Descending Subsequences of Random Permutations

SHAIY PILPEL

I.B.M. Thomas J. Watson Research Center,
P.O. Box 218, Yorktown Heights, New York 10598

Communicated by the Managing Editors

Received March 28, 1986

Given a random permutation of the numbers 1, 2, ..., n, let $L_n$ be the length of the longest descending subsequence of this permutation. Let $F_n$ be the minimal header (first element) of the descending subsequences having maximal length. It is known that $E L_n / \sqrt{n} \to c$ and that $c = 2$. However, the proofs that $c = 2$ are far from elementary and involve limit processes.

Several relationships between these two random variables are established, namely, $E L_n = \sum_{j=1}^{n-1} P(F_j = j)$ and $P(F_{n+1} = n+1) = 1 - EF_n / n + 1$. Some other combinatorial identities regarding the distribution of the bivariate random variable $(L_n, F_n)$ are also proved. The definition of $F_n$ is generalized, characterizing the elements appearing at the first row and first column of the Young tableau corresponding to a given permutation. As a result, an elementary proof for $c \leq 2$ is constructed.

1. DEFINITIONS AND NOTATIONS

The origin of the problem we are dealing with lies in one of the most famous examples of the pigeon hole principle that in any permutation of the first $n^2 + 1$ natural numbers there exists a monotone subsequence of length at least $n$.

Ulam [8] was the first to look at the more general problem of the distribution of the longest descending subsequence of a random permutation. Using simulation for small values of $n$, Ulam came to the correct conclusion that the mean behaves as a linear function of $\sqrt{n}$ but had the wrong constant. Following Ulam, this problem was studied by Baer and Brock [1], Schented [7], and most extensively by Hammersley [2].

Let $S_n$ be the symmetric group of permutations of the set $\{1, 2, ..., n\}$. For $\pi$ in $S_n$ write

$$\pi = (\pi_1, ..., \pi_n).$$

1 Present address: TradeLink Corporation, 175 West Jackson, Suite A1235, Chicago, Ill. 60604.

0097-3165/90 $3.00

Copyright © 1990 by Academic Press, Inc.
All rights of reproduction in any form reserved.
**DEFINITION.** The integers $i_1, i_2, \ldots, i_k$ form a descending subsequence in $\pi$ if $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and

$$\pi_{i_1} > \pi_{i_2} > \cdots > \pi_{i_k}.$$ 

**DEFINITION.** Let $L_n(\pi)$ be the length of the longest descending subsequence of $\pi \in S_n$,

$$L_n(\pi) = \max\{m: i_1, i_2, \ldots, i_m \text{ form a descending subsequence in } \pi\}.$$ 

Assigning equal probabilities to all the permutations $\pi$ in $S_n$, $L_n$ is a well-defined random variable. J. M. Hammersley [2], using sub-additive processes, proved the following theorem.

**THEOREM.** There exists a constant $c > 0$ such that

$$\frac{L_n}{\sqrt{n}} \rightarrow c$$

and

$$\frac{E L_n}{\sqrt{n}} \rightarrow c.$$ 

Hammersley conjectured that $c = 2$. This conjecture had been proved by Shepp and Logan [4] proving that $c \geq 2$ and Kerov and Versik [3] proving that $c \leq 2$. In view of the second part of Hammersley's theorem it is interesting to study the behavior of $E L_n$. The following random variables will establish a recursion relation for $E L_n$.

**DEFINITION.** For $\pi \in S_n$, define $F_n(\pi)$ to be the minimal header of all the descending subsequences of $\pi$ having length $L_n(\pi)$ (i.e., having maximal length).

**EXAMPLE.** Let $\pi = (3, 4, 2, 1)$. There are two descending subsequences of length 3,

$$(3, 2, 1)$$

and

$$(4, 2, 1).$$

So

$$L_n(\pi) = 3.$$
and

\[ F_4(\pi) = 3. \]

In this paper some relationships between these random variables are studied. It turns out that in order to compute \( EL_n \), all that one has to know is \( P(F_n = n) \). This might lead to an elementary proof that \( c = 2 \). Indeed we prove here that \( c \leq 2 \).

2. The Random Variables \( L_n \) and \( F_n \)

To get the feeling of the distribution of the random variables under discussion, we tabulate the values of the first 10 terms (Tables I and II). A table containing the complete distribution of \( L_n \) for \( n = 1, \ldots, 36 \) was calculated by Baer and Brock [11]. The recursive relationship between \( EL_n \) and \( F_n \) is established in the following theorem

**Theorem 1.** \( EL_n = EL_{n-1} + P(F_n = n) \).

**Proof.** It is easy to verify for \( n = 1, 2, 3 \). For \( n \geq 4 \), write down all the permutations of length \( n \) and sum up the lengths of their longest descending subsequences:

\[ J_n = \sum_{\pi \in S_n} L_n(\pi). \]

Clearly,

\[ EL_n = J_n/n!. \]

**Table I**

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>31</td>
</tr>
<tr>
<td>7</td>
<td>42</td>
</tr>
<tr>
<td>8</td>
<td>142</td>
</tr>
<tr>
<td>9</td>
<td>486</td>
</tr>
<tr>
<td>10</td>
<td>1679</td>
</tr>
</tbody>
</table>

\( n \)th row should be divided by \( n! \).
DESCENDING SEQUENCES

TABLE II
Distribution of $F_n$, $n = 1, \ldots, 10$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Note. Entries at the $n$th row should be divided by $n!$.

