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On Finite Limit Sets for Transformations on the Unit Interval

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An infinite sequence of finite or denumerable limit sets is found for a class of many-to-one transformations of the unit interval into itself. Examples of four different types are studied in some detail; tables of numerical results are included. The limit sets are characterized by certain patterns; an algorithm for their generation is described and established. The structure and order of occurrence of these patterns is universal for the class.

1. Introduction. The iterative properties of 1-1 transformations of the unit interval into itself have received considerable study, and the general features are reasonably well understood. For many-to-one transformations, however. the situation is less satisfactory, only special and fragmentary results having been obtained to date [1, 2]. In the present paper we attempt to bring some coherence to the problem by exhibiting an infinite sequence of finite limit sets whose structure is common to a broad class of non 1-1 transformations of [0, 1] into itself. Generally speaking, the limit sets we shall construct are not the only possible ones belonging to an arbitrary transformation in the underlying class. Nevertheless, our sequence—which we shall call the "U-sequence"—constitutes perhaps the most interesting family of finite limit sets in virtue of the universality of their structure and of their order of occurrence. With regard to infinite limit sets we shall have little to say. There is reason to believe, however, that for a non-vacuous (in fact, infinite) subset of the class of transformations considered here, our construction—suitably extended to the limit of "periods of infinite length"—is exhaustive in the

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probabilistic sense, namely, that with "probability 1" every limit set belongs to the U-sequence.

2. We begin by describing the class of transformations to which our construction applies, but make no claim that the conditions imposed are strictly necessary. The description is complicated by our attempt to exclude, insofar as is possible, certain finite limit sets, not belonging to the U-sequence, whose existence and structure depend on detailed properties of the particular transformation in question.

All our transformations will be of the form

$$T_\lambda(x) : x' = \lambda f(x),$$

where x' denotes the first iterate of x (*not* the derivative!) and λ varies in a certain open interval to be specified below. The fundamental properties of $f(x)$ will be:

A.1. $f(x)$ is continuous, single-valued, and piece-wise $C^{(1)}$ on $[0, 1]$, and strictly positive on the open interval, with $f(0) = f(1) = 0$.

A.2. $f(x)$ has a unique maximum, $f_{\max} \leqslant 1$, assumed either at a point or in an interval. To the left or right of this point (or interval) $f(x)$ is strictly increasing or strictly decreasing, respectively.

A.3. At any x such that $f(x) = f_{\max}$, the derivative exists and is equal to zero.

We allow the possibility that $f(x)$ assumes its maximum in an interval so as to include certain broken-linear functions with a "flat top" (cf. example (3.4) below).

In addition to the properties (A) we need some further conditions which will serve to define the range of the parameter λ:

B. Let $\lambda_{\max} = 1/f_{\max}$. Then there exists a λ_0 such that, for $\lambda_0 < \lambda < \lambda_{\max}$, $\lambda f(x)$ has only two fixed points, the origin and $x_F(\lambda)$, say, both of which are repellent. For functions of class $C^{(1)}$ this means simply that

$$\left.\begin{aligned} \lambda \frac{df}{dx}\bigg|_{x=0} &> 1 \\ \lambda \frac{df}{dx}\bigg|_{x=x_F(\lambda)} &< -1 \end{aligned}\right. \qquad (\lambda_0 < \lambda \leqslant \lambda_{\max}).$$

For piece-wise $C^{(1)}$ functions the generalization of these conditions is obvious.

The above conditions are sufficient to guarantee the existence of the

U-sequence; that they are not necessary can be shown by various examples, but we shall not pursue this matter here.

$f(x)$ as defined above clearly has the property that its piecewise derivative is less than 1 in absolute value in some interval N which includes that for which $f(x) = f_{\max}$. In order to exclude certain unwanted finite limit sets, we append the following condition:

C. $f(x)$ is convex in the interval N; at every point $x \notin N$, the piece-wise derivative of $f(x)$ is greater than 1 in absolute value.

Unfortunately, property (C) is not sufficient to exclude all finite limit sets not given by our construction; to achieve this end it might be necessary to restrict the underlying class of transformations rather drastically. We shall return to this point in Section 4 below.

It will simplify the subsequent discussion to make the non-essential assumption that $f(x)$ assumes its maximum at the point $x = \frac{1}{2}$ (or, if the function assumes its maximum in an interval, that the interval includes $x = \frac{1}{2}$). In the sequel we shall make this assumption without further comment. A particular iterate x' will then be said to be of "type L" or of "type R" according as $x' < \frac{1}{2}$ or $x' > \frac{1}{2}$, respectively. Given an initial point x_0, the minimum distinguishing information about the sequence of iterates $T_\lambda^{(k)}(x_0)$, $k = 1, 2, \ldots$, will consist in a "pattern" of R's and L's, the k-th letter giving the relative position of the k-th iterate of x_0 with respect to the point $x = \frac{1}{2}$. The patterns turn out to play a fundamental role in our construction; they will be discussed in detail in the following sections.

3. Let us give some simple examples of the class of transformations we are considering:

$$Q_\lambda(x): \quad x' = \lambda x(1 - x)$$
$$3 < \lambda < 4 \tag{3.1}$$

$$S_\lambda(x): \quad x' = \lambda \sin \pi x$$
$$\lambda_0 < \lambda < 1 \qquad (\text{with } .71 < \lambda_0 < .72) \tag{3.2}$$

$$C_\lambda(x): \quad x' = \lambda W(3 - 3W + W^2), \qquad W \equiv 3x(1 - x)$$
$$\lambda_0 < \lambda < \frac{64}{63} \qquad (\text{with } .872 < \lambda_0 < .873) \tag{3.3}$$

In the last two examples, more precise limits for λ_0 are available, but they are not important for our discussion. All these examples are convex functions of class $C^{(\infty)}$ which are, moreover, symmetric about $x = \frac{1}{2}$.

