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Nonstratifiability of topological vector spaces

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Abstract

It is shown that a compact κ-metrizable space with a dense monotonically normal subspace is metrizable. It is deduced that if a Banach space, in its weak topology, is stratifiable, then it is metrizable. Also, it is shown that $C_p(X)$ is stratifiable if and only if X is countable. © 1998 Published by Elsevier Science B.V.

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1. Introduction

It is a highly convenient property of metrizable locally convex topological vector spaces that every closed convex subspace is a retract. However, closed convex subspaces of non-metrizable locally convex topological vector spaces frequently fail to be retracts. Happily, there is a topological property, called stratifiability (originally defined in [4], and intensively studied subsequently, see the survey articles [16] and [19]), weaker than metrizability, such that closed convex subspaces of stratifiable locally convex topological vector spaces are retracts [2]. Since stratifiable spaces share many other properties with metrizable spaces (they are indeed, probably the most successful of the so called generalized metric properties) it is not surprising that various authors have asked when specific types of locally convex topological vector spaces are stratifiable. For example, Wheeler [20] asked if a separable Banach space in its weak topology is stratifiable; while Arhangelskii [1] asked when $C_p(X)$, the space of continuous real-valued functions on X with the topology of pointwise convergence, is stratifiable.

In this paper we show that for both $C_p(X)$ and Banach spaces in their weak topology, stratifiability implies metrizability. Indeed we show that any stratifiable locally convex topological vector space in its weak topology is metrizable. (Independently, Yashenko [21]

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has shown that X is countable if $C_p(X)$ is stratifiable.) For comparison it is important to note that there are many examples of stratifiable non-metrizable locally convex topological vector spaces (see [9,16,18]). In the interests of fuller generality it will be convenient to examine monotonically normal spaces. A space X is *monotonically normal* if there is an operator $V(.,.)$ assigning to pairs of points, x, and open neighborhoods, U, an open neighbourhood $V(x,U)$ of x contained in U so that whenever $V(x,U)\cap V(x',U')\neq\emptyset$, we have either $x\in U'$ or $x'\in U$. Monotonically normal spaces are of considerable interest in their own right, but for our purposes it is necessary to know that stratifiable spaces are monotonically normal. In the context of locally convex topological vector spaces we observe that the converse is true.

Lemma 0. *Let L be a locally convex topological vector space. Then L is stratifiable if and only if it is monotonically normal.*

Proof. To see this, let L be a locally convex topological vector space. As L is locally convex there is a nontrivial linear functional, f say, on L. Hence L can be factored into $L'\times\mathbb{R}$, where $L'=\ker(f)$. Since $L'\times\mathbb{R}=L$ is monotonically normal, and $\mathbb{R}$ contains convergent sequences, L' is stratifiable. Thus L, as the product of two stratifiable spaces, is stratifiable. (For details about the product theory of monotonically normal and stratifiable spaces, see [13,16].) □

There are examples (see [8]) of monotonically normal topological vector spaces which are not stratifiable (or even κ-stratifiable for some cardinal κ). (The reader is referred to the survey articles [10,11] by Gruenhage for further information on stratifiable and monotonically normal spaces.)

In the next section we prove the key theorem. Then Section 3 is devoted to applying this to general topological groups. A question of Heath is answered. The final section turns to locally convex topological vector spaces, and the solutions to the problems of Arhangelskii and Wheeler given.

2. The key theorem

First some definitions and related basic facts. A subspace Y of a space X is said to be K_1 *embedded* if there is a map k from τY, the topology on Y, to τX, the topology on X, such that: (1) $k(U)\cap Y=U$ and (2) $k(U)\cap k(V)\neq\emptyset$ implies $U\cap V\neq\emptyset$, for any $U,V\in\tau Y$. Dense subspaces are always K_1 embedded. It is also easy to check that if a subspace is a retract, then it is K_1 embedded. Every subspace of a monotonically normal space is K_1 embedded (see [7,10], where monotone normality is characterized in terms of K_1 embeddings).

