{f \circ \pi, g \circ \pi\}_W.
\]
defines a Poisson structure \( w \) on \( M \). ✷

Hereafter, we write \( f \) for both \( f \in C^{\infty}(M) \), and \( f \circ \pi \in C^{\infty}(TM) \). The bracket \( \{ , \}_W \) will be denoted simply by \( \{ , \} \).
Proposition 2.2 tells us that the polynomially graded Poisson structures $W$ of $TM$ (if any) are lifts of Poisson structures $w$ of $M$, i.e., $\pi: (TM, W) \rightarrow (M, w)$ is a Poisson mapping. We suggest that the general problem of looking for lifts of Poisson structures of a manifold to its tangent bundle is an interesting problem.

The polynomially graded Poisson structure $W$ is completely determined if, along with the brackets $\{f, g\}$, we also define the brackets $\{l(\alpha), f\}$ and $\{l(\alpha), l(\beta)\}$, where $\alpha, \beta \in \Omega^1(M)$.

By (2.1), the bracket $\{l(\alpha), f\} \in \mathcal{P}_1(M)$, i.e.,

$$\{l(\alpha), f\} = X_\alpha f + l(\gamma_\alpha f),$$

(2.3)

where $X_\alpha f \in C^\infty(M)$ and $\gamma_\alpha f \in \Omega^1(M)$.

Since $\{l(\alpha), \cdot\}|_{C^\infty(M)}$ is a derivation of $C^\infty(M)$, it follows that $X_\alpha$ is a vector field on $M$, and the mapping $\gamma_\alpha: C^\infty(M) \rightarrow \Omega^1(M)$ also is a derivation. Therefore, $\gamma_\alpha f$ only depends on $df$.

The Leibniz rule implies

$$\{l(h\alpha), f\} = h(X_\alpha f) + l((X_\alpha f)\alpha + h(\gamma_\alpha f)).$$

(2.4)

Hence $\gamma$ must satisfy

$$\gamma(h\alpha) = h\gamma\alpha + \left(X_\alpha f\right)\alpha.$$ (2.5)

Similarly, the bracket $\{l(\alpha), l(\beta)\}$ must have an expression of the form

$$\{l(\alpha), l(\beta)\} = U(\alpha, \beta) + l(\Phi(\alpha, \beta)) + s(\Psi(\alpha, \beta)),$$

(2.6)

where

$$U(\alpha, \beta) \in C^\infty(M), \quad \Phi(\alpha, \beta) \in \Omega^1(M), \quad \Psi(\alpha, \beta) \in S_2(T^*M)$$

are skew-symmetric operators. A replacement of $\beta$ by $f\beta$ in (2.5) leads to

$$U(\alpha, f\beta) = fU(\alpha, \beta),$$

whence, $U$ is a bivector field on $M$, and

$$\Phi(\alpha, f\beta) = f\Phi(\alpha, \beta) + (X_\alpha f)\beta, \quad \Psi(\alpha, f\beta) = f\Psi(\alpha, \beta) + \gamma_\alpha f \odot \beta.$$ (2.7)

On the other hand, if $(x^i)$ are local coordinates on $M$, if $y^i = l(dx^i)$, and if $w$ is the Poisson structure introduced by Proposition 2.2, Definition 2.1 tells us that the local coordinate expression of $W$ must be of the form

$$W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + (\varphi^{ij}(x) + y^a A_a^{ij}(x)) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2} (\eta^{ij}(x) + y^a \chi_a^{ij}(x) + y^a y^b B_a^{ij}(x)) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j},$$

(2.8)

where $w, \varphi, A, \eta, \chi, B$ are local functions on $M$.

**Definition 2.3.** A polynomially graded Poisson structure $W$ on $TM$ is said to be a graded structure if $\forall F \in \mathcal{H}_h, \forall G \in \mathcal{H}_k, \{F, G\}_W \in \mathcal{H}_{h+k}$. 


The conditions for a polynomially graded structure on $TM$ to be graded are $X_\alpha = 0$, $U = 0$, $\Phi = 0$, and then (2.7) reduces to

$$W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + y^a A^{ij}_a(x) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2} y^a y^b B^{ij}_{ab}(x) \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b}. \quad (2.8)$$

For a later utilization, we also give

**Definition 2.4.** A bivector field $W$ on $TM$ which is locally of the form (2.7) (respectively, (2.8)) is called a polynomially graded (respectively, graded) bivector field.

In this case we may speak of a skew-symmetric bracket

$$\{F, G\}_W := W(df, dG) \quad (F, G \in C^\infty(TM)) \quad (2.9)$$

which satisfies the Leibniz rule, but, generally, not the Jacobi identity.

**Proposition 2.5.** If $W$ is a graded Poisson structure on $TM$, the equality

$$\{l(\alpha), f\} = -l(D_{df} \alpha), \quad \alpha \in \Omega^1(M), \ f \in C^\infty(M) \quad (2.10)$$

defines a flat contravariant connection on the Poisson manifold $(M, w)$.

**Proof.** By a contravariant connection on $(M, w)$ we understand a contravariant derivative on the bundle $T^*M$ with respect to the Poisson structure [12].

With (2.3), condition (2.10) means that

$$D_{df} \alpha := -\gamma_a f, \quad (2.11)$$

and (2.4) has the equivalent form

$$D_{df}(h\alpha) = hD_{df} \alpha + (df)^* h\alpha, \quad \alpha \in \Omega^1(M), \ f, h \in C^\infty(M),$$

which is the characteristic property of a contravariant connection on a Poisson manifold.

Let us extend (2.11) by

$$D_{g(df)} \alpha := g D_{df} \alpha, \quad \forall g \in C^\infty(M). \quad (2.12)$$

The extension is correct because it is compatible with definition (2.10): if $g(df) = dh \quad (h \in C^\infty(M))$, then

$$-l(D_{dh} \alpha) = \{l(\alpha), h\} = W(dl(\alpha), dh) = g W(dl(\alpha), df) = g \{l(\alpha), f\} = -g l(D_{df} \alpha),$$

hence $D_{dh} \alpha = g D_{df} \alpha$ as needed.

