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A Hermitian least squares solution of the matrix equation $AXB = C$ subject to inequality restrictions[☆]

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ABSTRACT

This paper gives some closed-form formulas for computing the maximal and minimal ranks and inertias of $P - X$ with respect to X, where $P \in \mathbb{C}_H^n$ is given, and X is a Hermitian least squares solution to the matrix equation $AXB = C$. We derive, as applications, necessary and sufficient conditions for $X \geqslant (\leqslant, >, <) P$ in the Löwner partial ordering. In addition, we give necessary and sufficient conditions for the existence of a Hermitian positive (negative, nonpositive, nonnegative) definite least squares solution to $AXB = C$.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ and $\mathbb{C}_H^m$ stand for the sets of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively; the symbols A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[A, B]$ denotes a row block matrix consisting of A and B. We write $A > 0$ $(A \geqslant 0)$ if A is Hermitian positive (nonnegative) definite. Two Hermitian matrices A and B of the same size are said to satisfy the inequality $A > B$ $(A \geqslant B)$ in the Löwner partial ordering if $A - B$ is positive (nonnegative) definite. The Moore–Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four matrix equations:

$$\text{(i) } AXA = A, \qquad \text{(ii) } XAX = X, \qquad \text{(iii) } (AX)^* = AX, \qquad \text{(iv) } (XA)^* = XA.$$

Further, define $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$. The ranks of E_A and F_A are given by $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. The inertia of a Hermitian matrix A is defined to be the triplet $\operatorname{In}(A) = \{i_+(A), i_-(A), i_0(A)\}$, where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $i_+(A)$ and $i_-(A)$ are usually called the partial inertias of A. For a matrix $A \in \mathbb{C}_H^m$, we have $r(A) = i_+(A) + i_-(A)$ and $i_0(A) = m - r(A)$.

Linear matrix equations play a very important role in matrix theory and other disciplines, such as statistics and control theory. For a given matrix equation, one always wants to know the consistency condition, the general solution or least squares solution, the properties of (least squares) solutions and so on. For a linear matrix equation

$$AXB = C, \tag{1.1}$$

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where $X \in \mathbb{C}^{n\times p}$ is an unknown matrix, and $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$ are three given matrices, there have been many results given in the literature; see, e.g., [1–8]. For nonnegative and positive definite solutions, many specialists and researchers have discussed the problems concerning symmetrical linear matrix equations. For instance, Khatri and Mitra [1], Gross [9] and Zhang and Zhang [10] studied nonnegative and positive definite solutions to the matrix equation $AXA^* = B$, respectively. Dai and Lancaster [11] presented a condition for the existence of a symmetric, positive definite, nonnegative definite real solution and derived a formula for the general solution of the matrix equation $AXA^* = B$. Zhang [12] deduced a necessary and sufficient condition for the matrix equation $AXA^* = BB^*$ and $CXC^* = DD^*$ to have a common Hermitian nonnegative definite solution and observed its applications in statistics. Aleksandar [13] investigated the existence question of nonnegative definite solutions of the matrix equation $AX + XA = B$ where A is a given positive definite matrix and B is nonnegative definite.

For Hermitian, nonnegative and positive definite solutions of a non-symmetrical matrix equation, there have relatively few results in the literature. Guo and Huang [8] studied extremal ranks of the matrix expression $C - AXB$ with respect to Hermitian matrix X, and then we can obtain necessary and sufficient conditions for the existence of a Hermitian solution to $AXB = C$. Mitra [6] and Navarra et al. [7] provided conditions for the existence of a Hermitian solution and a representation of equation $AXB = C$. Tian [5] deduced a necessary and sufficient condition for $AXB = C$ to have a Hermitian solution and presented a general Hermitian solution expression. Khatri and Mitra [1] derived conditions for the existence of a Hermitian solutions of the equations $AX = B$, $AXB = C$ and $(AX, XB) = (E, F)$ using generalized inverses.