A space X has *calibre* (ω_1,ω,ω) if every point finite collection of open sets is countable. A space X has the *countable chain condition* (ccc) if every family of pairwise disjoint open sets is countable. Every monotonically normal space with the countable chain condition has calibre $(\omega_1, \omega, \omega)$ (see [7]). It is known that any product of separable metrizable spaces has calibre $(\omega_1, \omega, \omega)$.

Let X be a space, and let x be a point in X. A collection $\mathcal{P}$ of pairs of subsets of X is said to be a *local pairbase* at x if whenever U is an open neighbourhood of x there is a $P = (P_1, P_2) \in \mathcal{P}$ such that P_1 is open and $x \in P_1 \subseteq P_2 \subseteq U$. A local pairbase at a point x is *σ-cushioned* if we can write $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$ where for every $n \in \omega$ and $\mathcal{P}' \subseteq \mathcal{P}_n$, we have

$$\overline{\bigcup\{P_1\colon (P_1, P_2) \in \mathcal{P}'\}} \subseteq \bigcup\{P_2\colon (P_1, P_2) \in \mathcal{P}'\}.$$

A point in a space which has a σ-cushioned local pairbase is called a *σ-m_3 point*. It can easily be shown that every point of a monotonically normal space is a σ-m_3 point. (See [3] for more information about σ-m_3 points.)

A compact space X is said to be *κ-metrizable* if it can be K_1 embedded in some Tychonoff cube I^κ. (This is not the original definition but was shown to be equivalent to it by Shirokov [17].) From this it easily follows that compact metrizable spaces are κ-metrizable, and that an arbitrary product of compact κ-metrizable spaces is again κ-metrizable. It is also known that a regular closed subspace of a κ-metrizable space is κ-metrizable. (This follows immediately from the original definition of κ-metrizability.) For further information on compact κ-metrizable spaces, and related spaces, the reader is referred to Section 7 of Shakmatov's excellent survey article in [16].

Theorem 1a. *Let K be a compact κ-metrizable space, and let X be a K_1 embedded subspace of K which has calibre $(\omega_1, \omega, \omega)$. Then every σ-m_3 point of X is a point of first countability.*

Proof. As X is K_1 embedded in K, and K is K_1 embedded in some Tychonoff cube, we may suppose our space X is a subspace of I^S (for some set S), and there is a K_1 operator $k : \tau X \to \tau I^S$. Recall that a basic neighbourhood of a point x in I^S is of the form

$$B(x, F, \varepsilon) = \{x'\colon |x(s) - x'(s)| < \varepsilon,\ s \in F\},$$

where F is a finite subset of S, and $\varepsilon > 0$.

Let x be point of X, and let $\mathcal{P}$ is a local pairbase at x, where $\mathcal{P} = \bigcup_{m\in\omega} \mathcal{P}_m$, and each $\mathcal{P}_m$ is cushioned. Suppose, for a contradiction, that the character of x in X is uncountable.

We may suppose, for each $P \in \mathcal{P}$, that P_2 is closed in X. For each $P \in \mathcal{P}$ pick $F(P) \subseteq S$ and $n(P) \geqslant 1$ such that $B(x, F(P), 1/n(P)) \subseteq k(P_1)$. By transfinite induction we may find $\{P^\alpha\}_{\alpha\in\omega_1} \subseteq \mathcal{P}$ such that

$$\alpha < \beta < \omega_1 \Rightarrow P_2^\alpha \neq P_2^\beta \qquad \text{and} \qquad B(x, F(P^\alpha), 1/n(P^\alpha)) \cap X \not\subseteq P_2^\beta. \qquad (*)$$

Consider $\{F(P^\alpha)\}_{\alpha\in\omega_1}$. By the Pigeon Hole Principle, there are $m, n \in \omega$ and $\Lambda_1 \subseteq \omega_1$ such that $|\Lambda_1| = \omega_1$, and $\forall\alpha \in \Lambda_1$ $(n(P^\alpha) = n$ and $P^\alpha \in \mathcal{P}_m)$. Applying the Δ-system lemma, there is a $\Lambda_2 \subseteq \Lambda_1$ and finite $R \subseteq S$, such that $|\Lambda_2| = \omega_1$ and $F(P^\alpha) \cap F(P^\beta) = R$ for distinct $\alpha, \beta \in \Lambda_2$.