The curvature of this connection is [12]

$$C_{P}(df, dg) \alpha = D_{df} D_{dg} \alpha - D_{dg} D_{df} \alpha - D_{[df, dg]} \alpha,$$

and it is easy to see that its annulation is equivalent to the Jacobi identity

$$\{\{l(\alpha), f\}, g\} + \{\{f, g\}, l(\alpha)\} + \{\{g, l(\alpha)\}, f\} = 0. \quad \Box \quad (2.13)$$
Remark 2.6. For any polynomially graded bivector field $W$ such that the first term of (2.7) is a Poisson bivector on $M$, it follows similarly that (2.11) and (2.12) define a contravariant connection $D$ on $(M, w)$ but, generally, its curvature is not zero.

Now, let us make some remarks concerning the operator $\Psi$ of a graded Poisson structure on $TM$, where
\[
\{l(\alpha), l(\beta)\} = s(\Psi(\alpha, \beta)). \tag{2.14}
\]
With (2.10), the second relation (2.6) becomes
\[
\Psi(\alpha, f\beta) = f\Psi(\alpha, \beta) - \frac{1}{2} (D_{df}\alpha \otimes \beta + \beta \otimes D_{df}\alpha). \tag{2.15}
\]
Hence, $\Psi : T^*M \times T^*M \to \otimes^2 T^*M$ is a bidifferential operator of the first order.

The relation (2.15) allows us to derive the local coordinate expression of $\Psi$. Put
\[
D_{dx^i} dx^j = \Gamma^j_{ki} dx^k, \quad \alpha = \alpha_i dx^i, \quad \beta = \beta_j dx^j. \tag{2.16}
\]
It follows that
\[
\Psi(\alpha, \beta) = \alpha_i \beta_j \Psi(\alpha_i, \alpha_j) + \left( \Gamma^k_{ij} \partial_{\alpha_i} \partial_{\beta_j} - \Gamma^k_{ij} \partial_{\alpha_j} \partial_{\beta_i} \right) dx^k \odot dx^q
+ w_{kh} \partial_{\alpha_p} \partial_{\beta_q} dx^p \odot dx^q. \tag{2.17}
\]

Proposition 2.7. If $G$ is a symmetric covariant tensor field on $M$ and $\tilde{G} = G_{\hat{i}_1...\hat{i}_k} y^{\hat{i}_1} ... y^{\hat{i}_k}$ is its corresponding polynomial (see (1.14)) then, for any graded Poisson bivector field $W$ on $TM$, one has
\[
\{\tilde{G}, f\}_W = -D_{df} G. \tag{2.18}
\]

Proof. Here, $D_{df}$ is the extension of the operator of contravariant derivative $D$ to $S(T^*M)$, i.e.,
\[
(D_{df} G)(X_1, ..., X_k) = X_f^w (G(X_1, ..., X_k)) - \sum_{i=1}^{k} G(X_1, ..., D_{df} X_i, ..., X_k),
\]
where $X_1, ..., X_k \in \mathcal{V}^*(M)$, and $D_{df}X$ is defined by
\[
\langle D_{df} X, \lambda \rangle = X_f^w (X, \lambda) - \langle X, D_{df} \lambda \rangle, \quad \lambda \in \Omega^1(M).
\]
Using the Leibniz rule, we have
\[
\{\tilde{G}, f\} = \{G_{\hat{i}_1...\hat{i}_k}, f\} y^{\hat{i}_1} ... y^{\hat{i}_k} + \sum_{i=1}^{k} \{y^{\hat{i}_i}, f\} G_{\hat{i}_1...\hat{i}_k} y^{\hat{i}_1} ... y^{\hat{i}_k}.
\]

But,
\[
\{y^i, f\} = \{l(dx^i), f\} = -l(D_{df} dx^i) = -\frac{\partial f}{\partial x^i} \Gamma^a_{hi} y^h.
\]
Hence
\[
\{\tilde{G}, f\} = -y^1 \ldots y^q \left( X^q_f (G_{i_1 \ldots i_k}) + \sum_{l=1}^k \frac{\partial f}{\partial x^a} \Gamma_{a}^{l} G_{i_1 \ldots i_l h_{i_1+1} \ldots i_k} \right).
\]

The same expression is found for \(-\tilde{D}_{df} G\). □

**Proposition 2.8.** If we define an operator \(D_{df}\) which acts on the operator \(\Psi\) of (2.5) by
\[
(D_{df}\Psi)(\alpha, \beta) := D_{df}\left(\Psi(\alpha, \beta)\right) - \Psi(D_{df}\alpha, \beta) - \Psi(\alpha, D_{df}\beta),
\] (2.19)
the Jacobi identity
\[
\{\{l(\alpha), l(\beta)\}, f\} + \{\{l(\beta), f\}, l(\alpha)\} + \{\{f, l(\alpha)\}, l(\beta)\} = 0
\] (2.20)
is equivalent to
\[
(D_{df}\Psi)(\alpha, \beta) = 0, \quad \forall \alpha, \beta \in \Omega^1(M).
\] (2.21)

**Proof.** Express (2.20) by means of (2.10), (2.14) and (2.18) for \(G = \Psi(\alpha, \beta)\). □

Notice that
\[
(D_{df}\Psi)(\alpha, h\beta) = h(D_{df}\Psi)(\alpha, \beta) - \left[ C_D(df, dh)\alpha \right] \circ \beta.
\] (2.22)
Hence \(D_{df}\Psi\) is a bidifferential operator of the second order. Furthermore, from (2.22) we can see that (2.21) is invariant by \(\alpha \mapsto f\alpha, \beta \mapsto g\beta\) \((f, g \in C^\infty(M))\) iff the curvature \(C_D = 0\).

In order to discuss the Jacobi identity
\[
\sum_{(\alpha, \beta, \gamma)} \{\{l(\alpha), l(\beta)\}, l(\gamma)\} = 0
\] (2.23)
(putting indices between parentheses denotes that summation is on cyclic permutations of these indices), let us remark the existence of an operator \(\Xi\) such that
\[
\{s(G), l(\gamma)\} = \widetilde{\Xi}(G, \gamma),
\] (2.24)
where \(G \in S_2(T^*M), \gamma \in \Omega^1(M), \Xi(G, \gamma)\) is a symmetric 3-covariant tensor field on \(M\), and tilde is the isomorphism (1.15).