With regard to the existence of the U-sequence, these restrictions are in no way essential. As will be remarked in Section 4, however, these examples happen to belong to the subclass for which our construction does exhaust all finite limit sets.

As a further example, consider the broken-linear mapping:

$$\begin{aligned} L_\lambda(x; e):\quad & x' = \frac{\lambda}{e} x, && 0 \leqslant x \leqslant e, \\ & x' = \lambda, && e \leqslant x \leqslant 1 - e, \\ & x' = \frac{\lambda}{e}(1 - x), && 1 - e \leqslant x \leqslant 1, \\ & && \text{with} \quad 1 - e < \lambda < 1. \end{aligned} \tag{3.4}$$

Here e is a parameter characterizing the width 1-2e of the maximum, and may be chosen to have any value in the range $0 < e < \frac{1}{2}$. It remains fixed as λ is varied, and different choices of e yield distinct transformations.

4. The finite limit sets of our class of transformations—and, in particular, of the four special transformations given above—are attractive periods of order $k = 2, 3, \ldots$. (We exclude the case $k = 1$ by invoking property (B).) The reader will recall that an "attractive period of order k" is a set of k periodic points x_i, $i = 1, 2, \ldots, k$, with $T_\lambda(x_i) = x_{i+1}$ (in some order). Each of these is a fixed point of the k-th power of $T_\lambda : T_\lambda^{(k)}(x_i) = x_i$, for which, moreover, the (piece-wise) derivative satisfies

$$\left| \frac{dT_\lambda^{(k)}}{dx} \right|_{x=x_i} < 1.$$

(By the chain rule, the slope is the same at all points in the period.) As a consequence of this slope condition, there exists for each x_i an attractive neighborhood $n(x_i)$ such that for any $x^* \in n(x_i)$ the sequence of iterates $T_\lambda^{(jk)}(x^*), j = 1, 2, \ldots,$ will converge to x_i. Periodic points which do not satisfy this slope condition (more precisely, for which the absolute value of the derivative is greater than 1) have no attractive neighborhood; they are consequently termed repellent (or unstable). These points belong to what is sometimes called "the set of exceptional points," a set of measure zero in the interval which plays no role in a discussion of limit sets.

The sequence of finite periods which we shall exhibit will be characterized *inter alia* by the following property:

J. For every period belonging to the U-sequence there is a period point whose attractive neighborhood includes the point $x = \frac{1}{2}$.

Now it follows from a theorem of G. Julia [3] that, if $T_\lambda(x)$ is the restriction to [0, 1] of some function analytic in the complex plane whose derivative vanishes at a single point in the interval, then the only possible finite limit sets ($k > 1$) are those with the property (J). The transformations (3.1) through (3.3) are clearly of this type, so that, with respect to finite limit sets, the U-sequence will exhaust all possibilities for them. That Julia's criterion is not necessary is shown by example (3.4); in this case there cannot be any attractive periods which do not have a period point lying in the region $e \leqslant x \leqslant 1 - e$. Such a period, however, clearly is of the type described by property (J), and hence belongs to the U-sequence.

Taking our clue from property (J), we now investigate the solutions λ of the equation:

$$T_\lambda^{(k)}(\tfrac{1}{2}) = \tfrac{1}{2}. \tag{4.1}$$

The corresponding periodic limit sets will be attractive, since the slope of $\lambda f(x)$ at $x = \frac{1}{2}$ is zero by hypothesis (property (A.3)). By way of example, we choose $k = 5$. Then for each of the four transformations of Section 3 there are precisely three distinct solutions of equation (4.1). The three patterns—common to all four transformations—are:

$$\tfrac{1}{2} \to R \to L \to R \to R \to \tfrac{1}{2},$$
$$\tfrac{1}{2} \to R \to L \to L \to R \to \tfrac{1}{2},$$
$$\tfrac{1}{2} \to R \to L \to L \to L \to \tfrac{1}{2}.$$

Omitting the initial and final points as understood, we write these patterns in the simplified form:

$$\begin{aligned} &\mathrm{RLR}^2, \\ &\mathrm{RL}^2\mathrm{R}, \\ &\mathrm{RL}^3. \end{aligned} \tag{4.2}$$

In accordance with this convention, a pattern with $k - 1$ letters R or L will be said to be of "length k."

These solutions are clearly ordered on the parameter λ. In Table I we give the full set of solutions of (4.1), through $k = 7$, for all four special transformations; in the broken-linear case we choose $e = .45$. The numerical values of λ were found by a simple iterative technique (the "binary chopping process"); although they are given to only seven decimal digits, they are actually known to approximately twice that precision. Of course, once a particular λ has been found, the corresponding pattern can be generated by direct iteration.

TABLE I

			Values of λ_i			
i	k_i	P_i	$Q_\lambda(x)$	$S_\lambda(x)$	$C_\lambda(x)$	$L_\lambda(x; .45)$
1	2	R	3.2360680	.7777338	.9325336	.6581139
2	4	RLR	3.4985617	.8463822	.9764613	.7457329
3	6	RLR^3	3.6275575	.8811406	.9895107	.7806832
4	7	RLR^4	3.7017692	.9004906	.9955132	.8031673
5	5	RLR^2	3.7389149	.9109230	.9990381	.8180892
6	7	RLR^2LR	3.7742142	.9213346	1.0024311	.8318799
7	3	RL	3.8318741	.9390431	1.0073533	.8645337
8	6	RL^2RL	3.8445688	.9435875	1.0083134	.8858150
9	7	RL^2RLR	3.8860459	.9568445	1.0111617	.8977794
10	5	RL^2R	3.9057065	.9633656	1.0123766	.9085993
11	7	RL^2R^3	3.9221934	.9687826	1.0132699	.9187692
12	6	RL^2R^2	3.9375364	.9735656	1.0140237	.9278274
13	7	RL^2R^2L	3.9510322	.9782512	1.0146450	.9361518
14	4	RL^2	3.9602701	.9820353	1.0149542	.9462185
15	7	RL^3RL	3.9689769	.9857811	1.0152122	.9564172
16	6	RL^3R	3.9777664	.9892022	1.0154974	.9635343
17	7	RL^3R^2	3.9847476	.9919145	1.0156711	.9702076
18	5	RL^3	3.9902670	.9944717	1.0157727	.9775473
19	7	RL^4R	3.9945378	.9966609	1.0158320	.9846165
20	6	RL^4	3.9975831	.9982647	1.0158621	.9903134
21	7	RL^5	3.9993971	.9994507	1.0158718	.9957404