The purpose of this paper is to consider a Hermitian least square solution to (1.1) subject to inequality restrictions. In particular, necessary and sufficient conditions for the existence of a Hermitian positive (negative, nonpositive, nonnegative) definite least squares solution to (1.1) are derived. As far as we are aware, there has been no report concerning this problem up to the present.

We shall use the following results on ranks and inertias of matrices in the latter part of this paper.

Lemma 1.1 ([3]). *Let $\mathcal{S}$ be a set consisting of matrices over $\mathbb{C}^{m\times n}$, and let $\mathcal{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_H^m$. Then:*

(a) *For $m = n$, $\mathcal{S}$ has a nonsingular matrix if and only if $\max_{X\in\mathcal{S}} r(X) = m$.*
(b) *For $m = n$, all $X \in \mathcal{S}$ are nonsingular if and only if $\min_{X\in\mathcal{S}} r(X) = m$.*
(c) *$0 \in \mathcal{S}$ if and only if $\min_{X\in\mathcal{S}} r(X) = 0$.*
(d) *$\mathcal{H}$ has a matrix $X > 0$ $(X < 0)$ if and only if $\max_{X\in\mathcal{H}} i_+(X) = m$ $(\max_{X\in\mathcal{H}} i_-(X) = m)$.*
(e) *$\mathcal{H}$ has a matrix $X \geqslant 0$ $(X \leqslant 0)$ if and only if $\min_{X\in\mathcal{H}} i_-(X) = 0$ $(\min_{X\in\mathcal{H}} i_+(X) = 0)$.*

Lemma 1.2 ([14]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{l\times k}$. Then,*

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.2}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.3}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.4}$$

The following formulas follow from (1.2) to (1.4):

$$r\begin{bmatrix} A & BF_P \\ E_Q C & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & Q \\ 0 & P & 0 \end{bmatrix} - r(P) - r(Q), \tag{1.5}$$

$$r\begin{bmatrix} M & N \\ E_P A & E_P B \end{bmatrix} = r\begin{bmatrix} M & N & 0 \\ A & B & P \end{bmatrix} - r(P), \tag{1.6}$$

$$r\begin{bmatrix} M & AF_P \\ N & BF_P \end{bmatrix} = r\begin{bmatrix} M & A \\ N & B \\ 0 & P \end{bmatrix} - r(P). \tag{1.7}$$

The following results are well-known.

Lemma 1.3. *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$, $Q \in \mathbb{C}^{m\times n}$. Then,*

$$i_\pm\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_\pm(A) + i_\pm(B), \tag{1.8}$$

$$i_\pm\begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q). \tag{1.9}$$

Lemma 1.4 ([3]). *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}^{m\times n}$, and define*

$$U = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then,

$$i_{\pm}(U) = r(B) + i_{\pm}(E_B A E_B), \tag{1.10}$$

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^{\dagger} B \end{bmatrix}. \tag{1.11}$$

The following formula follows from (1.10) and (1.11):

$$i_{\pm}\begin{bmatrix} A & BF_P \\ F_P B^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P). \tag{1.12}$$

Concerning the consistency and general solutions of $AXB = C$, the following result is well-known; see, e.g., [1–3].

Lemma 1.5. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$ be given. Then, the matrix equation (1.1) has a solution for $X \in C^{n\times p}$ if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$. In this case, the general solution to (1.1) can be written in the following parametric form:*

$$X = A^{\dagger} C B^{\dagger} + F_A V_1 + V_2 E_B, \tag{1.13}$$

where $V_1, V_2 \in \mathbb{C}^{n\times p}$ are arbitrary.

The necessary and sufficient conditions for (1.1) to have a Hermitian solution and the general Hermitian solution expression are as follows.