By replacing \(G\) by \(fG\) and \(\gamma\) by \(h\gamma\), where \(f, h \in C^\infty(M)\), we get
\[
\Xi(fG, h\gamma) = fh\Xi(G, \gamma) - f(D_{dh}G) \circ \gamma + hG \circ D_{df}\gamma + \{f, h\} wG \circ \gamma,
\] (2.25)
which can be used to get the local coordinate expression
\[
\Xi(G, \gamma) = G_{ij} \gamma_k \Xi(dx^i \circ dx^j, dx^k)
\]
\[
+ \frac{1}{3} \sum_{(i,j,k)} \left( -G_{ij} \frac{\partial \gamma_k}{\partial x^a} \Gamma_{a}^{m} - G_{hi} \frac{\partial \gamma_k}{\partial x^a} \Gamma_{a}^{m} + \gamma_{h} \frac{\partial G_{ij}}{\partial x^a} \Gamma_{a}^{m} + w^{ab} \frac{\partial G_{ij}}{\partial x^a} \frac{\partial \gamma_k}{\partial x^b} \right) dx^i \circ dx^j \circ dx^k,
\] (2.26)
where \(\Gamma\) are the local coefficients of the contravariant connection \(D\) defined by (2.16).
Using the operator $\Xi$, the Jacobi identity (2.23) becomes
\begin{equation}
\sum_{(\alpha, \beta, \gamma)} \Xi(\Psi(\alpha, \beta), \gamma) = 0. \tag{2.27}
\end{equation}

We may summarize our analysis by

**Proposition 2.9.** The graded bivector field $W$ on $TM$ is a Poisson bivector iff:

(a) the induced bivector field $w$ on $M$ is Poisson;
(b) the associated contravariant connection $D$ is flat;
(c) the equalities (2.21) and (2.27) hold.

In this case, the projection $\pi : (TM, W) \to (M, w)$ is a Poisson mapping.

To get examples, we consider the following situation.

Suppose that the symplectic foliation $S$ of an $n$-dimensional Poisson manifold $(M, w)$ is contained in a regular foliation $\mathcal{F}$ on $M$, such that $T\mathcal{F}$ is a foliated bundle, i.e., there are local bases $\{Y_u\}$ $(u = 1, \ldots, p, p = \text{rank} \mathcal{F})$ of $T\mathcal{F}$ with transition functions constant along the leaves of $\mathcal{F}$. Consider a decomposition
\begin{equation}
TM = T\mathcal{F} \oplus \nu\mathcal{F}, \tag{2.28}
\end{equation}
where $\nu\mathcal{F}$ is a complementary subbundle of $T\mathcal{F}$, and $\mathcal{F}$-adapted local coordinates $(x^a, y^u)$ $(a = 1, \ldots, n - p)$ on $M$ [11]. Then,
\begin{align*}
T\mathcal{F} &= \text{span} \left\{ \frac{\partial}{\partial y^u} \right\} = \text{span}\{Y_u\}, \\
\nu\mathcal{F} &= \text{span} \left\{ X_a := \frac{\partial}{\partial x^a} - t^u_a \frac{\partial}{\partial y^u} \right\}, \tag{2.29}
\end{align*}
for some local function $t^u_a = t^u_a(x, y)$. Furthermore, if $\{Y_u\}, \{\tilde{Y}_v\}$ are local bases of the foliated structure of $T\mathcal{F}$ over the open neighborhoods $U, \tilde{U} \subseteq M$, then
\begin{equation}
\tilde{Y}_v = a^v_u(x)Y_u \ (u, v = 1, \ldots, p) \tag{2.30}
\end{equation}
over the connected components of $U \cap \tilde{U}$.

Since $S \subseteq \mathcal{F}$, the Poisson bivector $w$ is of the form
\begin{equation}
w = \frac{1}{2} w^{uv}(x, y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v} \ (w^{uv} = -w^{vu}). \tag{2.31}
\end{equation}

Now, $\forall V \in TM, V = \xi^aX_a + \eta^uY_u$, and we may consider $(x^a, y^u, \xi^a, \eta^u)$ as distinguished local coordinates on $TM$. The transition functions of these coordinates over the connected components of intersections of coordinate neighborhoods are of the form
\begin{align*}
\tilde{x}^a &= \tilde{x}^a(x), & \tilde{y}^u &= \tilde{y}^u(x, y), & \tilde{\xi}^a &= \frac{\partial \tilde{x}^a}{\partial x^b} \xi^b, & \tilde{\eta}^u &= b^u_a(x)\eta^v, \tag{2.32}
\end{align*}

where $b^u_a = \delta^u_w$ and $a, b = 1, \ldots, n - p; u, v = 1, \ldots, p$. 
Proposition 2.10. Under the previous hypotheses, the tangent bundle $TM$ has a graded Poisson bivector $W$, which has the expression (2.31) with respect to the distinguished local coordinates.

Proof. It follows from (2.32) that

$$W = \frac{1}{2} w^{uv}(x,y) \frac{\partial}{\partial y^u} \wedge \frac{\partial}{\partial y^v}$$

is a global tensor field on $TM$. Moreover, since $[W, W]$ has the same expression as on $M$, $W$ is a Poisson bivector.

To prove that $W$ is graded we also consider natural coordinates $(\tilde{x}^a, \tilde{y}^u, z^a, z^u)$ on $TM$, where $(z^a, z^u)$ are the vector coordinates with respect to the bases $\{\partial/\partial \tilde{x}^a, \partial/\partial \tilde{y}^u\}$. The transition functions to these coordinates are of the following local form

$$\tilde{x}^a = x^a, \quad \tilde{y}^u = y^u, \quad z^a = \xi^a, \quad z^u = -t^u_a(x,y)\xi^a + \alpha^v_u(x,y)\eta^v,$$

where the coefficients $\alpha^v_u$ are defined by $Y_u = \sum_v \alpha^v_u(\partial/\partial y^v)$. Accordingly,

$$\frac{\partial}{\partial y^u} = \frac{\partial}{\partial \tilde{y}^u} + \left( -\frac{\partial t^v_a}{\partial y^u} \xi^a + \frac{\partial \alpha^v_u}{\partial y^u} \eta^v \right) \frac{\partial}{\partial z^v} \quad (a = 1, \ldots, n - p; \ u, v, t = 1, \ldots, p),$$

and (2.34) shows that (2.33) turns into an expression of type (2.8). $\square$

Proposition 2.10 has the following interesting particular cases:

(a) The Poisson structure $w$ of $M$ is regular, and the bundle $TS$ is a foliated bundle; in this case, we take $\mathcal{F} = S$.