Let $H_n$ be the number of permutations $\pi \in S_n$ for which $F_n = n$. Thus

$$H_n = \sum_{\pi \in S_n} 1_{\{F_n = n\}}.$$

Clearly,

$$P(F_n = n) = \frac{H_n}{n!}.$$

Now look at permutations $\pi$ of length $n + 1$. For each such permutation one of the following holds:

(i) $F_{n+1}(\pi) = n + 1$ or

(ii) $F_{n+1}(\pi) < n$.

For each permutation $\pi \in S_{n+1}$, denote by $\pi^*$ the permutation in $S_n$ resulting from $\pi$ after deleting the number $n + 1$.

In case (i) the minimal header $(n + 1)$ is deleted, and since there is no other number to replace it, we get

$$L_n(\pi^*) = L_{n+1}(\pi) - 1.$$

While in case (ii),

$$L_n(\pi^*) = L_{n+1}(\pi).$$

Performing this deletion operation over all $\pi \in S_{n+1}$ one gets each $\pi^* \in S_n$ exactly $n + 1$ times. Therefore,

$$\sum_{\pi \in S_{n+1}} L_n(\pi^*) = (n + 1) \cdot J_n.$$
By definition, case (i) applies exactly $H_{n+1}$ times. Hence,

$$J_{n+1} = \sum_{\pi \in S_{n+1}} L_{n+1}(\pi)$$

$$= \sum_{\pi \in S_{n+1}} [L_{n+1}(\pi) + 1] + \sum_{\pi \in S_{n+1}} L_n(\pi^*)$$

$$= \sum_{\pi \in S_{n+1}} L_n(\pi^*) + \sum_{\pi \in S_{n+1}} 1$$

$$= (n+1) \cdot J_n + H_{n+1}.$$

Dividing by $(n+1)!$ we get

$$\frac{J_{n+1}}{(n+1)!} = \frac{J_n}{n!} + \frac{H_{n+1}}{(n+1)!},$$

i.e.,

$$EL_{n+1} = EL_n + P(F_{n+1} = n+1).$$

The following corollary is an immediate application of Theorem 1.

**Corollary.** $EL_n = \sum_{j=1}^n P(F_j = j)$.

Thus a "good" formula for this part of the distribution might enable us to compute $c$ directly.

Given a permutation $\pi \in S_n$, its descending subsequences can be characterized either by value, i.e., $-\pi_{i_1} > -\pi_{i_2} > \cdots > -\pi_{i_k}$, or by the position of their elements, i.e., $-i_1, i_2, \ldots, i_k$.

**Definition.** For a permutation $\pi \in S_n$, let

$$D_n(\pi) =$$ The position (counted from the right) of the rightmost first element of a descending subsequence of $\pi$ having maximal length.

**Example.** Let $\pi = (3, 1, 4, 2)$, then

$$L_4(\pi) = 2.$$

There are three descending subsequences of length 2,

$$(3, 1), \quad (3, 2), \quad (4, 2),$$
therefore,

\[ F_d(\pi) = 2. \]

The rightmost sequence is \((4, 2)\), and 4 appears in the second position from the right. Thus

\[ D_d(\pi) = 2. \]

**Theorem 2.** The joint distribution function of \((L_n, F_n)\) is the same as the joint distribution function of \((L_n, D_n)\).

**Proof.** Fix \(n\); we have to prove that for each \(j, k = 1, 2, ..., n\),

\[ P(D_n = j, L_n = k) = P(F_n = j, L_n = k). \]

Define the mappings

\[ P_i: S_n \to \{1, 2, ..., n\}, \quad i = 1, 2, ..., n \]

by

\[ P_i((\pi_1, ..., \pi_n)) = \text{the position from the right of } i \text{ in this } \pi. \]

Define

\[ f: S_n \to S_n \]

by

\[ f(\pi) = (P_n(\pi), P_{n-1}(\pi), ..., P_1(\pi)); \]

properties of \(f\):

1. \(f\) is a one to one and onto transformation.
2. \(f^2 = I\)
3. \(L_n(\pi) = L_n(f(\pi))\)
4. \(F_n(\pi) = D_n(f(\pi))\)
5. \(D_n(\pi) = F_n(f(\pi)).\)