We note that the set of 21 patterns and its λ-ordering is common to all four transformations. This remains true when we extend our calculations through $k = 15$. As k increases, the total number of solutions of (4.1) becomes large, as indicated in Table II. Thus for $k \leqslant 15$ there is a total of 2370 distinct solutions of equation (4.1). In the appendix we give a complete list of ordered patterns for $k \leqslant 11$.

The fact that these patterns and their λ-ordering are a common property of four apparently unrelated transformations (note that they are not connected by ordinary conjugacy, a relation which will be discussed in Section (6) suggests that the pattern sequence is a general property of a wide class of mappings. For this reason we have called this sequence of patterns the U-sequence where "U" stands (with some exaggeration) for

TABLE II

k ...	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of solutions ...	1	1	2	3	5	9	16	28	51	93	170	315	585	1091

"universal." In the next section we shall state and prove a logical algorithm which generates the U-sequence for any transformation having the properties (A) and (B) of Section 2. In the present section we confine ourselves to describing what might be called the "λ-structure" of the limit sets associated with the patterns of the U-sequence. No proofs are included, since the results given here will not be used in the proof of our main theorem.

As constructed, the patterns of the U-sequence correspond to distinct solutions of equation (4.1); they are attractive k-periods containing the point $x = \frac{1}{2}$ and possessing the property (J). It is clear by continuity that, given any solution λ (with finite k) and its associated pattern $P_k(\lambda)$, then for sufficiently small $\epsilon > 0$ there will exist periodic limit sets with the same pattern for all $\bar{\lambda}$ in the interval $\lambda - \epsilon \leqslant \bar{\lambda} \leqslant \lambda + \epsilon$. In other words, each period has a finite "λ-width." It is also clear that there exist critical values $m_1(\lambda)$ and $m_2(\lambda)$ such that, for $\bar{\lambda} < \lambda - m_1$ and $\bar{\lambda} > \lambda + m_2$, the pattern $P_k(\lambda)$ does *not* correspond to an attractive period of $T_{\bar{\lambda}}(x)$.

Consider now for simplicity the case in which the transformation is $C^{(1)}$, and take m_1 and m_2 to be boundary values such that for $\lambda - m_1 < \bar{\lambda} < \lambda + m_2$ the periodic limit set with pattern $P_k(\lambda)$ is attractive. As $\bar{\lambda}$ varies in this interval, the slope $dT_{\bar{\lambda}}^{(k)}/dx$ at a period point varies continuously from $+1$ to -1, the values ± 1 being assumed at the boundary points. It is natural to ask: what happens if $\bar{\lambda}$ lies just to the left or just to the right of the above interval? The question as to what the limit sets look like if $\bar{\lambda} = \lambda - m_1 - \delta$ (δ small) is a difficult one; the conjectured behavior will be described in Section 6, but rigorous proof is lacking. For $\bar{\lambda} = \lambda + m_2 + \delta$ we are in better case. As shown in Section 5, corresponding to any solution λ of (4.1) and its associated pattern there exists an infinite sequence of solutions $\lambda < \lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(\infty)}$ with associated patterns $H^{(1)}(\lambda^{(1)}), H^{(2)}(\lambda^{(2)}),...,$ called "harmonics," with the property that they exhaust all possible solutions λ^* in the interval $\lambda \leqslant \lambda^* \leqslant \lambda^{(\infty)}$. The sequence of harmonics of a given solution is a set of periods of order $2^m k$, $m = 1, 2,...,$ with contiguous λ-widths and well-defined pattern structure; no other periods of the U-sequence can exist for any λ^* in the given interval (harmonics have been encountered before in a more restrictive context: cf. reference 4). From the construction of Section 5 it will be obvious that $\lambda^{(\infty)}$ exists as a right-hand limit; the question as to the nature of the limit sets for $\lambda^* = \lambda^{(\infty)} + \epsilon$ remains open (but cf. Conjecture 2 of Section 6).

In order to prepare the ground for the discussion in the next section, we give here the following formal

DEFINITION 1. Let $P = \mathrm{RL}^{\alpha_1}\mathrm{R}^{\alpha_2}\mathrm{L}^{\alpha_3}\cdots$ be a pattern corresponding to some solution of (4.1). Then the (first) harmonic of P is the pattern $H = P\mu P$, where $\mu = L$ if P contains an odd number of R's, and $\mu = R$ otherwise.

For example, the pattern RLR^2 has the harmonic $H = RLR^2LRLR^2$, while for RL^2R we have $H = RL^2R^3L^2R$.

Naturally, the construction of the harmonic can be iterated, so that one may speak of the second, third,..., m-th harmonic, etc. When necessary, we shall write $H^{(j)}$ to denote the j-th harmonic.

In addition to the harmonic H of a pattern P, there is another formal construct which will be used in the sequel:

DEFINITION 2. The "antiharmonic" A of a pattern P is constructed analogously to the harmonic H except that $\mu = \mathrm{L}$ when P contains an even number of R's, while $\mu = \mathrm{R}$ otherwise.