(b) $S$ is contained in a leaf-wise, locally affine, regular foliation $\mathcal{F}$. This means that we have $\mathcal{F}$-adapted, local coordinates $(x^a, y^u)$ with local transition functions

$$\tilde{y}^v = p^v_u(x) y^u + q^v_u(x),$$

and we may use the local vector fields $Y_u = \partial/\partial y^u$.

(c) The Poisson manifold $(M, w)$ has a flat linear connection $\nabla$, possibly with torsion. Then, we may take as leaves of $\mathcal{F}$ the connected components of $M$, and the vector fields $Y_u$ to be local $\nabla$-parallel vector fields. (Then, in (2.30) we have locally constant coefficients $a^v_u$.)

In particular, the result applies for a locally affine manifold $M$ (where $\nabla$ has no torsion), and for a parallelizable manifold $M$ (where we have global vector fields $Y_u$).

As a consequence, we see that Proposition 2.10 holds for the Lie–Poisson structure of any Lie coalgebra $G^*$ [12], which means that $TG^* = G^* \times G^*$ has a graded Poisson structure.

3. Transversal Poisson structures of foliations and graded bivector fields on tangent bundles

The results of the previous section indicate that the conditions for the existence of a graded Poisson structure on a tangent bundle $TM$ are rather restrictive. On the other hand, we will show in this section that more general, but still interesting, graded bivector fields always exist.

We begin with the following general definition. Let $\mathcal{F}$ be an arbitrary regular foliation, with $p$-dimensional leaves, on an $n$-dimensional manifold $N$. We denote by $C^{\infty}_{\mathrm{fol}}(N)$ the space of differentiable functions on $N$ which are constant along the leaves of $\mathcal{F}$ (foliated functions).
Definition 3.1. A transversal Poisson structure of \((N, \mathcal{F})\) is a bivector field \(w\) on \(N\) such that
\[
\{f, g\} := w(df, dg), \quad f, g \in C^\infty(N)
\]
restricts to a Lie algebra bracket on \(C^\infty_{\text{fol}}(N)\).

Proposition 3.2. The bivector field \(w \in \mathcal{V}^2(N)\) defines a transversal Poisson structure of the foliation \(\mathcal{F}\) iff
\[
(L_Yw)|_{\text{Ann } T \mathcal{F}} = 0, \quad [w, w]|_{\text{Ann } T \mathcal{F}} = 0,
\]
for all \(Y \in \Gamma(T \mathcal{F})\).

Proof. The annihilator space \(\text{Ann } T \mathcal{F} \subseteq \Omega^1(N)\) is
\[
\text{Ann } T \mathcal{F} = \text{span}\{df \mid f \in C^\infty_{\text{fol}}(N)\},
\]
i.e., \(f \in C^\infty_{\text{fol}}(N)\) iff, \(\forall Y \in \Gamma(T \mathcal{F}), Yf = 0\).

Accordingly, if \(f, g \in C^\infty_{\text{fol}}(N)\) one has
\[
(L_Yw)(df, dg) = Y(w(df, dg)) = Y\{f, g\},
\]
and we see that the first condition (3.2) is equivalent with \(\{f, g\} \in C^\infty_{\text{fol}}(N), \forall f, g \in C^\infty_{\text{fol}}(M)\).

The second condition (3.2) is a direct consequence of the formula (e.g., [7]):
\[
[w, w](df, dg, dh) = 2 \sum_{(f, g, h)} \{\{f, g\}, h\}. \quad \square
\]

Consider again a decomposition (2.28), and \(\mathcal{F}\)-adapted local coordinates \((x^a, y^u)(a = 1, \ldots, n - p,\ u = 1, \ldots, p)\) on \(N\) such that (2.29) holds (with no reference to any fields \(Y_u\) this time). Then
\[
w = \frac{1}{2} w^{ab} X_a \wedge X_b + w^{mn} \frac{\partial}{\partial y^m} \wedge \frac{\partial}{\partial y^n}, \tag{3.4}
\]
and the first condition (3.2) means that, locally, \(w^{ab} = w^{ab}(x)\).

Although this is not our main subject, we will derive some more facts about transversal Poisson structures of foliations.

Proposition 3.3. The Hamiltonian vector field \(X_f := i(df)w\) of a foliated function \(f\) is a foliated vector field (i.e., projectable on the space of leaves).

Proof. A vector field \(Z \in \Gamma TN\) is foliated if \(\forall Y \in \Gamma(T \mathcal{F}), L_Y Z \in \Gamma(T \mathcal{F})\). But, if \(Y \in \Gamma(T \mathcal{F})\) and \(g \in C^\infty_{\text{fol}}(N)\) then, by (3.2),
\[
(L_Y X_f) g = \partial g(L_Y (\sharp w(df))) = (L_Y w)(df, dg) + w(d(Yf), dg) = 0.
\]
Therefore \(L_Y X_f \in \Gamma(T \mathcal{F})\). \(\square\)

Definition 3.4. The generalized distribution \(\mathcal{D}\) defined by
\[
\mathcal{D} = \text{span}\{Y(x), X_f(x) \mid Y \in \Gamma(T \mathcal{F}), f \in C^\infty_{\text{fol}}(N)\} \quad (x \in N)
\]
is called the characteristic distribution of \(w\) on \((N, \mathcal{F})\).
Proposition 3.5. The characteristic distribution $D$ of a transversal Poisson structure of a foliation is completely integrable, and each leaf $\Sigma$ of $D$ is a presymplectic manifold, with a presymplectic 2-form of kernel $T\mathcal{F}|_\Sigma$.

Proof. Brackets of the form $[Y_1, Y_2], [Y, X_f], Y_1, Y_2, Y \in \Gamma(T\mathcal{F}), f \in C^\infty_{\text{fol}}(N)$ belong to $D$ because $\mathcal{F}$ is a foliation, and because of Proposition 3.3. The latter also shows that $\forall f, g, h \in C^\infty_{\text{fol}}(N)$,

$$dh([X_f, X_g] - X_{\{f,g\}}) = 0,$$

whence

$$[X_f, X_g] = X_{\{f,g\}} + Y, \quad Y \in \Gamma(T\mathcal{F}).$$

Thus, the distribution $D$ is involutive.