Define $U_\alpha = (B(x,R,1/n)\cap X)\cap(X\setminus P_2^\alpha)$. Clearly U_α is open in X, and for distinct α and β, $U_\alpha \neq U_\beta$. Note that $U_\beta = \emptyset$ if and only if $B(x,R,1/n)\cap X \subseteq P_2^\beta$. However, for any $\alpha \in \Lambda_2$,

$$B(x,F(P^\alpha),1/n)\cap X \subseteq B(x,R,1/n)\cap X.$$

Thus by (∗), there is a $\Lambda_3 \subseteq \Lambda_2$ such that $|\Lambda_3| = \omega_1$ and $U_\alpha \neq \emptyset$ for all $\alpha \in \Lambda_3$.

As X has calibre (ω_1,ω,ω), there is an infinite $\Lambda_4 \subseteq \Lambda_3$ and $z \notin \bigcap_{\alpha\in\Lambda_4} U_\alpha$. Observe that $z \notin P_2^\alpha$, for all $\alpha \in \Lambda_4$. Take any basic neighbourhood of z in X, say $B(z,F,\varepsilon)\cap X$ (we may assume $R \subseteq F$). Pick a basic $B(z,\widehat{F},\hat{\varepsilon}) \subseteq k(B(z,F,\varepsilon)\cap X)$. As Λ_4 is infinite, and $\{F(P^\alpha)\setminus R\colon \alpha\in\Lambda_4\}$ is a pairwise disjoint family, there is a α_0 in Λ_4 such that $(F(P^{\alpha_0})\setminus R)\cap(\widehat{F}\setminus R) = \emptyset$.

Then, $B(x,F(P^{\alpha_0}),1/n)\cap B(z,\widehat{F},\hat{\varepsilon})$ is open and nonempty. Thus $k(P_1^{\alpha_0})\cap k(B(z,F,\varepsilon)) \neq \emptyset$, and so $P_1^{\alpha_0}\cap B(z,F,\varepsilon)\cap X \neq \emptyset$. Therefore, $z \notin \bigcup\{P_2^\alpha\colon \alpha\in\Lambda_4\}$, but $z \in \overline{\bigcup\{P_1^\alpha\colon \alpha\in\Lambda_4\}}$—contradicting $\mathcal{P}_m$ cushioned. □

Theorem 1b. *A compact κ-metrizable space, K, with a dense monotonically normal subspace, X say, is metrizable.*

Proof. Tychonoff cubes have the countable chain condition, and it is easy to check that a K_1 embedded subspace of a ccc space also has the countable chain condition. Thus X has the countable chain condition. As X is monotonically normal it follows that X has calibre (ω_1,ω,ω), and also that every point of X is a σ-m_3 point. From Theorem 1a we deduce that X is first countable. But a compact κ-metrizable with a dense first countable subspace is metrizable [20] (see also [16]). □

We observe that in the above result we can not weaken 'monotonically normal' to 'hereditarily normal' or even 'hereditarily Lindelof' because 2^{ω_1} has a countable dense subspace. The author conjectures that the result will hold if we replace 'κ-metrizable' with 'dyadic' (that is to say, the continuous image of 2^κ for some cardinal κ) or even their common generalization 'Shirokov' (see [16] for the definition).

3. Topological groups

The link between topological groups and κ-metrizable spaces is that every compact topological group is κ-metrizable. In fact it is known that a compact space K is κ-metrizable if (and only if) it can be K_1 embedded in some topological group [15].