Lemma 1.6 ([5–7]). *Given $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{m\times p}$, and assuming that the matrix equation $AXB = C$ is solvable for $X \in \mathbb{C}^{n\times n}$, the following statements are equivalent.*

(a) *The matrix equation $AXB = C$ has a Hermitian solution for X.*

(b)

$$R(C) \subseteq R(A), \qquad R(C^*) \subseteq R(B^*), \qquad r\begin{bmatrix} C & 0 & A \\ 0 & -C^* & B^* \\ B & A^* & 0 \end{bmatrix} = 2r[\,A^*, B\,]. \tag{1.14}$$

(c) *The pair of matrix equations*

$$AYB = C \quad \text{and} \quad B^* Y A^* = C^* \tag{1.15}$$

have a common solution for Y.

In this case, the general Hermitian solution to $AXB = C$ can be written as

$$X = \frac{1}{2}(Y + Y^*) \tag{1.16}$$

where Y is the common solution to (1.16), or equivalently,

$$X = \frac{1}{2}(Y_0 + Y_0^*) + E_G U_1 + (E_G U_1)^* + F_A U_2 F_A + E_B U_3 E_B, \tag{1.17}$$

where Y_0 is a special common solution to (1.16), $G = [\,A^*, B\,]$ and the three matrices $U_1 \in \mathbb{C}^{n\times n}$, $U_2, U_3 \in \mathbb{C}_H^n$ are arbitrary.

The following results are related to ranks and inertias of some matrix expressions.

Lemma 1.7 ([3]). *Given $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times p}$ and $D \in \mathbb{C}^{m\times q}$, define*

$$M = \begin{bmatrix} A & B & C & D \\ B^* & 0 & 0 & 0 \\ C^* & 0 & 0 & 0 \\ D^* & 0 & 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} A & B & C & D \\ B^* & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$\max_{X\in\mathbb{C}^{n\times m},Y\in\mathbb{C}_H^p,Z\in\mathbb{C}_H^q} r[\,A - BX - (BX)^* - CYC^* - DZD^*\,] = \min\{m, r(N)\}, \tag{1.18}$$

$$\min_{X\in\mathbb{C}^{n\times m},Y\in\mathbb{C}_H^p,Z\in\mathbb{C}_H^q} r[\,A - BX - (BX)^* - CYC^* - DZD^*\,] = 2r(N) - r(M) - 2r(B), \tag{1.19}$$

$$\max_{X\in\mathbb{C}^{n\times m},Y\in\mathbb{C}_H^p,Z\in\mathbb{C}_H^q} i_{\pm}[\,A - BX - (BX)^* - CYC^* - DZD^*\,] = i_{\pm}(M), \tag{1.20}$$

$$\min_{X\in\mathbb{C}^{n\times m},Y\in\mathbb{C}_H^p,Z\in\mathbb{C}_H^q} r[\,A - BX - (BX)^* - CYC^* - DZD^*\,] = r(N) - i_{\mp}(M) - r(B). \tag{1.21}$$

Lemma 1.8 (*[15,3]*).

(a) *Let $A_1, A_2, B_1, B_2, C_1, C_2$ and D be matrices such that the expression $D - C_1A_1^{\dagger}B_1 - C_2A_2^{\dagger}B_2$ is defined. Then*

$$r(D - C_1A_1^{\dagger}B_1 - C_2A_2^{\dagger}B_2) = r\begin{bmatrix} A_1^*A_1A_1^* & 0 & A_1^*B_1 \\ 0 & A_2^*A_2A_2^* & A_2^*B_2 \\ C_1A_1^* & C_2A_2^* & D \end{bmatrix} - r(A_1) - r(A_2). \tag{1.22}$$

(b) *Let A, B, C, D and P, Q be matrices such that expression $D - CP^{\dagger}AQ^{\dagger}B$ is defined. Then*

$$r(D - CP^{\dagger}AQ^{\dagger}B) = r\begin{bmatrix} P^*AQ^* & P^*PP^* & 0 \\ Q^*QQ^* & 0 & Q^*B \\ 0 & CP^* & -D \end{bmatrix} - r(P) - r(Q). \tag{1.23}$$