Furthermore, let $U$ be an $\mathcal{F}$-adapted coordinate neighborhood, and $p: U \to V; V := U/U \cap \mathcal{F}$ the submersion onto the corresponding space of slices. Because of Proposition 3.3, the distribution $p_*(D)$ exists on $V$, and, obviously, it precisely is tangent to the symplectic distribution of the Poisson structure induced by the first term of (3.4) on $V$. It follows that $p_*(D)$ has a constant dimension along the integral paths of the vector fields $p_*X_f$ $(f \in C^\infty_{\text{fol}}(N))$. Hence $D = p^{-1}_*(p_*(D))$ has a constant dimension along the integral paths of the vector fields $X_f$. $D$ also has a constant dimension along the integral paths of vector fields $Y \in \Gamma(T\mathcal{F})$ because $p_*(D)$ does not change along such paths.

Now, the complete integrability of $D$ follows from one of the versions of the Frobenius–Sussmann–Stefan theorem, Theorem 2.9′ of [12].

The leaves $\Sigma$ of the characteristic distribution $D$ are immersed submanifolds of $N$ which are foliated by the corresponding restriction of $\mathcal{F}$, and are sent by the submersion $p: U \to V := U/U \cap \mathcal{F}$ encountered above to symplectic manifolds, included in the symplectic leaves, say $\sigma$, of the projection of the first term of (3.4). It is obvious that the symplectic forms of $\sigma$ lift to a global, closed 2-form $\lambda$ on $\Sigma$, with the kernel $T\mathcal{F}|_\Sigma$. $\square$

As a matter of fact, we may notice that $w$ produces more than just a presymplectic structure on the leaves $\Sigma$ of $D$. It also produces the generalized distribution

$$E := \sharp_w \text{Ann}(T\mathcal{F}) = \text{span}\{X_f | f \in C^\infty_{\text{fol}}(N)\}$$

which has a restriction of constant rank on each leaf $\Sigma$, such that $T\Sigma = T(\mathcal{F}|_\Sigma) \oplus E|_\Sigma$.

Now we return to the tangent bundles $TM$. All of them have the vertical foliation $\mathcal{F}$ by fibers with the tangent distribution $V := T\mathcal{F}$.

The set of foliated functions on $TM$, may be identified with $C^\infty(M)$.

Proposition 3.6. Any polynomially graded bivector field $W$ on $TM$, which is $\pi$ related with a Poisson structure of $M$ is a transversal Poisson structure of $(TM, V)$.

Proof. $\pi$ is the projection $TM \to M$, and if we take $W$ as in (2.7), $W$ is $\pi$-related with the tensor $w$ defined on $M$ by the first term of (2.7). Then, (3.1) obviously holds. $\square$

Definition 3.7. A transversal Poisson structure of the vertical foliation of $TM$ will be called a semi-Poisson structure on $TM$. 
In particular, the structures of Proposition 3.6 are polynomially graded semi-Poisson structures.

In what follows, we will discuss a class of graded semi-Poisson structures of a tangent bundle $TM$ and show how to construct all the graded semi-Poisson bivector fields on $TM$ which have a given induced Poisson structure $w$ on the base manifold.

Let us consider a Poisson bivector $w$ on $M$. Recall that a semispray (a second order differential equation) \cite{9} on $M$ is a vector field $S$ on $TM$ such that $FS = E$, where $F = (\partial/\partial y^i) \otimes dx^i$ is the natural almost tangent structure and $E = y^i (\partial/\partial y^i)$ is the Euler vector field on $TM$. The local coordinate expression of $S$ is of the form

$$S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x, y) \frac{\partial}{\partial y^i}.$$  \hspace{1cm} (3.6)

Let $\nabla$ be a torsionless linear connection on $M$, with the local coefficients $\Gamma^k_{ij}$ and $S$ its associated semispray (the geodesic spray) given by

$$S = y^i \frac{\delta}{\delta x^i}, \quad \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - y^k \Gamma^i_{kj} \frac{\partial}{\partial y^j}.$$  

**Proposition 3.8.** If $(M, w)$ is a Poisson manifold then the bivector field

$$W = -\frac{1}{2} L_S w^C,$$  \hspace{1cm} (3.7)

where $w^C$ is the complete lift of $w$ to $TM$, defines a graded semi-Poisson structure on $TM$.

**Proof.** If the local coordinate expression of $w$ is (1.1), $w^C$ is given by (1.2), and we get

$$W = \frac{1}{2} w^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} - y^a w^{ij} \Gamma^i_{ka} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j} - \frac{1}{4} y^a y^b \left( \frac{\partial^2 w^{ij}}{\partial x^a \partial x^b} - w^{ij} \frac{\partial \Gamma^i_{ka}}{\partial x^k} + w^{ki} \frac{\partial \Gamma^i_{ab}}{\partial x^k} + 2 w^{kj} \frac{\partial \Gamma^i_{ab}}{\partial x^k} \Gamma^i_{ka} - 2 \frac{\partial w^{ki}}{\partial x^a} \Gamma^i_{ka} \right) \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}. \hspace{1cm} (3.8)$$

From (3.7), it follows that

$$\{F_1, F_2\}_w := W(dF_1, dF_2) = -\frac{1}{2} (L_S \{F_1, F_2\}_w - \{L_S F_1, F_2\}_w - \{F_1, L_S F_2\}_w).$$  \hspace{1cm} (3.9)

where $F_1, F_2 \in C^\infty(TM)$.

For further reference, we will say that $W$ of (3.7), (3.8) is the graded $\nabla$-lift of the Poisson structure $w$ of $M$. We are going to describe it in a different form below.