Theorem 2a. *Let G be a monotonically normal topological group, and let X be a locally compact subspace of G. Then X is metrizable.*

Proof. As G is monotonically normal, every subspace is K_1 embedded. In particular, every compact subspace is K_1 embedded, and hence is κ-metrizable. From Theorem 1b we deduce that every compact subspace of G is metrizable. It is shown in [16] that monotonically normal topological groups are hereditarily paracompact. The claim now follows from the well known fact that paracompact locally metrizable spaces are metrizable. □

A subset, S, of a topological group, G, is said to be totally bounded if for any neighbourhood, U, of the identity, there is a finite subset S_0 of S so that $S_0U \supseteq S$. It is well known that the completion, $\widehat{G}$, of a topological group, G, with a totally bounded open neighbourhood of the identity, is locally compact. Given this, the following result is evidently a variation on Theorem 2a.

Theorem 2b. *Let G be a topological group with a totally bounded open neighbourhood of the identity which is monotonically normal. Then G is metrizable.*

Proof. As $\widehat{G}$ is a locally compact group, it is homeomorphic to $\mathbb{R}^n \times K \times D$, where $n \in \omega$, K is a compact subgroup of $\widehat{G}$ and D is discrete [5]. Compact groups and compact metric spaces are compact κ-metrizable, as are their regular closed subspaces. Hence the identity of $\widehat{G}$ has an open neighbourhood basis of sets whose closures are compact and κ-metrizable.

From the given conditions on G, we can find one of these neighborhoods with a dense monotonically normal subspace. Now from Theorem 1b we deduce that $\widehat{G}$ is first countable, and thus $\widehat{G}$ (and, *a fortiori* G) is metrizable. □

Heath, attempting to construct monotonically normal non-metrizable subgroups of 2^κ (with coordinatewise multiplication), was only able to show their non-existence in certain cases, and left the problem (as stated in [18]) open in general. Example 3 provides a complete solution to his question.

Example 3. Let $\oplus$ be any compatible group operation on 2^κ (some cardinal κ), and let G be a monotonically normal subgroup of $(2^\kappa, \oplus)$. Then G is metrizable.

Proof. Consider the closure, $\overline{G}$, of G in $(2^\kappa, \oplus)$. This is a compact topological group, and hence is κ-metrizable, with a dense monotonically normal subgroup. That G is metrizable follows from Theorem 1b. □

The final result of this section shows that our assumption in Theorem 1a about calibre $(\omega_1, \omega, \omega)$ is necessary. The example is also of interest for a quite different reason. Elsewhere [16], the author has shown that a separable topological group any (hence all) of whose points are σ-m_3 is stratifiable. The example demonstrates that 'separable' can not be weakened to 'countable chain condition'.

Example 4. There is a ccc dense subgroup of a compact topological group (hence κ-metrizable), all of whose points are σ-m_3 but do not have countable character.

Proof. Let $G = \{x \in 2^{\omega_1}\colon |\{\alpha\colon x(\alpha) = 1\}| < \omega\}$ be considered as a topological subgroup of 2^{ω_1} (with its standard Tychonoff topology and coordinatewise multiplication).

Then G is dense in 2^{ω_1}, and so is ccc; and nowhere in G is there a point of first countability.

It remains to show that each point of G is σ-m_3. Since G is a topological group, taking translations if necessary, it is sufficient to show that the identity is a σ-m_3 point. A basic neighbourhood of $\boldsymbol{x}$ in G is $B(\boldsymbol{x}, F) = \{\boldsymbol{y} \in G\colon \boldsymbol{y}(\alpha) = \boldsymbol{x}(\alpha),\ \forall \alpha \in F\}$ where F is a finite subset of ω_1. Define $\mathcal{P} = \{\langle B(\mathbf{0}, F), B(\mathbf{0}, F)\rangle\colon \text{ finite } F \subseteq \omega_1\}$. We show that $\mathcal{P}$ is cushioned. To do this take any family $\mathcal{F}$ of finite subsets of ω_1 (corresponding to a subcollection of $\mathcal{P}$), and any point $\boldsymbol{x}$ in G with $\boldsymbol{x} \notin \bigcup_{F \in \mathcal{F}} B(\mathbf{0}, F)$. Let $F_x = \{\alpha \in \omega_1\colon \boldsymbol{x}(\alpha) = 1\}$. Then $B(\boldsymbol{x}, F_x) \cap B(\mathbf{0}, F) = \emptyset$ for all F in $\mathcal{F}$. □