Once these properties of \(f\) are established the proof is complete:

1. Immediate, since the functions \(P_i\) are the inverse coordinate functions (in reverse order) and characterize the permutation uniquely.
2. The following formula holds

\[ P_i(P_n(\pi), P_{n-1}(\pi), ..., P_1(\pi)) = \pi_i. \]
Since $\pi_i$ is moved to place $n + 1 - i$, applying $P_i$ returns it to the $i$th place as $(n + 1) - (n + 1 - i) = i$. Therefore,

$$f^2 = I.$$  

(3) Assume $\pi_i > \pi_j$ for $i > j$. Then, by the definition of $P_i$,

$$P_i(\pi) < P_j(\pi).$$

So, writing it in reverse order we get

$$\pi^*_{n-j} > \pi^*_{n-i} \quad (i < j),$$

where

$$\pi^* = (\pi^*_1, ..., \pi^*_n) = f(\pi).$$

Hence, any descending subsequence of length $l$ in $\pi$ is transformed into such a subsequence in $f(\pi)$. This stays true for $l = L_n(\pi)$.

Properties (1) and (2) guarantee that there exists no descending subsequence in $f(\pi)$ the length of which is greater then $L_n(\pi)$. Thus

$$L_n(\pi) = L_n(f(\pi)).$$

(4) Let $\pi_i$ be a header of a descending subsequence of maximal length in $\pi$. Then its place in $f(\pi)$ is the place from the right of this $\pi_i$. (This is, in fact, true for each $\pi_i$.) Now, if $\pi_i$ is the smallest header of such a subsequence, there cannot appear in $f(\pi)$ any number to its right which is the header of a descending subsequence of maximal length [by (3)].

Using, again, property (3), there exists a descending subsequence of maximal length starting at this point. Hence

$$F_n(\pi) = D_n(f(\pi)).$$

(5) Immediate, using properties (2) and (4).

Combining (1) through (5) the proof is established.

As we are more interested in the marginal distributions $F_n$ and $D_n$ we get

**Corollary.** $D_n \sim F_n$.

The usefulness of this theorem lies in the fact that $D_n$ is sometimes simpler to work with than $F_n$. $F_n$ depends on the specific values of the members of the permutation while $L_n$ and $D_n$ do not. If one wants to construct a Hammersley’s type equivalent random variable, one can do so for $D_n$ (or $L_n$), while that might be impossible for $F_n$. 
EXAMPLE. Let $\pi = (1, 4, 2, 3)$. Then

\[
\begin{align*}
P_1(\pi) &= 4 \\
P_2(\pi) &= 2 \\
P_3(\pi) &= 1 \\
P_4(\pi) &= 3.
\end{align*}
\]

So

\[
f(\pi) = (3, 1, 2, 4),
\]

\[
\begin{array}{c}
\begin{array}{cccc}
1 & 4 & 2 & 3 \\
\end{array} & \xrightarrow{f} & \begin{array}{cccc}
3 & 1 & 2 & 4 \\
\end{array}
\end{array}
\]

Now

\[
\begin{align*}
L_4(\pi) &= 2 \quad [(4, 2), (4, 3)] \\
F_4(\pi) &= 3 \\
D_4(\pi) &= 3
\end{align*}
\]

and

\[
\begin{align*}
L_4(f(\pi)) &= 2 \quad [(3, 2), (3, 1)] \\
F_4(f(\pi)) &= 3 \\
D_4(f(\pi)) &= 4.
\end{align*}
\]

The first use of Theorem 2 is in proving the following:

**Theorem 3.** \(P(F_{n+1} = n+1) = 1 - EF_n/(n+1).\)

**Proof.** We will actually prove that

\[
P(F_{n+1} = n+1) = 1 - \frac{ED_n}{n+1}
\]

and use the fact that \(EF_n = ED_n\).

Write down all the permutations of length \(n\) and copy this list another \(n\) times (so we have \(n+1\) such lists). Insert the number \(n+1\) into the first place of the first \(n!\) permutations, into the second place of the following \(n!\) permutations, etc. Thus we get all \((n+1)!\) permutations of \(S_{n+1}\).
For each $\pi \in S_{n+1}$ let $\pi^* \in S_n$ be the resulting permutation after deleting $n+1$. Clearly,

$$F_{n+1}(\pi) = n + 1 \quad \text{iff} \quad n + 1 \text{ was inserted to the left of the rightmost header of a descending subsequence of maximal length of } \pi^*.$$ 

$$\quad \text{iff} \quad n + 1 \text{ was inserted left to } D_n(\pi^*).$$

If $F_{n+1}(\pi) = n + 1$ there is only one header to all descending subsequences of $\pi$ of length $L_{n+1}(\pi)$, namely $n + 1$. So the longest descending subsequences in $\pi^*$ is of length $L_{n+1}(\pi) - 1$.