First, $\forall H \in S_k(T^*M)$, define $\nabla H \in S_{k+1}(T^*M)$ by

$$\nabla H(X_1, \ldots, X_{k+1}) = \frac{1}{k+1} \sum_{i=1}^{k+1} (\nabla X_i, H)(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}).$$

Then, with the notation of (1.15), it follows easily that

$$L_S \widehat{H} = \nabla \widehat{H}.$$  \hspace{1cm} (3.10)
If $G_1, G_2 \in S(T^*M)$, using (1.16) and (3.10), we get the explicit formula
\begin{equation}
\{\tilde{G}_1, \tilde{G}_2\}_W = -\frac{1}{2} \iota \left( \langle t^\nu \langle G_1, G_2 \rangle - \langle t^\nu G_1, G_2 \rangle - \langle G_1, t^\nu G_2 \rangle \rangle \right),
\end{equation}
where $(\cdot, \cdot)$ is the Schouten–Nijenhuis bracket of symmetric tensors, and $\iota$ is the isomorphism (1.15).

**Proposition 3.9.** The graded $\nabla$-lift $W$ of $w$ is characterized by:

(i) the Poisson structure induced by $W$ on the base manifold coincides with the given Poisson structure $w$ on $M$, i.e.,
\begin{equation}
\{f, g\}_W = \{f, g\}_w, \quad \forall f, g \in C^\infty(M);
\end{equation}

(ii) for every $f \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$
\begin{equation}
\{l(\alpha), f\}_W = -l(\nabla_X \alpha);
\end{equation}

(iii) for any Pfaff forms $\alpha$ and $\beta$ of $M$ we have
\begin{equation}
\{l(\alpha), l(\beta)\}_W = -\frac{1}{2} \iota \left( \langle t^\nu \langle \alpha, \beta \rangle - \langle t^\nu \nabla \alpha, \beta \rangle - \langle \alpha, t^\nu \nabla \beta \rangle \rangle \right).
\end{equation}

**Proof.** (i) If $f, g \in C^\infty(M)$, from (3.9) and the definition of $w^C$ in Section 1, we get
\begin{equation}
\{f, g\}_w = \frac{1}{2} \left( \{L_S f, g\}_w + \{f, L_S g\}_w \right) = \frac{1}{2} \left( \{l(df), g\}_w - \{l(dg), f\}_w \right)
\end{equation}
\begin{equation}
= \frac{1}{2} \left( X^w f g - X^w g f \right) = \{f, g\}_w.
\end{equation}

(ii) For $f \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$, (3.11) becomes
\begin{equation}
\{l(\alpha), f\}_W = -\frac{1}{2} \iota \left( \langle t^\nu \langle \alpha, \beta \rangle - \langle t^\nu \nabla \alpha, \beta \rangle - \langle \alpha, t^\nu \nabla \beta \rangle \rangle \right).
\end{equation}

Here, we have
\begin{equation}
\{l(\alpha), f\}_w = -\alpha(X_f), \quad L_S \{l(\alpha), f\}_w = -l(d(\alpha(X_f)));
\end{equation}
and, with (3.10), (1.13) and (1.16),
\begin{equation}
\{L_S l(\alpha), f\}_w = -l(i_X (t^\nu \nabla \alpha)).
\end{equation}

Finally, we have
\begin{equation}
\{l(\alpha), L_S f\}_w = \{l(\alpha), l(df)\}_w = l(\{\alpha, df\} = -l(L_X \alpha) = -l(d(\alpha(X_f)) + i_X d\alpha).
\end{equation}

With these results (3.15) gives
\begin{equation}
\{l(\alpha), f\}_W = -\frac{1}{2} \iota \left[ i_X (d\alpha + t^\nu \nabla \alpha) \right].
\end{equation}

Since the torsion of $\nabla$ vanishes we have
\begin{equation}
2(d\alpha)(X, Y) = (\nabla_X \alpha)Y - (\nabla_Y \alpha)X, \quad X, Y \in \chi(M),
\end{equation}
and
\[ d\alpha + \nabla\alpha = \nabla\alpha, \]
where \( \nabla\alpha \) is the 2-covariant tensor field defined by \( \nabla\alpha(X, Y) = (\nabla_X\alpha)(Y), \forall X, Y \in \mathcal{V}^1(M) \), and (3.16) exactly becomes (3.13).

(iii) (3.14) is a direct consequence of (3.11). \( \square \)

**Remark 3.10.** Comparing the relation (3.13) with (2.10) we see that the contravariant derivative \( D \) associated to the graded semi-Poisson structure \( W \) is the contravariant derivative induced by the linear connection \( \nabla \) (see [12]).

**Remark 3.11.** The relation (3.14) provides us the expression of the operator \( \Psi_W \) associated to \( W \) (see (2.14)):
\[
\Psi_W(\alpha, \beta) = -\frac{1}{2}(\nabla\langle\alpha, \beta\rangle - \langle\nabla\alpha, \beta\rangle - \langle\alpha, \nabla\beta\rangle).
\] (3.17)

Now, we will prove

**Proposition 3.12.** Let \((M, w)\) be a Poisson manifold. The graded semi-Poisson structures \( W \) on \( TM \) for which the canonical projection \( \pi : (TM, W) \to (M, w) \) is a Poisson mapping are defined by the relations
\[
\{f, g\}_W = \{f, g\}_w, \quad \{l(\alpha), f\}_W = -l(D_{df}\alpha), \quad \{l(\alpha), l(\beta)\}_W = s(\Psi(\alpha, \beta)),
\]
\( f, g \in C^\infty(M), \alpha, \beta \in \Omega^1(M) \), where \( D \) is an arbitrary contravariant connection of \((M, w)\) and the operator \(\Psi\) is given by
\[
\Psi = \Psi_0 + A + T,
\] (3.18)
with terms as follows: \(\Psi_0\) is the operator \(\Psi\) of a fixed graded semi-Poisson structure \( W_0 \), \( A : T^*M \times T^*M \to \bigotimes^2 T^*M \) is a skew-symmetric, first order, bidifferential operator with the property
\[
A(\alpha, f\beta) = fA(\alpha, \beta) - \tau(df, \alpha) \odot \beta,
\] (3.19)
where \(\tau\) is a tensor field of type \((2, 1)\) on \(M\), and \(T \in \Gamma((\wedge^2 TM) \otimes (\bigotimes^2 T^*M))\).