Thus in passing from a pattern to its harmonic the "R-parity" changes, while for the antiharmonic the parity remains the same. The antiharmonic is a purely formal construct and never corresponds to any periodic limit set; the reason for this will become clear in the next section. Note that, like that of the harmonic, the antiharmonic construction can be iterated to any desired order.

5. We begin by defining the "extension" of a pattern:

DEFINITION 3. The H-extension of a pattern P is the pattern generated by iterating the harmonic construction applied j times to P, where j increases indefinitely.

DEFINITION 4. The A-extension of P is the pattern $A^{(j)}(P)$, where j increases indefinitely. Here $A^{(j)}(P)$ denotes the j-th iterate of the antiharmonic.

In these definitions we avoid writing $j \to \infty$, in order to avoid raising questions concerning the structure of the limiting pattern. In practice, all that will be required is that j is "sufficiently large."

We are now in a position to state

THEOREM 1. *Let K be an integer. Consider the complete ordered sequence of solutions of* (4.1) *and their associated patterns for* $2 \leqslant k \leqslant K$. *Let* λ_1 *be any such solution with pattern* P_1 *and length* k_1, *and let* $\lambda_2 > \lambda_1$ be *the "adjacent" solution (i.e., the next in order) with pattern* P_2 *and length* k_2.

Form the H-extension of P_1 *and the A-extension of* P_2. *Reading from left to right, the two extensions* $H(P_1)$ *and* $A(P_2)$ *will have a maximal common leading subpattern* P^* *of length* k^*, *so that we may write*

$$H(P_1) = P^*\mu_1 \ldots, \; A(P_2) = P^*\mu_2 \ldots, \; \mu_1 \neq \mu_2,$$

where μ_i *stands for one of the letters* L *or* R.

Case 1. $k^* \geqslant 2k_1$. *Then the solution* λ^* *of lowest order such that* $\lambda_1 < \lambda^* < \lambda_2$ *is the harmonic of* P_1.

Case 2. $k^* < 2k_1$. *Then the solution* λ^* *of lowest order such that* $\lambda_1 < \lambda^* < \lambda_2$ *corresponds to the pattern* P^* *of length* k^* ($> K$ *necessarily*).

A simple consequence of this theorem is the following:

COROLLARY. *Let* $|k_1 - k_2| = 1$ *in Theorem* 1. *Then the lowest order solution* λ^* *with* $\lambda_1 < \lambda^* < \lambda_2$ *has length* $k^* = 1 + \max(k_1, k_2)$.

This follows from the theorem on noting that all patterns have the common leading subpatterns (not maximal!) RL; therefore, in forming the extensions, the first disagreement will indeed come at the indicated value of k^*.

We give some examples of the application of Theorem 1.

Example 1. Take $K = 9$. Reference to the table in the appendix shows that patterns #12 and #14 are adjacent. We have

$$P_1 = \text{RLR}^4, \; k_1 = 7, \; H(P_1) = \text{RLR}^4\text{LRLR}^4\ldots,$$
$$P_2 = \text{RLR}^4\text{LR}, \; k_2 = 9, \; A(P_2) = \text{RLR}^4\text{LRLRL}\ldots,$$

so $P^* = \text{RLR}^4\text{LRLR}$, with $k^* = 11$, as verified by the table.

Example 2. Again take $K = 9$. Patterns #16 and #19 are adjacent and $P_1 = \text{RLR}^2$ with $k_1 = 5$; here $k^* \geqslant 2k_1$. Therefore, by Case 1, the lowest order solution between the two patterns is the harmonic of P_1, namely, $\text{RLR}^2\text{LRLR}^2$, as given in the table (pattern #17).

To prove Theorem 1 we must first introduce some new concepts. Consider the transformation:

$$T_m(x): x' = \lambda_{\max} f(x) \tag{5.1}$$

This transformation maps [0, 1] *onto* itself; hence, for any point in the interval, the inverses of all orders exist. Let us restrict ourselves for the moment to the point $x = \frac{1}{2}$ and its set (2^k in number) of k-th order inverses. At each step in constructing a k-th order inverse we have the free choice of taking a point on the right or on the left. For example, designating the point $x = \frac{1}{2}$ by the letter O, a possible inverse of order 5 would be represented by the sequence of letters

$$\mathrm{RLR^2O}, \tag{5.2}$$

which is to be read from *right to left*. Let us call a sequence like (5.2), when read from right to left, a "5-th order inverse path of the point $x = \frac{1}{2}$." Note that (5.2) is precisely the pattern associated with the first solution of equation (4.1) for $k = 5$. Another possible inverse path of the same order would be $\mathrm{L^2R^2O}$, but this clearly does not correspond to any solution of (4.1).

Choosing a particular k-th order inverse path of $x = \frac{1}{2}$, let us call the numerical value of the corresponding k-th inverse the "coordinate" of the path. Obviously, no path whose coordinate is less than $\frac{1}{2}$ can correspond to a pattern associated with a solution of (4.1). In order to achieve a 1-1 correspondence between a subclass of inverse paths and our periodic patterns we introduce the concept of a "legal inverse path," which we abbreviate as "l.i.p."

DEFINITION 5. For the transformation $T_m(x)$ (cf. (5.1)), an l.i.p. of order k is a k-th order inverse path of $x = \frac{1}{2}$ whose coordinate x_k has the greatest numerical value of any point on the path.