Hence, summing up over all permutations we get (as in Theorem 1)

$$P(F_{n+1} = n + 1) = 1 - \frac{ED_n}{n + 1}.$$ 

**Corollary.** $EL_n = \sum_{j=1}^{n} (1 - EF_{j-1}/j) = n - \sum_{j=1}^{n} (EF_{j-1}/j)$.

**Note.** Let

$$D^*_n(\pi) = (n + 1) - D_n(\pi)$$

(i.e., counting the place from the left instead of the right). Then

$$EL_n = \sum_{j=1}^{n} \frac{ED^*_j}{j}.$$ 

Theorem 2 proves symmetry between the value of the minimal header (of a descending subsequence of maximal length) and its position. The same function $f$ can be used to prove the analogous symmetry between the value of the rightmost header and the position of the minimal header.

**Definition.** For a permutation $\pi \in S_n$, let

$$V_n(\pi) = \text{The value of the rightmost header of a descending subsequence of } \pi \text{ having maximal length}.$$ 

$$- \pi_{n+1} - D_n(\pi)$$
and let

\[ W_n(\pi) = \text{The position (counted from the right) of the minimal header of a descending subsequence of } \pi \text{ having maximal length.} \]

i.e.,

\[ F_n(\pi) = \pi_{n+1} - W_n(\pi). \]

**Theorem 4.** \( V_n \) has the same distribution as \( W_n \).

**Example.** Let \( \pi = (3, 1, 4, 2) \); then

\[
\begin{align*}
L_4(\pi) &= 2 \\
F_4(\pi) &= 3 \\
D_4(\pi) &= 2 \\
V_4(\pi) &= 4 \\
W_4(\pi) &= 4.
\end{align*}
\]

### 3. Young Tableaux

**Definition.** A standard Young tableau of order \( n \) is an arrangement of \( n \) distinct natural numbers in rows and columns so that the numbers in each row and in each column form increasing sequences. There is an element in each row in the first column and an element in each column in the first row. No gaps between the number are allowed.

**Definition.** The shape of a standard tableau is an arrangement of squares with one square replacing each number in the standard tableau.

Schensted [7] established a one to one correspondence between permutations \( \pi \in S_n \) and a pair of standard Young tableaux of order \( n \), called the \( P \)-symbol and the \( Q \)-symbol. The correspondence takes place via the following algorithm.

**Schensted's Algorithm.** For a standard Young tableau \( S \), and a number \( x \), \( S+\ x \) is defined as the array obtained from \( S \) applying the following steps:

1. Insert \( x \) in the first row of \( S \) either by displacing the smallest number which is larger than \( x \), or, if no number is larger than \( x \), by adding \( x \) at the end of the first row.

2. If \( x \) displaced a number \( y \) in the first row, insert \( y \) in the second row of \( S \) either by displacing the smallest number which is larger than \( y \), or if no number in the second row is larger than \( y \), by adding \( y \) a the end of the second row.

3. Repeat this process row by row until some number is added to the end to the row.
DEFINITION. For a permutation \( \pi \in S_n \) the corresponding \( P \)-symbol is the array generated by

\[
(\cdots((\pi_1 \leftarrow \pi_2) \leftarrow \pi_3)\cdots) \leftarrow \pi_n).
\]

The \( Q \)-symbol corresponding to this permutation is the array obtained by putting \( k \) in the square which is added to the shape of the \( P \)-symbol when \( \pi_k \) is inserted in the \( P \)-symbol.

Schensted’s main results are:

**Theorem** (Schensted [7, Lemma 3]). There is a one to one correspondence between permutations \( \pi \in S_n \) and ordered pairs of standard tableaux of the same shape both containing 1, 2, ..., \( n \). The corresponding pair is \((P, Q)\).

**Theorem** (Schensted [7, Theorem 3]). The number of permutations \( \pi \in S_n \) having a longest increasing subsequence of length \( \alpha \) and a longest decreasing subsequence of length \( \beta \), is the sum of the squares of the number of standard tableaux with shapes having \( \alpha \) columns and \( \beta \) rows.

For a given standard shape, the number of standard tableaux having this shape is given by the Frame–Robinson–Thrall formula [7, Theorem 1]. In view of these theorems and the Frame–Robinson–Thrall formula, Schensted’s algorithm is the key tool for computing the distribution of \( L_n \).

The next observation shows that this algorithm can be used for computing the distribution of \( F_n \).

If \( \pi \in S_n \) corresponds to the symbol pair \((P, Q)\) then the entry in the lowest leftmost place in \( P \) is \( F_n(\pi) \), and the corresponding entry in \( Q \) is \( D_n(\pi) \).