**Proof.** If \( D \) is the contravariant derivative associated to \( W \) in Remark 2.6, then, to change it, means to use a connection \( D' = D + \tau \), where \(\tau\) is a tensor field of type \((2, 1)\) on \(M\). Accordingly, from (2.15) it follows that \(\Psi' - \Psi\) is a bidifferential operator with the property (3.19). Then, with the contravariant connection \( D \) chosen, we see from (2.15) again, that the only possible change of \(\Psi\) consist in adding a tensor \( T \). \( \square \)

**Remark 3.13.** An example of operator \(\Psi_0\) is provided by \(\Psi_W\) given by (3.17).

Notice that a given Poisson structure \( w \) on \(M\) may have no graded Poisson lift to \(TM\). In particular, a flat contravariant connection \( D \) may not exist. Indeed [12], one can mimic the Chern–Weil construction of characteristic classes and associate to each Poisson manifold \((M, w)\), Pontriagin–Poisson classes \(p_h(M, w)\) which are the image of the usual Pontriagin classes in the \(LP\)-cohomology by the
homomorphism
\[ \sharp : [\lambda] \in H^k_{DR}(M) \mapsto [\lambda^\sharp] \in H^k_{LP}(M). \]
If a flat \( D \) exists, all \( p_k(M, w) = 0 \). Thus, if a non zero Pontriagin–Poisson class exists, there is no flat connection \( D \).

4. Horizontal lifts of a Poisson structure

In this section, we define horizontal lifts of a Poisson bivector \( w \) to the tangent bundle of the Poisson manifold \( (M, w) \) and study the conditions for these lifts to be Poisson bivectors, and to be compatible with the complete lift \( w^C \).

Let \( M \) be an \( n \)-dimensional manifold and \( \pi : TM \to M \) its tangent bundle. On \( TM \), we consider a nonlinear connection, i.e., a distribution \( H \), called horizontal, such that \( T(TM) = H \oplus V \), where \( V \) denotes the vertical distribution tangent to the fibers of \( TM \) [9,10]. If \( (x^i) \) are local coordinates on \( M \) and \( (x^i, y^j) \) \((i, j = 1, \ldots, n)\) are the induced coordinates on \( TM \) (see Section 1), we have bases of the form
\[
V = \operatorname{span} \left\{ \frac{\partial}{\partial y^j} \right\}, \quad H = \operatorname{span} \left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j} \right\},
\]
and \( \Gamma^j_i \) are called the coefficients of the connection.

Equivalently, the nonlinear connection may be seen as an almost product structure \( \Gamma \) on \( TM \) such that the eigendistribution corresponding to the eigenvalue \(-1\) is the vertical distribution \( V \) [9]. Then
\[
h = \frac{1}{2} (\operatorname{Id} + \Gamma) : TM \to H
\]
is the horizontal projector of \( \Gamma \), and the curvature \( R \) of the connection is the Nijenhuis tensor
\[
R(X, Y) = -N_h(X, Y) = -[hX, hY] + h[hX, Y] + h[X, hY] - h[X, Y],
\]
where \( X, Y \in \mathcal{V}^1(TM) \). \( R \) vanishes if at least one argument is in \( V \), and always takes values in \( V \), hence, locally, we may write [9]
\[
R = \frac{1}{2} R^k_{ij} \, dx^i \wedge dx^j \otimes \frac{\partial}{\partial y^k}, \quad R^k_{ij} = \frac{\delta \Gamma^k_i}{\delta x^j} - \frac{\delta \Gamma^k_j}{\delta x^i}.
\]
Then, we get
\[
\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = -R^k_{ij} \frac{\partial}{\partial y^k}, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right] = \frac{\partial \Gamma^k_i}{\partial y^j} \frac{\partial}{\partial y^k}.
\]
In particular, \( H \) is involutive iff \( R = 0 \).

Let us consider a bivector \( w \) on the base manifold \( M \), having the local coordinate expression (1.1).

**Definition 4.1.** The horizontal lift of \( w \) to the tangent bundle \( TM \), with respect to the connection \( \Gamma \) is the (global) bivector field \( w^H \) defined by
\[
w^H = \frac{1}{2} w^{ij}(x) \frac{\delta}{\delta x^i} \wedge \frac{\delta}{\delta x^j}.
\]
Notice that the horizontal lift (4.4) is different from that of [16].

**Proposition 4.2.** Let \((M, w)\) be a Poisson manifold. If the horizontal distribution \(H\) is defined by a linear connection \(\nabla\) on \(M\), the bivector \(w^H\) defines a graded semi-Poisson structure on \(TM\).

**Proof.** With respect to the bases \((\partial/\partial x^i, \partial/\partial y^j)\), the expression of \(w^H\) is of the form (2.8). \(\square\)

**Proposition 4.3.** A horizontal lift \(w^H\) is a Poisson bivector on \(TM\) iff \(w\) is a Poisson bivector on the base manifold \(M\) and

\[
R(X^H_f, X^H_g) = 0, \quad \forall f, g \in C^\infty(M),
\]

where \(X_f\) denotes the \(w\)-Hamiltonian vector field of \(f\) and \(X^H_f\) is the horizontal lift of \(X_f\) [16].

**Proof.** A straightforward computation yields the Schouten–Nijenhuis bracket

\[
\left[ w^H, w^H \right] = \frac{1}{3} \left( \sum_{(i,j,k)} w^{h_{ik}} \frac{\partial w^{ij}}{\partial x^h} \frac{\delta}{\delta x^i} \wedge \frac{\delta}{\delta x^j} \wedge \frac{\delta}{\delta x^k} + w^{hi} w^{lj} R^k_{hl} \frac{\delta}{\delta x^i} \wedge \frac{\delta}{\delta x^j} \wedge \frac{\delta}{\delta y^k} \right).
\]

Since the vanishing of the first term of (4.6) is equivalent to \([w, w] = 0\) on \(M\), \(w^H\) is a Poisson bivector on \(TM\) iff \(w\) is a Poisson bivector on \(M\) and

\[
w^{hi} w^{lj} R^k_{hl} = 0.
\]

The latter equation has the equivalent form

\[
R((\sharp w^H)_{\alpha}, (\sharp w^H)_{\beta}) = 0, \quad \forall \alpha, \beta \in \Omega^1(M),
\]

which is also equivalent to (4.5). \(\square\)

**Remark 4.4.**

(i) If \(w^H\) is a Poisson bivector, the projection \(\pi : (TM, w^H) \to (M, w)\) is a Poisson mapping.