(c) *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{q\times n}$, $P \in \mathbb{C}^{q\times m}$ and $D \in \mathbb{C}_H^n$. Then*

$$i_{\pm}(D - B^*(P^{\dagger})^*AP^{\dagger}B) = i_{\pm}\begin{bmatrix} -PAP^* & PP^*P & 0 \\ P^*PP^* & 0 & P^*B \\ 0 & B^*P & D \end{bmatrix} - r(P). \tag{1.24}$$

2. The Hermitian least square solution of the matrix equation $AXB = C$ subject to inequality restrictions

Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times q}$ and $C \in \mathbb{C}^{m\times q}$ be given. It is well-known that the least squares solution to (1.1) is the solution of its normal equation and the normal equation corresponding to

$$\min_{X\in\mathbb{C}_H^n} \|AXB - C\| \tag{2.1}$$

is

$$A^*AXBB^* = A^*CB^*. \tag{2.2}$$

Obviously, $R(A^*CB^*) \subseteq R(A^*) = R(A^*A)$, $R(BC^*A) \subseteq R(B) = R(BB^*)$, i.e. (2.2) is consistent. Assuming that

$$r\begin{bmatrix} A^*CB^* & 0 & A^*A \\ 0 & -BC^*A & BB^* \\ BB^* & A^*A & 0 \end{bmatrix} = 2r(G), \tag{2.3}$$

from Lemma 1.6, we know that (2.2) has a Hermitian solution, and its general Hermitian solution can be written as

$$X = \frac{1}{2}(Y_0 + Y_0^*) + E_GU_1 + (E_GU_1)^* + F_AU_2F_A + E_BU_3E_B, \tag{2.4}$$

where $G = [\,A^*A, BB^*\,]$, Y_0 is a special common solution to

$$A^*AYBB^* = A^*CB^* \quad \text{and} \quad BB^*YA^*A = BC^*A, \tag{2.5}$$

and the three matrices $U_1 \in \mathbb{C}^{n\times n}$, $U_2, U_3 \in \mathbb{C}_H^n$ are arbitrary.

Lemma 2.1. *Let A, B, C, D and P, Q be matrices such that the expression $D - CP^{\dagger}AQ^{\dagger}B - (CP^{\dagger}AQ^{\dagger}B)^*$ is defined, and D is Hermitian. Define*

$$S = \begin{bmatrix} 0 & -P^*AQ^* & 0 & P^*PP^* & 0 \\ -QA^*P & 0 & QQ^*Q & 0 & 0 \\ 0 & Q^*QQ^* & 0 & 0 & Q^*B \\ PP^*P & 0 & 0 & 0 & PC^* \\ 0 & 0 & B^*Q & CP^* & D \end{bmatrix}.$$

Then

$$i_{\pm}(D - CP^{\dagger}AQ^{\dagger}B - (CP^{\dagger}AQ^{\dagger}B)^*) = i_{\pm}(S) - r(P) - r(Q). \tag{2.6}$$

Proof. Applying Lemma 1.8(c),

$$\begin{aligned}
&i_{\pm}(D - CP^{\dagger}AQ^{\dagger}B - (CP^{\dagger}AQ^{\dagger}B)^*) \\
&= i_{\pm}\left(D - [\,C, B^*\,]\begin{bmatrix}0 & Q^*\\ P & 0\end{bmatrix}^{\dagger}\begin{bmatrix}0 & A^*\\ A & 0\end{bmatrix}\begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix}^{\dagger}\begin{bmatrix}C^*\\ B\end{bmatrix}\right) \\
&= i_{\pm}\begin{bmatrix}
-\begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix}\begin{bmatrix}0 & A^*\\ A & 0\end{bmatrix}\begin{bmatrix}0 & Q^*\\ P & 0\end{bmatrix} & \begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix}\begin{bmatrix}0 & Q^*\\ P & 0\end{bmatrix}\begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix} & 0\\
\begin{bmatrix}0 & Q^*\\ P & 0\end{bmatrix}\begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix}\begin{bmatrix}0 & Q^*\\ P & 0\end{bmatrix} & 0 & \begin{bmatrix}0 & Q^*\\ P & 0\end{bmatrix}\begin{bmatrix}C^*\\ B\end{bmatrix}\\
0 & [\,C, B^*\,]\begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix} & D
\end{bmatrix} \\
&\quad - r\begin{bmatrix}0 & P^*\\ Q & 0\end{bmatrix} \\
&= i_{\pm}(S) - r(P) - r(Q). \qquad \square
\end{aligned} \tag{2.7}$$