**Example.** For \( \pi = (2, 4, 3, 7, 6, 1, 5) \) then

\[
P = 1 \quad 3 \quad 5 \\
\quad 1 \quad 2 \quad 4
\]

\[
Q = 2 \quad 6 \\
\quad 3 \quad 5
\]

\[
4 \quad 7 \\
\quad 6 \quad 7
\]

and

\[
F_7(\pi) = 4 \\
D_7(\pi) = 6.
\]

This phenomenon is a special case of the next theorem characterizing the entries to the first row and column of the corresponding \( P \)-symbol.
DEFINITION. For a permutation $\pi \in S_n$, let

$$M_n^k(\pi) = \min \{ \pi_{i_1}, ..., \pi_{i_k} : \text{form a descending subsequence in } \pi \}.$$ 

In other words, $M_n^k(\pi)$ is the minimal header of all descending subsequences of length $k$.

This definition is valid for $k = 1, 2, ..., L_n(\pi)$, and clearly

$$M_n^1(\pi) = 1$$

and

$$M_n^{L_n(\pi)}(\pi) = F_n(\pi).$$

Similarly, define

$$N_n^k(\pi) = \min \{ \pi_{i_1}, ..., \pi_{i_k} : \text{form an ascending subsequence in } \pi \}.$$ 

A simple argument shows that all the $M_n^k(\pi)$, $k = 1, 2, ..., L_n(\pi)$, are distinct. The same holds for $N_n^k(\pi)$, $k = 1, 2, ..., L_*(\pi)$, where $L_*(\pi)$ is the length of the longest ascending subsequence of $\pi$.

The corollary following the theorem shows that this is true for both sets as well.

THEOREM 5. Let $\pi \in S_n$ and let $S = (s_{i,j})$ be the P-symbol corresponding to the permutation $\pi$. Then, for $j = 1, 2, ..., L_n(\pi)$ and $k = 1, 2, ..., L_*(\pi)$,

$$s_{j,1} = M_n^j(\pi)$$

and

$$s_{1,k} = N_n^k(\pi).$$

Proof. First prove the following two lemmas.

LEMMA 1. Each $s_{r,j}$ $(r = 1, 2, ..., L_n(\pi), j = 1, 2, ..., \text{length of row } r)$ is the header of at least one descending subsequence of length $r$.

Proof of Lemma 1. Simple induction on $r$. For $r = 1$ it is trivial.

For $r = 2$, any $\pi_i$ appears in the second row only if it was pushed down by a number $\pi_s < \pi_i$ appearing after $\pi_i$. Therefore, $\pi_i$ is the header of a subsequence of length 2.

Assume the lemma is true up to $r - 1$ and prove for $r$. (The exact induction hypothesis should read—if $\pi_\xi$ appears in row $r$ during the construction of $S$, then it is the header of a descending subsequence of length $r$.)

As in the case of $r = 2$, $\pi_i$ appears in the $r$th row only if it was pushed down from the $(r - 1)$th row by some $\pi_s$. By the induction hypothesis this
\[ \pi_s \] is the header of a descending subsequence of length \( r - 1 \), and as \( \pi_i > \pi_s \) and \( \pi_i \) appears prior to \( \pi_s \), \( \pi_i \) is the header of a descending subsequence of length \( r \).

**Lemma 2.** The number \( M^j_n(\pi) \) is pushed down at least \( j - 1 \) times.

**Proof of Lemma 2.** As the number \( M^j_n(\pi) \) appears in the input sequence \( \pi_1, ..., \pi_n \) to Schensted’s algorithm, it is placed somewhere in the first row. Let \( \pi_i \) be the first number to appear in \( (\pi_1, ..., \pi_n) \) which belongs to a descending subsequence of length \( j \) starting with \( M^j_n(\pi) \). When this \( \pi_i \) appears it will push \( M^j_n(\pi) \) down from the first row to the second if it was not pushed down before that. (Otherwise, \( \pi_i \) displaces some \( \pi_s \) appearing after \( M^j_n(\pi) \) and \( M^j_n(\pi) > \pi_s > \pi_i \), so \( \pi_s \) is the header of a descending subsequence of length \( j \), contradicting the minimality of \( M^j_n(\pi) \).) The same argument shows that if \( \pi_u \) is the second number to appear in \( (\pi_1, ..., \pi_n) \) which belongs to a descending subsequence of length \( j \) starting with \( M^j_n(\pi) \), then when this \( \pi_u \) appears as an input to Schensted’s algorithm it will displace \( \pi_i \) or some other number that may serve as a first element in such a subsequence, say \( \pi_v \). This \( \pi_v \) will push \( M^j_n(\pi) \) down from the second row had it not been pushed down before. Repeat this argument \( j - 1 \) times to get that \( M^j_n(\pi) \) is pushed down at least \( j - 1 \) times.