4. Locally convex topological vector spaces

Let L be a vector space over a topological field F. A topological vector space topology τ on L is said to be a *weak* topology on L if it is the same topology as that induced by the set L^* of all continuous linear functionals on (L, τ). Evidently, a Banach space with its weak topology has a weak topology, in the sense just defined. But a vector space may admit many compatible weak topologies. To help identify which topological vector space topologies are weak topologies, and to assist in the proof of Theorem 6a, we have the following lemma. (This lemma is probably folklore, at least for locally convex topological vector spaces. Unfortunately, the author has been unable to find a suitable reference.)

Lemma 5. *Let (L, τ) be a topological vector space over F. Then the following are equivalent*:

(1) *τ is a weak topology,*
(2) *(L, τ) can be embedded as a dense linear subspace of $F^{\mathcal{H}}$, where $\mathcal{H}$ is a Hamel basis for L^*,*
(3) *(L, τ) can be embedded as a linear subspace of a power of F.*

Theorem 6a. *Let L be a topological vector space over a separable metrizable topological field F, with a weak topology. Then the following are equivalent*:

(1) *L is monotonically normal,*
(2) *L is stratifiable,*
(3) *L is metrizable, and*
(4) *L^* has countable (algebraic) dimension.*

Proof. It clearly suffices to show that (4) implies (3), and (1) implies (4).

(4) *implies* (3). Let $\mathcal{H}$ be a countable Hamel basis for L^*. Then, by Lemma 5, L can be embedded in $F^{\mathcal{H}}$. So L is metrizable.

(1) *implies* (4). Let L be monotonically normal, and let $\mathcal{H}$ be a Hamel basis for L^*. Then, by Lemma 5, L is a dense subspace of $F^{\mathcal{H}}$. As F is separable metrizable, it has a metrizable compactification, K say. Compact metrizable spaces are κ-metrizable, as are arbitrary products of compact κ-metrizable spaces. Thus L is a dense monotonically

normal subspace of the compact κ-metrizable space $K^{\mathcal{H}}$, and so the claim follows from Theorem 1b. □

Similarly to the proof of (1) *implies* (4) above, but using Theorem 1a in place of Theorem 1b, we may deduce a related result.

Theorem 6b. *If a monotonically normal space can be K_1 embedded (in particular, if it is a retract) in a topological vector space, over a separable metrizable field, with a weak topology, then it is metrizable.*

Now we consider particular cases. Evidently we can answer Wheeler's question, but we can also give a parallel result for the dual of a Banach space in its weak* topology. To be clear, let B be a Banach space. Write B_w for B with the weak topology induced by the dual space B^*. Additionally to the usual weak topology on the Banach space B^* (induced by B^{**}), B^* is a locally convex topological vector space when considered as a linear subspace of $\mathbb{R}^B$, in which case it is denoted $B^*_{w^*}$ (this topology is called the weak* topology). By Lemma 5, $B^*_{w^*}$ has a weak topology. It is known that metrizability of either a Banach space in its weak topology, or of the dual in the weak* topology forces the Banach space to be finite-dimensional.

Theorem 7. *Let B be a Banach space. Then the following are equivalent:*
(1) *B_w is monotonically normal,*
(2) *B_w is metrizable, and*
(3) *B is finite dimensional.*

Theorem 8. *Let B be a Banach space. Then the following are equivalent:*
(1) *$B^*_{w^*}$ is monotonically normal,*
(2) *$B^*_{w^*}$ is metrizable, and*
(3) *$B^*(= B)$ is finite dimensional.*

The dual of $C_p(X)$ is customarily written, $L_p(X)$, and regarded as a linear subspace of $\mathbb{R}^{C_p(X)}$. The spaces $C_p(X)$ and $L_p(X)$ are locally convex topological vector spaces. The natural copy of X in $L_p(X)$, is a Hamel basis; while $L_p(X)$ is metrizable if and only if X is finite. Thus we resolve Arhangelskii's problem.