Theorem 2.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{m\times q}$ and $P \in \mathbb{C}_H^n$ be given. Assume that (2.3) holds, an X is a Hermitian least squares solution to (1.1). Define*

$$T_1 = \begin{bmatrix} BB^* & 0 & 0 & BC^*A & 0\\ 0 & A^*A & 0 & 0 & A^*CB^*\\ 0 & A & -A & 0 & APBB^*\\ B^* & 0 & B^* & B^*PA^*A & 0\end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & -\frac{1}{2}BC^*A & 0 & -\frac{1}{2}BB^*PA^* & -BB^*\\ -\frac{1}{2}A^*CB^* & 0 & A^*A & 0 & 0\\ 0 & A^*A & 0 & A^* & 0\\ -\frac{1}{2}APBB^* & 0 & A & 0 & -A\\ -BB^* & 0 & 0 & -A^* & 0\end{bmatrix}.$$

Then,

$$\max_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} r(P - X) = \min\{n, 2n + r(T_1) - 2r(A) - 2r(B) - r(G)\}, \tag{2.8}$$

$$\min_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} r(P - X) = 2r(T_1) - r(T_2) - 2r(B), \tag{2.9}$$

$$\max_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} i_{\pm}(P - X) = n + i_{\pm}(T_2) - 2r(A) - r(B), \tag{2.10}$$

$$\min_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} i_{\pm}(P - X) = r(T_1) - i_{\mp}(T_2) - r(B). \tag{2.11}$$

Proof. Substituting (2.4) into $P - X$ yields

$$P - X = P - \frac{1}{2}(Y_0 + Y_0^*) - E_GU_1 - (E_GU_1)^* - F_AU_2F_A - E_BU_3E_B. \tag{2.12}$$

Define

$$M = \begin{bmatrix} P - \frac{1}{2}(Y_0 + Y_0^*) & E_G & F_A & E_B\\ E_G & 0 & 0 & 0\\ F_A & 0 & 0 & 0\\ E_B & 0 & 0 & 0\end{bmatrix}, \qquad N = \begin{bmatrix} P - \frac{1}{2}(Y_0 + Y_0^*) & E_G & F_A & E_B\\ E_G & 0 & 0 & 0\end{bmatrix}.$$

Applying (1.19)–(1.22) to (2.12) yields

$$\begin{aligned}\max_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} r(P-X) &= \max_{U_1\in\mathbb{C}^{n\times n}, U_2, U_3\in\mathbb{C}_H^n} r\left[P-\frac{1}{2}(Y_0+Y_0^*)-E_GU_1-(E_GU_1)^*-F_AU_2F_A-E_BU_3E_B\right]\\ &= \min\{n,\ r(N)\},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\min_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} r(P-X) &= \min_{U_1\in\mathbb{C}^{n\times n}, U_2, U_3\in\mathbb{C}_H^n} r\left[P-\frac{1}{2}(Y_0+Y_0^*)-E_GU_1-(E_GU_1)^*-F_AU_2F_A-E_BU_3E_B\right]\\ &= 2r(N)-r(M)-2r(E_G),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\max_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} i_\pm(P-X) &= \max_{U_1\in\mathbb{C}^{n\times n}, U_2, U_3\in\mathbb{C}_H^n} i_\pm\left[P-\frac{1}{2}(Y_0+Y_0^*)-E_GU_1-(E_GU_1)^*-F_AU_2F_A-E_BU_3E_B\right]\\ &= i_\pm(M),\end{aligned} \tag{2.15}$$

$$\begin{aligned}\min_{\min\|AXB-C\|, X\in\mathbb{C}_H^n} i_\pm(P-X) &= \min_{U_1\in\mathbb{C}^{n\times n}, U_2, U_3\in\mathbb{C}_H^n} i_\pm\left[P-\frac{1}{2}(Y_0+Y_0^*)-E_GU_1-(E_GU_1)^*-F_AU_2F_A-E_BU_3E_B\right]\\ &= r(N)-i_\mp(M)-r(E_G),\end{aligned} \tag{2.16}$$