Combining Lemmas 1 and 2 we see that \( M^j_n(\pi) \) appears in the \( j \)th row. The numbers in each row form an ascending sequence. Therefore, \( M^j_n(\pi) \) occupies the first column of this row. A symmetric argument proves the second identity. \( \square \)

**Note.** Lemma 1 is a generalization of Lemmas 4 and 5 in [7].

The inverse to Lemma 1, namely—that if \( \pi_i \) is the header of a descending subsequence of length \( r \) and of no descending subsequence of length \( r + 1 \) then it appears in row \( r \) in the corresponding \( P \)-symbol, is not true, as can be seen in the following example.

**Example.** \( \pi = (2, 4, 3, 6, 5, 1) \) and its corresponding \( P \)-symbol is

\[
P = \begin{matrix}
1 & 3 & 5 \\
2 & 6 \\
4
\end{matrix}
\]

6 appears in the second row, 5 in the first row, etc.

**Corollary.** 1. If \( \pi_i \neq 1 \) is a minimal header of a descending subsequence of length \( k \), then it cannot be the minimal last element of an ascending subsequence of any length.
2. If $\pi_i \neq 1$ is a minimal last element of an ascending subsequence of length $k$, then it cannot be the minimal header of a descending subsequence of any length.

As noted before, $F_n(\pi)$ is the number occupying the lowest leftmost box of the $P$-symbol corresponding to $\pi$. Since the numbers in each row form an ascending sequence, $F_n(\pi) = n$ if and only if the $P$-symbol corresponding to $\pi$ has only one box in the lowest row and this box contains $n$.

Denote by $\lambda_n$ any standard shape containing $n$ boxes, and by $\lambda_n^*$ the shape generated from $\lambda_n$ by adding an extra box at the first column of an extra row.

**Example.** If

$$\lambda_n = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}$$

then

$$\lambda_n^* = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
 & & & \\
\end{array}$$

Let $K(\lambda_n)$ denote the number of standard Young tableaux having shape $\lambda_n$.

**Lemma.** $P(F_n = n) = (1/n!) \sum_{\lambda_{n-1}} K(\lambda_{n-1}) K(\lambda_n^*)$.

**Proof.** Combining Sch Ested's correspondence results with the previous argument one can write

$$P(F_n = n) = \sum_{\lambda_{n-1}^*} \left( \frac{\text{number of } P\text{-symbols having shape } \lambda_{n-1}^* \text{ with } n \text{ in the lowest leftmost box}}{n!} \times \frac{\text{number of } Q\text{-symbols having shape } \lambda_{n-1}^*}{n!} \right)$$

But, it easy to verify that

the number of $P$-symbols having shape $\lambda_{n-1}^*$ with $n$ in the lowest leftmost box $- K(\lambda_{n-1})$
while

the number of Q-symbols having shape $\lambda_{n-1}^* = K(\lambda_{n-1}^*)$.

Thus, the required identity is obtained. 

Don Coppersmith noted that this Lemma yields an upper bound for the term $P(F_n = n)$.

**Lemma.** $P(F_n = n) \leq 1/\sqrt{n}$.

**Proof.** By Schested's theorem

$$(n-1)! = \sum_{\lambda_{n-1}} K(\lambda_{n-1})^2.$$ 

Hence

$$n! = \sum_{\lambda_{n-1}} \left[ \sqrt{n} \cdot K(\lambda_{n-1}) \right]^2.$$ 

Also,

$$n! = \sum_{\lambda_n} K(\lambda_n)^2 \geq \sum_{\lambda_{n-1}} K(\lambda_{n-1}^*)^2.$$ 

Using Cauchy–Schwarz inequality

$$n! \geq \sum_{\lambda_{n-1}} \sqrt{n} \cdot K(\lambda_n) \cdot K(\lambda_{n-1}^*).$$ 

Thus using the previous lemma,

$$\frac{1}{\sqrt{n}} \geq \frac{1}{n!} \sum_{\lambda_{n-1}} K(\lambda_n) \cdot K(\lambda_{n-1}^*)$$

$$= P(F_n = n).$$ 

Summing up we get

**Theorem 6.** $c \leq 2$.

**Proof.** By Theorem 1,

$$E L_n = \sum_{j=1}^{n} P(F_j = j) \leq \sum_{j=1}^{n} \frac{1}{\sqrt{j}}.$$
Use the elementary inequality

\[ \frac{1}{\sqrt{k}} \leq 2 \cdot (\sqrt{k} - \sqrt{k - 1}) \]

to get

\[ EL_n \leq \sum_{k=1}^{n} 2 \cdot (\sqrt{k} - \sqrt{k - 1}) = 2 \cdot \sqrt{n}. \]

So, in fact, we get the stronger result that for every \( n \),

\[ EL_n / \sqrt{n} \leq 2. \]

Hence

\[ \lim_{n \to \infty} \frac{EL_n}{\sqrt{n}} \leq 2. \]