In other words, of all the inverses constituting the path, the coordinate (i.e., the k-th inverse) lies farthest to the right. Note that any inverse path of $x = \frac{1}{2}$ can be inversely extended to an l.i.p. by appending on the left some suitable sequence, e.g., the sequence RL^α with α sufficiently large. Now consider the transformation $T_\lambda(x)$ corresponding to $T_m(x)$ with $\lambda < \lambda_{\max}$. As λ decreases, the original l.i.p. is deformed into an inverse path with varying coordinate $x_k(\lambda)$, but with the same pattern. By continuity, there clearly exists a λ^* for which

$$T_{\lambda^*}(\tfrac{1}{2}) = x_k(\lambda^*);$$

this in turn implies that for $\lambda = \lambda^*$ there exists a solution of equation (4.1) with the same pattern as that of the original l.i.p. On the other hand, for an inverse path (with, say, an R-type coordinate) which is *not* an l.i.p. the cycle will close on some intermediate point of the path (farther to the right than $x_k(\lambda)$), so that the path cannot be further inverted; this means that the original pattern cannot correspond to a solution of (4.1). Thus we have proved

LEMMA 1. *There is a* 1-1 *correspondence between the set of l.i.p.'s and the patterns associated with the solutions of equation* (4.1).

We note that the l.i.p.'s are naturally ordered on the values of their coordinates. By Lemma 1, any true statement about the pattern structure and coordinate ordering of the set of l.i.p.'s corresponds to a true statement about the pattern structure and λ-ordering of the set of solutions of (4.1).

Given some l.i.p. of order k, we construct an *inverse extension* $I(P)$ of the path according to the prescription $I(P) = P\mu PO$, where μ is R or L. Obviously, one choice corresponds to the harmonic, the other to the antiharmonic (Definitions 1 and 2). We can therefore speak of the harmonic or antiharmonic of an l.i.p. as well as of a pattern. Now, because of the monotonicity property (A.2) it follows that, given any two points, taking the left-hand inverse of both points preserves their relative order, while taking the right-hand inverse reverses it. A simple argument shows that $x_A < x < x_H$, where x is the coordinate of some l.i.p. and x_A, x_H are the coordinates of its antiharmonic and harmonic, respectively. This explains why the harmonic of an l.i.p. is again an l.i.p.. while the antiharmonic is not (and hence can never correspond to an attractive period of the U-sequence).

One final concept, the "projection" of an interval, will be of value in the subsequent discussion.

DEFINITION 6. Choose any two points x_1, x_2 in (0, 1); they define some interval I. Let $\bar{P}$ be an arbitrary sequence of R's and L's with $k - 1$ letters in all. Now, for some $T_m(x)$, construct the inverse paths $\bar{P}x_1$ and $\bar{P}x_2$. The coordinate x_1^* and x_2^* of these two paths define a new interval I^*, called the "projection under $\bar{P}$ of I." Because the defining inverse paths are of length k, we refer to it as a k-th order projection. (If we wish to exhibit explicitly the end-points x_1, x_2 of the interval I, we write I(x_1, x_2); in contrast to the usual notation for an interval, no ordering is implied.)

Proof of Theorem 1. It is clear that, if two intervals I, I^* are related by a k-th order projection, then for any point $x \in I^*$ we have $T_m^{(k)}(x) \in I$.

Consider now any l.i.p. with pattern PO and coordinate x_1. Its harmonic is again an l.i.p., with pattern PμPO and coordinate x_H, μ being either R or L depending on the R-parity of P. If x_μ is the point corresponding to the choice μ, then this construction shows that the interval $I^*(x_1, x_H)$ is the (k-th order) projection under P of the interval $I(\frac{1}{2}, x_\mu)$. Now any point x in the interior of I^* must map into the interior of I, and the end-points must map into end-points. Thus no inverse path of $x = \frac{1}{2}$—which is one of the end-points of I—can have a coordinate x^* satisfying $x_1 < x^* < x_H$. Precisely the same argument can be made for the antiharmonic. This proves

LEMMA 2. *Let* PO *be some l.i.p. with coordinate* x_1. *Form the* $H^{(j)}$-*extension of P, with coordinate* $x_H^{(j)}$, *and the* $A^{(j)}$-*extension of P with coordinate* $x_A^{(j)}$. *We then have* $x_A^{(j)} < x_1 < x_H^{(j)}$. *The intervals* $I^*(x_1, x_H^{(j)})$ *and* $I(x_1, x_A^{(j)})$ *do not contain the coordinate of any inverse path of* $x = \frac{1}{2}$.

The left-hand interval $I^*(x_1, x_A^{(j)})$ is of no significance for the limit sets of $T_\lambda(x)$; in fact, for λ a solution of (4.1) this interval shrinks to zero (and for $\lambda^* < \lambda$, neither the harmonic nor the antiharmonic exists). The right-hand, interval, however, is important. Using Lemma 2 and Lemma 1 we immediately derive

LEMMA 3. *If* λ_1 *is a solution of equation* (4.1) *and* λ_H *is the solution corresponding to its harmonic, then there does not exist any solution* λ^* *of* (4.1) *with the property* $\lambda_1 < \lambda^* < \lambda_H$.

Iterating this argument, we verify the statement of Section 4 that the sequence of harmonics is contiguous, i.e., that harmonics are always adjacent.

The adjacency property of harmonics serves to prove Case 1 of Theorem 1. We now proceed to Case 2.

Given some K, let (P_1, x_1, k_1) and (P_2, x_2, k_2) be two adjacent l.i.p.'s with $x_1 < x_2$ and $K + 1 < 2k_1$. Form the H-extension of P_1 and the A-extension of P_2; these can be written in the form

$$\begin{aligned} H(P_1) &= P^*\mu_1 \cdots \\ A(P_2) &= P^*\mu_2 \cdots \end{aligned} \qquad (\mu_1 \neq \mu_2).$$

The coordinates x_H and x_A define an interval I^* which is a projection of $I(x_{\mu_1}, x_{\mu_2})$; clearly, I^* is contained in the original interval $I(x_1, x_2)$. Since I contains the point $x = \frac{1}{2}$, there must exist an inverse path of $\frac{1}{2}$, P*O, with coordinate x^* satisfying $x_1 < x^* < x_2$. But P*O must be an l.i.p. since it is a leading subpattern of the interated harmonic of P_1. Moreover, by the adjacency assumption, its length k^* must necessarily be greater than k_1 or k_2. On the other hand, P*O is the shortest pattern for which an interval with non-zero content exists. Invoking Lemma 1, we see that the proof of the theorem is complete.