We will simplify $r(N)$ and $i_\pm(M)$ by applying three types of elementary block matrix operation, elementary block congruence matrix operations and (1.6) and (1.13).

We can prove $R(E_G)\subseteq R(E_B)$ easily, then

$$\begin{aligned}
r(N) &= r\begin{bmatrix} P-\frac{1}{2}(Y_0+Y_0^*) & F_A & E_B\\ E_G & 0 & 0\end{bmatrix}\\
&= r\begin{bmatrix} P-\frac{1}{2}(Y_0+Y_0^*) & I_n & I_n & 0\\ I_n & 0 & 0 & G\\ 0 & A & 0 & 0\\ 0 & 0 & B^* & 0\end{bmatrix} - r(A)-r(B)-r(G)\\
&= r\begin{bmatrix} P-\frac{1}{2}(Y_0+Y_0^*) & I_n & I_n & 0 & 0\\ I_n & 0 & 0 & A^*A & BB^*\\ 0 & A & 0 & 0 & 0\\ 0 & 0 & B^* & 0 & 0\end{bmatrix} - r(A)-r(B)-r(G)\\
&= n + r\begin{bmatrix} I_n & 0 & A^*A & BB^*\\ -AP+\frac{1}{2}AY_0+\frac{1}{2}AY_0^* & -A & 0 & 0\\ 0 & B^* & 0 & 0\end{bmatrix} - r(A)-r(B)-r(G)\\
&= 2n + r\begin{bmatrix} -A & APA^*A-\frac{1}{2}AY_0A^*A-\frac{1}{2}AY_0^*A^*A & APBB^*-\frac{1}{2}AY_0BB^*-\frac{1}{2}AY_0^*BB^*\\ B^* & 0 & 0\end{bmatrix}\\
&\quad - r(A)-r(B)-r(G)\\
&= 2n + r\begin{bmatrix} -A & 0 & APBB^*-(A^\dagger)^*A^*CB^*\\ B^* & B^*PA^*A-B^\dagger BC^*A & 0\end{bmatrix} - r(A)-r(B)-r(G),
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
& r\begin{bmatrix} -A & 0 & APBB^*-(A^\dagger)^*A^*CB^*\\ B^* & B^*PA^*A-B^\dagger BC^*A & 0\end{bmatrix}\\
&= r\left(\begin{bmatrix} -A & 0 & APBB^*\\ B^* & B^*PA^*A & 0\end{bmatrix} - \begin{bmatrix}0\\ I_q\end{bmatrix} B^\dagger[\,0, BC^*A, 0\,] - \begin{bmatrix} I_m\\ 0\end{bmatrix}(A^*)^\dagger[\,0, 0, A^*CB^*,\,]\right)\\
&= \begin{bmatrix} B^*BB^* & 0 & 0 & B^*BC^*A & 0\\ 0 & AA^*A & 0 & 0 & AA^*CB^*\\ 0 & A & -A & 0 & APBB^*\\ B^* & 0 & B^* & B^*PA^*A & 0\end{bmatrix} - r(A)-r(B)\\
&= r(T_1)-r(A)-r(B).
\end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.17) yields