4. Constructing the Joint Distribution Function

No explicit formula is known for the term

\[ P(F_n = i, L_n = j). \]

We present expressions for some special entries of this distribution. The following tables present the values of these matrices for 2, 3, ..., 7.

\[
\begin{array}{c|cc}
F_2 & F_3 \\
\hline
1 & 2 \\
L_2 & 1 & 1 \\
& 2 & 1 \\
\end{array}
\begin{array}{c|cc}
F_3 & F_4 \\
\hline
1 & 2 & 3 \\
L_3 & 1 & 1 \\
& 2 & 2 \\
& 3 & 1 \\
\end{array}
\begin{array}{c|cccc}
F_4 & F_5 \\
\hline
1 & 2 & 3 & 4 \\
L_4 & 1 & 1 \\
& 2 & 5 & 5 \\
& 3 & 3 & 6 \\
& 4 & 1 \\
\end{array}
\]
The following statements summarize (proofs deleted) the formulas for the second row, the diagonal and the upper diagonal entries of the distribution matrices of \((F_n, L_n)\).

**Statements.** (1) For the second row:

\[
P(L_n = 2, F_n = n) = (n - 1)/n!,
\]

\[
P(L_n = 2, F_n = n - k) = P(L_n = 2, F_n = n - k + 1)
+ n \cdot P(L_{n-1} = 2, F_{n-1} = n - k - 1) \quad k = 1, 2, ..., n - 3;
\]

\[
P(L_n = 2, F_n = 2) = P(L_n = 2, F_n = 3).
\]

Simple induction shows that for \(j = 3, ..., n\) we get the generalized Catalan numbers:

\[
P(L_n = 2, F_n = j) = \frac{j - 1}{n} \cdot \binom{2n - j}{n - j + 1} \cdot \frac{1}{n!},
\]
As a special case we get
\[
P(L_n = 2, F_n = 2) = \frac{\binom{2(n-1)}{n-1}}{n!} \cdot \frac{1}{n!} \quad \text{(Catalan numbers)}.
\]

It is easy to verify that
\[
P(L_n = 2) = P(L_n = 2, F_n = 2)
\]
\[
= \frac{\binom{2(n-1)}{n-1}}{n} \cdot \frac{1}{n!}
\]
which was observed in [2, 6].

(2) For the diagonal:
\[
P(L_n = 1, F_n = 1) = \frac{1}{n!}
\]
\[
P(L_n = n, F_n = n) = \frac{1}{n!}
\]
\[
P(L_n = j, F_n = j) + P(L_n = j + 1, F_n = j + 1)
\]
\[
= P(L_{n-1} = j - 1, F_{n-1} = j - 1), \quad j = 3, \ldots, n - 1.
\]

As \(P(L_n = 2, F_n = 2)\) is known from (1) this yields a complete recursive formula for the diagonal.

(3) For the upper diagonal:
\[
P(L_n = j, F_n = j + 1) = (j - 1) \cdot P(L_n = j, F_n = j), \quad j = 2, \ldots, n - 1.
\]

A special case,
\[
P(L_n - n - 1) = P(L_n - n - 1, F_n - n - 1) + P(L_n - n - 1, F_n = n)
\]
\[
= \frac{n-1}{n!} + \frac{(n-1)(n-2)}{n!} \quad \text{(by (2) and (3))}
\]
\[
= \frac{(n-1)^2}{n!}.
\]

There is no known pattern by which we can construct, recursively, the joint distribution function of \((F_n, L_n)\) from the previous values of \((F_1, L_1), \ldots, (F_{n-1}, L_{n-1})\), or from the marginal distributions of \(F_1, \ldots, F_n, L_1, \ldots, L_n\). Following Hammersley's idea, if we restrict ourselves to the vector of the joint distribution with \(F_n = n\), we have the following theorem.
DEFINITION. Let
\[ \beta_i^{(n)} = P(L_n = i) \cdot n! \quad n = 1, 2, \ldots; i = 1, 2, \ldots, n. \]
\[ \gamma_i^{(n)} = P(L_n = i, F_n = n) \cdot n! \quad n = 1, 2, \ldots; i = 1, 2, \ldots, n. \]

Note. The following correspondence between the \( \gamma_p^{(n)} \) and the \( \alpha_{p,q}^{(n)} \) in [2] can be proved:
\[ \alpha_{p,q}^{(n)} = \frac{n!}{(n+q-1)} \gamma_p^{(n)}. \]

The following theorem establishes the relationship between the \( \gamma_i^{(n)} \)s and the \( \beta_i^{(n)} \)s.

THEOREM 7. For \( p \geq 2 \),
\[ \sum_{j=1}^{n-p+1} \frac{n!}{(n+1-j)} \cdot \gamma_p^{(n+1-j)} + \sum_{j=1}^{p-1} \beta_j^{(p)} = n!. \]

Proof. This theorem is written as a number theoretic identity since this form is more useful in later applications.