The formulation of a practical algorithm, using the results of Theorem 1, needs little comment. Given the complete U-sequence for $k \leqslant K$, one generates the sequence for $K + 1$ by inserting the appropriate pattern of length $K + 1$ between every two (non-harmonic) adjacent patterns whenever the theorem permits it. The pattern $R(k = 2)$ remains the lowest pattern; as is easily shown, for any k the last pattern is always of the form RL^{k-2}, and this is simply appended to the list. As previously mentioned, the algorithm has been checked (to $k \leqslant 15$) for the four special transformations of Section 3 by actually finding the corresponding solutions of equation (4.1)—a simple process in which there are no serious accuracy limitations.

We remark here that the combinatorial problem of enumerating all l.i.p.'s of a given length k has been solved [5]; the number of patterns turns out to be just the number of symmetry types of primitive periodic sequences (with two "values" or letters allowed) under the cyclic group C_k (so that the full symmetry group is $C_k \times S_2$, where S_2 is the symmetric group on two letters). For k a prime, this number is simple, and turns out to be given by the expression

$$\frac{1}{k}(2^{k-1} - 1).$$

We encountered this enumeration problem previously (cf. reference 4, Table 1); at that time we were not aware of the work of Gilbert and Riordan [5].

6. In this final section we collect some observations and conjectures concerning the nature of limit sets not belonging to the U-sequence, ending with a few remarks on the relation of conjugacy.

(1) *Other finite limit sets.* As remarked in the introduction, it does not seem possible to exclude "anomalous" limit sets without seriously restricting the underlying class of transformations. To convince the reader that such anomalous periods can in fact exist we give a simple example:

Let us alter the special transformation (3.4) in the following way (we take $e = .45$):

$$\begin{aligned} x' &= 4.5\lambda x, && 0 \leqslant x \leqslant .2 \\ x' &= \lambda(.4x + .82), && .2 \leqslant x \leqslant .45 \\ x' &= \lambda, && .45 \leqslant x \leqslant .55 \\ x' &= \frac{\lambda}{.45}(1 - x), && .55 \leqslant x \leqslant 1 \end{aligned} \qquad (.55 < \lambda < 1). \qquad (6.1)$$

Then, in addition to the U-sequence (with λ values different from those of the original transformation), there exists an attractive 2-period in the range $\lambda_1 < \lambda \leqslant 1$ with

$$\lambda_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{.19}.$$

Note that the 2-period remains attractive even for $\lambda = 1$. While the anomalous periods do not affect the existence of the U-sequence, they do cause additional partitioning of the unit interval because their existence implies that there is a set of points (with non-zero measure) whose sequence of iterated images will converge to the periods in question.

These anomalous periods, however, differ radically from those belonging to the U-sequence in that they do not possess the property (J). This in turn means that the slope at a period point is strongly bounded away from zero. Thus, at least for transformations with the property (C), it seems reasonable to conjecture that such periods cannot have arbitrary length and still remain attractive. Hence we make

CONJECTURE 1. *For transformations with properties* (A), (B) *and* (C), *the anomalous attractive periods constitute at most a finite sequence.*

(2) *Infinite limit sets.* For simplicity we consider the case in which there are no anomalous periods, e.g., functions covered by Julia's theorem (or some valid extension thereof). We assign to each period of the U-sequence a λ-measure equal to its λ-width. The question is then: is the λ-measure of the full U-sequence equal to $\lambda_{max} - \lambda_0$? Or, put otherwise, is there a set of non-zero measure in the interval $(\lambda_0, \lambda_{max})$ such that the sequence of iterates of $x = \frac{1}{2}$ does *not* converge to a member of the U-sequence? Numerical experiments with the four special transformations of Section 3 together with some heuristic arguments based on the iteration of the algorithm of Theorem 1 leads us to make the modest

CONJECTURE 2. *For an infinite subclass of transformations with properties* (A) *and* (B), *the* λ*-measure of the U-sequence is the whole* λ *interval.*

(3) *A limiting case.* Take the transformation $L_\lambda(x; e)$ of Section 3 and set $e = \frac{1}{2}$. We then have

$$L_\lambda(x; \tfrac{1}{2}): \begin{aligned} x' &= 2\lambda x, \quad 0 \leqslant x \leqslant \tfrac{1}{2} \\ x' &= 2\lambda(1 - x), \quad \tfrac{1}{2} \leqslant x \leqslant 1 \end{aligned} \qquad (\tfrac{1}{2} < \lambda < 1). \tag{6.2}$$

Although we cannot speak of attractive periods in this case (since the slope of the function is nowhere less than 1 in absolute value), it is still of interest to investigate the corresponding solutions of equation (4.1). These turn out to be a subset of the U-sequence in which the 2-period, all harmonics, and all patterns algorithmically generated from the harmonics and adjacent nonharmonics, are absent. The count through $k \leqslant 15$ is given in Table III, which may be compared with Table II.

TABLE III

k ...	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of solutions	1	1	3	4	9	14	27	48	93	163	315	576	1085

One can explain this behavior by saying that, as the width 1-2e of the flat-top shrinks to zero, the harmonics and harmonic-generated periods "coalesce" in structure with their fundamentals. This provides another illustration of the nature of the harmonics outlined in Section 4.