$$r(N) = 2n-2r(A)-2r(B)-r(G)+r(T_1) \tag{2.19}$$

$$
\begin{aligned}
i_{\pm}(M) &= i_{\pm}\begin{bmatrix} P-\frac{1}{2}(Y_0+Y_0^*) & F_A & E_B \\ F_A & 0 & 0 \\ E_B & 0 & 0 \end{bmatrix} \\
&= i_{\pm}\begin{bmatrix} P-\frac{1}{2}(Y_0+Y_0^*) & I_n & I_n & 0 & 0 \\ I_n & 0 & 0 & A^* & 0 \\ I_n & 0 & 0 & 0 & B \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & B^* & 0 & 0 \end{bmatrix} - r(A) - r(B) \\
&= n + i_{\pm}\begin{bmatrix} 0 & -A^* & B \\ -A & 0 & \frac{1}{2}APB - \frac{1}{4}AY_0B - \frac{1}{4}AY_0^*B \\ B^* & \frac{1}{2}B^*PA^* - \frac{1}{4}B^*Y_0^*A^* - \frac{1}{4}B^*Y_0A^* & 0 \end{bmatrix} - r(A) - r(B) \\
&= n + i_{\pm}\begin{bmatrix} 0 & -A^* & B \\ -A & 0 & \frac{1}{2}APB - \frac{1}{2}(A^\dagger)^*A^*CB^*(B^\dagger)^* \\ B^* & \frac{1}{2}B^*PA^* - \frac{1}{2}B^\dagger BC^*AA^\dagger & 0 \end{bmatrix} - r(A) - r(B), \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
& i_{\pm}\begin{bmatrix} 0 & -A^* & B \\ -A & 0 & \frac{1}{2}APB - \frac{1}{2}(A^\dagger)^*A^*CB^*(B^\dagger)^* \\ B^* & \frac{1}{2}B^*PA^* - \frac{1}{2}B^\dagger BC^*AA^\dagger & 0 \end{bmatrix} \\
&= i_{\pm}\left(\begin{bmatrix} 0 & -A^* & B \\ -A & 0 & \frac{1}{2}APB \\ B^* & \frac{1}{2}B^*PA^* & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ I_q \end{bmatrix} B^\dagger BC^*AA^\dagger[\,0, I_m, 0\,] - \frac{1}{2}\begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}(A^*)^\dagger A^*CB^*(B^*)^\dagger[\,0, 0, I_q\,]\right) \\
&= i_{\pm}\begin{bmatrix} 0 & -\frac{1}{2}B^*BC^*AA^* & 0 & B^*BB^* & 0 & 0 & 0 \\ -\frac{1}{2}AA^*CB^*B & 0 & AA^*A & 0 & 0 & 0 & 0 \\ 0 & A^*AA^* & 0 & 0 & 0 & A^* & 0 \\ BB^*B & 0 & 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & -A^* & B & \\ 0 & 0 & A & 0 & -A & 0 & \frac{1}{2}APB \\ 0 & 0 & 0 & B^* & B^* & \frac{1}{2}B^*PA^* & 0 \end{bmatrix} - r(A) - r(B) \\
&= i_{\pm}(T_2) - r(A). \qquad (2.21)
\end{aligned}
$$

Substituting (2.21) into (2.20) yields

$$i_{\pm}(M) = n + i_{\pm}(T_2) - 2r(A) - r(B). \qquad (2.22)$$

We can also obtain

$$r(M) = 2n + r(T_2) - 4r(A) - 2r(B). \qquad (2.23)$$

Substituting (2.19), (2.22) and (2.23) into (2.13)–(2.16) yields (2.8)–(2.11).
From Theorem 2.2 and Lemma 1.1, we have proved the result. □

Theorem 2.3. *Let $AXB = C$ and G, P, T_1, T_2, T_3 be stated as in Theorem 2.2. Then:*