(ii) If \(w\) is defined by a symplectic form on \(M\), condition (4.5) becomes \(R = 0\).

**Corollary 4.5.** If \((M, w)\) is a Poisson manifold and the connection \(\Gamma\) on \(TM\) is defined by a covariant derivative \(\nabla\) on \(M\), the bivector \(w^H\) defines a Poisson structure on \(TM\) iff the curvature \(C_D\) of the contravariant connection induced by \(\nabla\) on \(TM\) vanishes. In this case, \(w^H\) is a graded Poisson structure on \(TM\).

**Proof.** Remember that \(D\) is defined by \(D_{hf} = \nabla_{X^f}\), and we may see this operator as acting either on \(T^*M\) or on \(TM\) [12].

If \(\Gamma^k_i\) are the connection coefficients of \(\nabla\), \(\Gamma^k_i = \Gamma^k_i(x)y^j\) and \(R^k_{ij} = y^h R^k_{lij}\), where \(R^k_{lij}\) are the components of the curvature \(R_V\). Condition (4.7) becomes

\[
R_V(\sharp(w^H)_{\alpha}, \sharp(w^H)_{\beta}) Z = 0, \quad \forall \alpha, \beta \in \Omega^1(M), \forall Z \in \mathcal{V}^1(M);
\]

equivalently,

\[
R_V(X_f, X_g) Z = 0 \quad \forall f, g \in C^\infty(M), \forall Z \in \mathcal{V}^1(M).
\]

This condition is equivalent to \(C_D = 0\). \(\square\)
If the connection $\Gamma$ on $TM$ is defined by a covariant derivative $\nabla$ on $M$, the conditions for the graded bivector field $w^H$ to be Poisson are simpler than those of Proposition 2.9, and (2.21), (2.27) must be consequences of the conditions of Proposition 4.3. Furthermore, one can check that the operators $\Psi$ and $\Xi$ of $w^H$ (see (2.11), (2.24)) are given by

\begin{align}
\Psi_{\alpha, \beta}(X, Y) &= \frac{1}{2} w\left( (\nabla_\alpha)X, (\nabla_\beta)Y \right) + w\left( (\nabla_\alpha)Y, (\nabla_\beta)X \right), \\
\Xi(G, \gamma)(X, Y, Z) &= \frac{1}{3!} \sum_{(X, Y, Z)} w\left( (\nabla_G)(X, Y), (\nabla_\gamma)Z \right),
\end{align}

(4.10, 4.11)

where $\alpha, \beta, \gamma \in \Omega^1(M)$, $G \in S_2(T^*M)$, and $\nabla_\cdot$ means that we create a 1-form which is evaluated on $Z \in \mathfrak{v}^1(M)$ by the application of $\nabla_Z$.

Let us consider an arbitrary Poisson structure $w$ on $TM$. Following [13], we would like to know whether there are semisprays on $TM$ which are Hamiltonian vector fields with respect to $w^H$.

**Proposition 4.6.** If the Poisson bivector $w$ on $M$ is not defined by a symplectic structure, there are no $w^H$-Hamiltonian semisprays on $TM$.

**Proof.** If $F \in C^\infty(TM)$, then

$$X^H_F = w^{ij} \frac{\delta F}{\delta x^j} \frac{\partial}{\partial x^i} - w^{ik} \frac{\delta F}{\delta y^j} \Gamma^j_k \frac{\partial}{\partial y^i},$$

and (3.6) shows that $X^H_F$ is a semispray iff

$$w^{ij} \frac{\delta F}{\delta y^j} = y^j. \quad (4.12)$$

(4.12) implies $-w^{ij} \frac{\partial}{\partial y^j} (\frac{\delta F}{\delta x^i}) = \delta^j_k$, therefore, $(w^{ij})$ is a nonsingular matrix. \qed

Recall that two Poisson structures on a manifold $M$ are compatible if the bivector fields $w_1$ and $w_2$ satisfy the condition

$$[w_1, w_2] = 0, \quad (4.13)$$

or, equivalently, $w_1 + w_2$ also is a Poisson bivector field.

If $w^H$ is a Poisson bivector, it is natural to discuss its compatibility with the complete lift $w^C$ of $w$.

**Proposition 4.7.** Let $w$ be a Poisson structure, and $\nabla$ a symmetric linear connection on $M$ such that the associated contravariant connection of $TM$ has zero curvature. Then the Poisson bivector $w^H$ is compatible with the complete lift $w^C$ iff

$$i_X(\nabla^2 w) = 0, \quad \forall f \in C^\infty(M), \quad (4.14)$$

where $\nabla^2 w = \nabla \nabla w$ is the tensor field of type $(2, 2)$ on $M$ defined by

$$\left( \nabla^2 w \right)(X, Y) = \left( \nabla_X(\nabla_Y w) \right) - \nabla_{\nabla_X Y} w, \quad X, Y \in \mathfrak{v}^1(M).$$
Proof. We compute the bracket \([w^H, w^C]\) using the auxiliary notations

\[
a^{ij} := y^k \frac{\partial w^{ij}}{\partial x^k} + \Gamma^i_k w^{kj} - \Gamma^j_k w^{ki}, \quad t^{hi}_{hl} = \frac{\partial \Gamma^k_h}{\partial y^l} - \frac{\partial \Gamma^k_l}{\partial y^h}.
\]

By a straightforward computation, it follows that \([w^H, w^C] = 0\) is equivalent to

\[
w^{hi} w^{jli} t^{hl}_{hi} = 0, \quad w^{hi} \left( \delta a^{jk} \frac{\partial \Gamma^k_h}{\partial y^l} - a^{ij} \frac{\partial \Gamma^i_k}{\partial y^l} + a^{ik} \frac{\partial \Gamma^j_k}{\partial y^l} \right) = 0.
\]

If \(\Gamma\) comes from a symmetric linear connection \(\nabla\) on \(M\), the first condition (4.16) holds, and the second condition (4.16) is the coordinate expression of (4.14).

Remark 4.8. For any \(w \in \mathcal{V}^2(M)\), one can see that \(Q = w^H + w^C\) is a Poisson bivector iff \(w\) and \(w^H\) are Poisson bivectors and \(w^C\) is compatible with \(w^H\).

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