(4) *Conjugacy.* Two transformations $f(x)$, $g(x)$ on [0, 1] are said to be conjugate to each other if there exists a continuous, 1-1 mapping $h(x)$ of [0, 1] onto itself such that

$$g(x) = hf[h^{-1}(x)], \qquad x \in [0, 1]. \tag{6.3}$$

If $f(x)$ and $g(x)$ are themselves 1-1, the question of the existence of an $h(x)$ satisfying (6.3) is settled by a theorem of Schreier and Ulam [6]. When $f(x)$, $g(x)$ are not homeomorphisms, very little is known about the existence or nonexistence of a conjugating function $h(x)$.

The importance of (6.3) for our purpose is that the attractive nature of limit sets is preserved under conjugacy; in particular, if $T_\lambda(x)$ possess the U-sequence, then so does every conjugate of it. Clearly, our class of transformations must be invariant under conjugation by the set of all continuous, 1-1 functions $h(x)$ on [0, 1]. (Incidentally, we now see why our special choice of the point $x = \frac{1}{2}$ is no restriction, since it can be shifted by conjugation with an appropriate $h(x)$.)

It has long been known [7] that the parabolic transformation (3.1) with $\lambda = \lambda_{\max} = 4$ is conjugate to the broken-linear transformation (6.2) with $\lambda = 1$, the conjugating function being

$$h(x) = \frac{2}{\pi} \sin^{-1} (\sqrt{x}).$$

In general, no such pairwise conjugacy exists for the four special transformations of Section 3. For particular choices of the parameters this can be shown by making the following simple test. If $f(x_0) = x_0$ and $g(x_1) = x_1 (x_0, x_1 \neq 0)$, then a short calculation shows that

$$\left.\frac{df(x)}{dx}\right|_{x=x_0} = \left.\frac{dg(x)}{dx}\right|_{x=x_1}. \tag{6.4}$$

It is easily established that (6.4) does not hold in general for any pair of our special transformations.

In view of the existence of the U-sequence, it is of interest to speculate whether there is not some well-defined but less restrictive equivalence relation that will serve to replace conjugacy (for one such suggestion-due to S. Ulam—see the remarks in reference 1, p. 49). Of course, Theorem 1 itself provides such an equivalence relation, albeit not a very useful one:

Let $T_{1\lambda}(x)$, $T_{2\mu}(x)$ be two transformations with properties (A) and (B). Then there exists a mapping function M_{12} such that $M_{12}(\lambda) = \mu$, the domain of M being the union of the λ-widths of the U-sequence for T_1 and the range being the union of the μ-widths of the U-sequence for T_2.

Since at present nothing whatsoever is known about these mappings M_{ij}, the above correspondence amounts to nothing more than a restatement of the existence of the U-sequence itself.

Appendix

The following table gives the complete ordered set of patterns associated with the U-sequence for $K \leqslant 11$; i is a running index, K gives the pattern length, and $I(K)$ indicates the relative order of periods of given length K. The ordering corresponds to the λ-ordering of solutions of equation (4.1).

i	K	I(K)	Pattern	i	K	I(K)	Pattern
1	2	1	R	41	10	9	RL^2RLR^3L
2	4	1	RLR	42	7	3	RL^2RLR
3	8	1	RLR^3LR	43	10	10	$RL^2RLRLRL$
4	10	1	RLR^3LRLR	44	11	15	$RL^2RLRLRLR$
5	6	1	RLR^3	45	9	7	RL^2RLRLR
6	10	2	RLR^5LR	46	11	16	RL^2RLRLR^3
7	8	2	RLR^5	47	10	11	RL^2RLRLR^2
8	10	3	RLR^7	48	11	17	RL^2RLRLR^2L
9	11	1	RLR^8	49	8	5	RL^2RLRL
10	9	1	RLR^6	50	11	18	RL^2RLRL^2RL
11	11	2	RLR^6LR	51	5	2	RL^2R
12	7	1	RLR^4	52	10	12	$RL^2R^3L^2R$
13	11	3	RLR^4LRLR	53	11	19	$RL^2R^3L^2RL$
14	9	2	RLR^4LR	54	8	6	RL^2R^3L
15	11	4	RLR^4LR^3	55	11	20	$RL^2R^3LR^2L$
16	5	1	RLR^2	56	10	13	$RL^2R^3LR^2$
17	10	4	RLR^2LRLR^2	57	11	21	$RL^2R^3LR^3$
18	11	5	RLR^2LRLR^3	58	9	8	RL^2R^3LR
19	9	3	RLR^2LRLR	59	11	22	RL^2R^3LRLR
20	11	6	$RLR^2LRLRLR$	60	10	14	RL^2R^3LRL
21	7	2	RLR^2LR	61	7	4	RL^2R^3
22	11	7	RLR^2LR^3LR	62	10	15	RL^2R^5L
23	9	4	RLR^2LR^3	63	11	23	RL^2R^5LR
24	11	8	RLR^2LR^5	64	9	9	RL^2R^5
25	10	5	RLR^2LR^4	65	11	24	RL^2R^7
26	8	3	RLR^2LR^2	66	10	16	RL^2R^6
27	10	6	RLR^2LR^2LR	67	11	25	RL^2R^6L
28	11	9	$RLR^2LR^2LR^2$	68	8	7	RL^2R^4
29	3	1	RL	69	11	26	RL^2R^4LRL
30	6	2	RL^2RL	70	10	17	RL^2R^4LR
31	9	5	RL^2RLR^2L	71	11	27	$RL^2R^4LR^2$
32	11	10	$RL^2RLR^2LR^2$	72	9	10	RL^2R^4L
33	10	7	RL^2RLR^2LR	73	11	28	$RL^2R^4L^2R$
34	11	11	RL^2RLR^2LRL	74	6	3	RL^2R^2
35	8	4	RL^2RLR^2	75	11	29	$RL^2R^2LRL^2R$
36	11	12	RL^2RLR^4L	76	9	11	RL^2R^2LRL
37	10	8	RL^2RLR^4	77	11	30	$RL^2R^2LRLR^2$
38	11	13	RL^2RLR^5	78	10	18	RL^2R^2LRLR
39	9	6	RL^2RLR^3	79	11	31	RL^2R^2LRLRL
40	11	14	RL^2RLR^3LR	80	8	8	RL^2R^2LR