Theorem 9. *Let X be a space. Then the following are equivalent:*
(1) *$C_p(X)$ is monotonically normal,*
(2) *$C_p(X)$ is metrizable, and*
(3) *X is countable.*

Theorem 10. *Let X be a space. Then the following are equivalent:*
(1) *$L_p(X)$ is monotonically normal,*
(2) *$L_p(X)$ is metrizable, and*
(3) *X is finite.*

Uspenskii (who relayed Arhangelskii's question to the author) specifically asked whether $C_p(\mathbb{R})$ is stratifiable. From Theorem 9 it follows that it is not. This can be improved. Let $\mathcal{Q}_p$ be the set of all polynomials with rational coefficients. Then $\mathcal{Q}_p$ is a countable topological subgroup of $C_p(\mathbb{R})$, and is a topological vector space over the rationals. As $\mathcal{Q}_p$ is dense in $C_p(\mathbb{R})$, by Theorem 6a, it cannot be monotonically normal.

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References

[1] A.V. Arhangelskii, A survey of C_p-theory, Questions Answers Gen. Topology 5 (1987).
[2] C.J.K. Borges, On stratifiable spaces, Pacific J. Math. 17 (1966) 1–16.
[3] R. Buck, Some weaker monotone separation and basis properties, Topology Appl. 69 (1996) 1–12.
[4] J.G. Ceder, Some generalizations of metric spaces, Pacific J. Math. 11 (1961) 105–125.
[5] J. Cleary and S.A. Morris, Topologies on locally compact topological groups, Bull. Aust. Math. Soc. 38, 105–111.
[6] P.M. Gartside, DPhil. Thesis, University of Oxford, UK (1993).
[7] P.M. Gartside, Cardinal invariants of monotonically normal spaces, Topology Appl. 77 (1997) 303–314.
[8] P.M. Gartside, Monotonically normal topological groups, Preprint.
[9] P.M. Gartside and E. Reznichenko, Near metric properties of function spaces, Preprint.
[10] G. Gruenhage, Generalized metric spaces, in: K. Kunen and J. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 423–501.
[11] G. Gruenhage, Generalized metric spaces and metrization, in: M. Hušek and J. van Mill, eds., Recent Progress in General Topology (Elsevier, Amsterdam, 1992) 242.
[12] R.W. Heath, Monotone normality in topological groups, Zb. Rad. (1984) 13–18.
[13] R.W. Heath, D. Lutzer and P. Zenor, Monotonically normal spaces, Trans. Amer. Math. Soc. 178 (1973) 481–493.
[14] E.V. Schepin, On κ-metrizable spaces, Math. USSR Izvestija 14 (1980) 407–440.
[15] D.B. Shakmatov, Dugundji spaces and topological groups, Comm. Math. Univ. Carolin. 31 (1990) 129–143.
[16] D.B. Shakmatov, Compact spaces and their generalizations, in: M. Hušek and J. van Mill, eds., Recent Progress in General Topology (Elsevier, Amsterdam, 1992) 597.
[17] L.V. Shirokov, An external characterization of Dugundji spaces and κ-metrizable compact Hausdorff spaces, Sov. Math. Dokl. 25 (1982) 507–510.
[18] S. Shkarin, Private communication.
[19] K. Tamano, Generalized metric spaces II, in: K. Morita and J. Nagata, eds., Topics in General Topology (North-Holland, Amsterdam, 1989) 368–409.
[20] R.F. Wheeler, The retraction property, in: Measure Theory, Oberwolfach, Lecture Notes in Mathematics 945 (Springer, Berlin, 1981).
[21] I. Yaschenko, Preprint.