Divide both sides of the equation by \( n! \). Then the theorem actually states that for \( p \geq 2 \),
\[ 1 = P(L_n = 1) + P(L_n = 2) + \ldots + P(L_n = p-1) \]
\[ + P(L_n = p, F_n = n) + P(L_{n-1} = p, F_{n-1} = n-1) \]
\[ + \ldots + P(L_p = p, F_p = p) \]
or, in other words,
\[ P(L_n \geq p) = P(L_n = p, F_n = n) + P(L_{n-1} = p, F_{n-1} = n-1) \]
\[ + \ldots + P(L_p = p, F_p = p) \]
using Theorem 2:
\[ P(L_n = i, F_n = j) = P(L_n = i, D_n = j). \]

Thus, we have to prove
\[ P(L_n \geq p) = P(L_n = p, D_n = n) + P(L_{n-1} = p, D_{n-1} = n-1) \]
\[ + \ldots + P(L_p = p, D_p = p). \]
Write down all $n!$ permutations $\pi \in S_n$. For each permutation $\pi = (\pi_1, \ldots, \pi_n)$, start from the right side and ask whether

$$L_{n+1-i}(\pi_i, \pi_{i+1}, \ldots, \pi_n) \geq p.$$ 

This question is legitimate, since neither the value of $L_n$ nor that of $D_n$ depends on the specific representation of the $\pi_i$'s.

Define the index of $\pi$,

$$I_p(\pi) = \begin{cases} \text{the first time } L_{n+1-i}(\pi_i, \pi_{i+1}, \ldots, \pi_n) = p \\ \text{undefined, if it does not occur.} \end{cases}$$

Clearly,

$$L_n(\pi) \geq p \iff \text{an index for this } \pi \text{ exists.}$$

For permutations the index of which exists, the following is true:

$$D_{I_p(\pi)}(\pi_i\downarrow_{I_p(\pi)}) = I_p(\pi)$$

$$L_{I_p(\pi)}(\pi_i\downarrow_{I_p(\pi)}) = p$$

where $\pi_{i\downarrow_{I_p(\pi)}}$ is the rightmost part of the permutation $\pi$ starting at its index $I_p(\pi)$. The last two events are independent, assuming we stop at the first index $p$ of $\pi$ and do not resume the search for this $\pi$ any more. Thus, the last equivalence states that

$$P(L_n \geq p) = \sum_{k=p}^{n} P(L_k = p, D_k = k),$$

proving the theorem.

Applying this theorem in the spirit of Hammersley we have

**Corollary.** 1. **Knowing the distribution of** $L_1, \ldots, L_n$ (that is, of the $\beta_i^{(n)}$'s) **we can construct the** $\gamma_i^{(n)}$'s.

2. **Knowing the** $\gamma_i^{(k)}$'s, $k = 1, 2, \ldots, n$; $i = 1, 2, \ldots, k$, **we can construct the** $\beta_i^{(k)}$'s.

**Proof.** It is clear how to do this (both ways) for $n = 1, 2, 3$. Assume it is true up to $n - 1$ and prove for $n$.

1. Knowing $\beta_i^{(k)}$, $k = 1, 2, \ldots, n-1$, we can construct, by the induction hypothesis, all the $\gamma_i^{(k)}$, $k = 1, 2, \ldots, n-1$; $i = 1, 2, \ldots, k$. Clearly,

$$\gamma_i^{(n)} = 0 \quad \text{for all } n.$$
To construct the $\gamma_p^{(n)}$ for $p \geq 2$, use the identity established in Theorem 7 (for this $p$) and note that only one unknown quantity appears in that expression, namely $\gamma_p^{(n)}$. Hence, this $\gamma_p^{(n)}$ can be solved.

2. Knowing all the $\gamma_i^{(n)}$, $k = 1, 2, \ldots, n$; $i = 1, 2, \ldots, k$, then clearly

$$\beta_1^{(n)} = 1$$

and

$$\beta_n^{(n)} = 1.$$

Use, again, the identity of Theorem 7, this time in a step by step mode. Evaluate first

$$\beta_1^{(n)} + \beta_2^{(n)} \quad \text{for } p = 2.$$

As $\beta_1^{(n)}$ is known, we get $\beta_2^{(n)}$.

Next evaluate

$$\beta_1^{(n)} + \beta_2^{(n)} + \beta_3^{(n)} \quad \text{for } p = 3.$$

As now both $\beta_1^{(n)}$ and $\beta_2^{(n)}$ are known, we get $\beta_3^{(n)}$.

Applying this procedure to $p = 2, 3, \ldots, n$, all the $\beta_i^{(n)}$ are uniquely determined.

\section*{References}