i	K	I(K)	Pattern	i	K	I(K)	Pattern
81	11	32	$RL^2R^2LR^3L$	121	9	17	RL^3R^3L
82	10	19	$RL^2R^2LR^3$	122	11	50	$RL^3R^3LR^2$
83	11	33	$RL^2R^2LR^4$	123	10	30	RL^3R^3LR
84	9	12	$RL^2R^2LR^2$	124	11	51	RL^3R^3LRL
85	11	34	$RL^2R^2LR^2LR$	125	8	11	RL^3R^3
86	10	20	$RL^2R^2LR^2L$	126	11	52	RL^3R^5L
87	7	5	RL^2R^2L	127	10	31	RL^3R^5
88	10	21	$RL^2R^2L^2RL$	128	11	53	RL^3R^6
89	11	35	$RL^2R^2L^2RLR$	129	9	18	RL^3R^4
90	9	13	$RL^2R^2L^2R$	130	11	54	RL^3R^4LR
91	11	36	$RL^2R^2L^2R^3$	131	10	32	RL^3R^4L
92	10	22	$RL^2R^2L^2R^2$	132	11	55	$RL^3R^4L^2$
93	11	37	$RL^2R^2L^2R^2L$	133	7	7	RL^3R^2
94	4	2	RL^2	134	11	56	$RL^3R^2LRL^2$
95	8	9	RL^3RL^2	135	10	33	RL^3R^2LRL
96	11	38	$RL^3RL^2R^2L$	136	11	57	RL^3R^2LRLR
97	10	23	$RL^3RL^2R^2$	137	9	19	RL^3R^2LR
98	11	39	$RL^3RL^2R^3$	138	11	58	$RL^3R^2LR^3$
99	9	14	RL^3RL^2R	139	10	34	$RL^3R^2LR^2$
100	11	40	RL^3RL^2RLR	140	11	59	$RL^3R^2LR^2L$
101	10	24	RL^3RL^2RL	141	8	12	RL^3R^2L
102	11	41	$RL^3RL^2RL^2$	142	11	60	$RL^3R^2L^2RL$
103	7	6	RL^3RL	143	10	35	$RL^3R^2L^2R$
104	11	42	$RL^3RLR^2L^2$	144	11	61	$RL^3R^2L^2R^2$
105	10	25	RL^3RLR^2L	145	9	20	$RL^3R^2L^2$
106	11	43	RL^3RLR^2LR	146	11	62	$RL^3R^2L^3R$
107	9	15	RL^3RLR^2	147	5	3	RL^3
108	11	44	RL^3RLR^4	148	10	36	RL^4RL^3
109	10	26	RL^3RLR^3	149	11	63	RL^4RL^3R
110	11	45	RL^3RLR^3L	150	9	21	RL^4RL^2
111	8	10	RL^3RLR	151	11	64	$RL^4RL^2R^2$
112	11	46	$RL^3RLRLRL$	152	10	37	RL^4RL^2R
113	10	27	RL^3RLRLR	153	11	65	RL^4RL^2RL
114	11	47	RL^3RLRLR^2	154	8	13	RL^4RL
115	9	16	RL^3RLRL	155	11	66	RL^4RLR^2L
116	11	48	RL^3RLRL^2R	156	10	38	RL^4RLR^2
117	10	28	RL^3RLRL^2	157	11	67	RL^4RLR^3
118	6	4	RL^3R	158	9	22	RL^4RLR
119	10	29	$RL^3R^3L^2$	159	11	68	RL^4RLRLR
120	11	49	$RL^3R^3L^2R$	160	10	39	RL^4RLRL

i	K	I(K)	Pattern
161	11	69	RL^4RLRL^2
162	7	8	RL^4R
163	11	70	$RL^4R^3L^2$
164	10	40	RL^4R^3L
165	11	71	RL^4R^3LR
166	9	23	RL^4R^3
167	11	72	RL^4R^5
168	10	41	RL^4R^4
169	11	73	RL^4R^4L
170	8	14	RL^4R^2
171	11	74	RL^4R^2LRL
172	10	42	RL^4R^2LR
173	11	75	$RL^4R^2LR^2$
174	9	24	RL^4R^2L
175	11	76	$RL^4R^2L^2R$
176	10	43	$RL^4R^2L^2$
177.	11	77	$RL^4R^2L^3$
178	6	5	RL^4
179	11	78	RL^5RL^3
180	10	44	RL^5RL^2
181	11	79	RL^5RL^2R
182	9	25	RL^5RL
183	11	80	RL^5RLR^2
184	10	45	RL^5RLR
185	11	81	RL^5RLRL
186	8	15	RL^5R
187	11	82	RL^5R^3L
188	10	46	RL^5R^3
189	11	83	RL^5R^4
190	9	26	RL^5R^2
191	11	84	RL^5R^2LR
192	10	47	RL^5R^2L
193	11	85	$RL^5R^2L^2$
194	7	9	RL^5
195	11	86	RL^6RL^2
196	10	48	RL^6RL
197	11	87	RL^6RLR
198	9	27	RL^6R
199	11	88	RL^6R^3
200	10	49	RL^6R^2
201	11	89	RL^6R^2L
202	8	16	RL^6
203	11	90	RL^7RL
204	10	50	RL^7R
205	11	91	RL^7R^2
206	9	28	RL^7
207	11	92	RL^8R
208	10	51	RL^8
209	11	93	RL^9

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