(a) *There exist $X \geqslant P$ such that X is a Hermitian least squares solution to $AXB = C$ if and only if*

$$r(T_1) = i_-(T_2) + r(B).$$

(b) *There exist $X \leqslant P$ such that X is a Hermitian least squares solution to $AXB = C$ if and only if*

$$r(T_1) = i_+(T_2) + r(B).$$

(c) *There exist $X > P$ such that X is a Hermitian least squares solution to $AXB = C$ if and only if*

$$i_-(T_2) = 2r(A) + r(B).$$

(d) *There exist $X < P$ such that X is a Hermitian least squares solution to $AXB = C$ if and only if*

$$i_+(T_2) = 2r(A) + r(B).$$

(e) *There exists a nonsingular matrix $P - X$ such that X is a Hermitian least squares solution to $AXB = C$ if and only if*

$$n + r(T_1) \geqslant 2r(A) + 2r(B) + r(G).$$

(f) *P is a Hermitian least squares solution to $AXB = C$ if and only if*

$$2r(T_1) = r(T_2) + 2r(B).$$

If P is the zero matrix in Theorem 2.3, we have the following conclusions.

Corollary 2.4. *Let $AXB = C$ be stated as in Theorem 2.2. Define*

$$S_1 = \begin{bmatrix} BB^* & 0 & 0 & BC^*A & 0 \\ 0 & A^*A & 0 & 0 & A^*CB^* \\ 0 & A & -A & 0 & 0 \\ B^* & 0 & B^* & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & -\frac{1}{2}BC^*A & 0 & 0 & -BB^* \\ -\frac{1}{2}A^*CB^* & 0 & A^*A & 0 & 0 \\ 0 & A^*A & 0 & A^* & 0 \\ 0 & 0 & A & 0 & -A \\ -BB^* & 0 & 0 & -A^* & 0 \end{bmatrix}.$$

Then:

(a) *$AXB = C$ has a Hermitian nonnegative definite least squares solution if and only if*

$$r(S_1) = i_-(S_2) + r(B).$$

(b) *$AXB = C$ has a Hermitian nonpositive definite least squares solution if and only if*

$$r(S_1) = i_+(S_2) + r(B).$$

(c) *$AXB = C$ has a Hermitian negative definite least squares solution if and only if*

$$i_+(S_2) = 2r(A) + r(B).$$

(d) *$AXB = C$ has a Hermitian positive definite least squares solution if and only if*

$$i_-(S_2) = 2r(A) + r(B).$$

(e) *$AXB = C$ has a nonsingular Hermitian least squares solution if and only if*

$$n + r(S_1) \geqslant 2r(A) + 2r(B) + r(G).$$

(f) *0 is a least squares solution of $AXB = C$ if and only if*

$$2r(S_1) = 2r(B) + r(S_2).$$

(g) *All Hermitian least squares solutions of $AXB = C$ are nonsingular if and only if*

$$2r(S_1) = n + 2r(B) + r(S_2).$$

Finally, we will present a numerical example to verify our conclusion.

Numerical example. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -5 & -1 \\ 6 & 3 \end{bmatrix}$. Then

$$G = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & -1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} A^*CB^* & 0 & A^*A \\ 0 & -BC*A & BB^* \\ BB^* & A^*A & 0 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 & 0 & 2 & -1 \\ -5 & 9 & 0 & 0 & -1 & 2 \\ 0 & 0 & -4 & 6 & 1 & -1 \\ 0 & 0 & 5 & -9 & -1 & 2 \\ 1 & -1 & 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 2 & 0 & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0 & 0-2 & 2.5 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \\ 0 & 0 & 3 & -4.5 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ -2 & 3 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2.5 & -4.5 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix},$$

and we can compute $r(A), r(B), r(G), r(F)$ and $i_-(S_2)$ by using Matlab, obtaining results as follows:

$$r(A) = 2, \qquad r(B) = 2, \qquad r(G) = 2, \qquad r(F) = 4, \qquad i_-(S_2) = 6,$$

and

$$r(F) = 2r(G), \qquad i_-(S_2) = 2r(A) + r(B).$$

From Corollary 2.4(d), we know that there exist Hermitian positive definite least squares solution to the equation $AXB = C$. In fact, we can verify that $X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is a Hermitian positive definite least squares solution to $AXB = C$.

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