JOURNAL OF ALGEBRA **189,** 58–89 (1997)
ARTICLE NO. JA966876

Automorphism Subgroups of Finite Index in Algebraic Mapping Class Groups

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Communicated by Susan Montgomery

Received April 2, 1996

We give an algebraic proof of the Birman–Bers theorem—an algebraic result whose previous proofs used topology or analysis, and which says that a certain subgroup of finite index in the (algebraic) mapping class group of an oriented punctured surface is isomorphic to a certain group of automorphisms. The index 2 case gives rise to an automorphism of the group consisting of those automorphisms of a free group that stabilize the normal subgroup generated by an oriented-surface relator, and we analyze this curious automorphism. © 1997 Academic Press

1. INTRODUCTION

1.1. *Notation.* Let G be a group. For a subset X of G, $X^{\pm 1}$ will denote the set consisting of the elements of X and their inverses.

Let a, b, c be elements of G. The product of a and b will be denoted by ab and also by $a \cdot b$. We write $[a, b]$ to denote $aba^{-1}b^{-1}$ and write $a(b)$ to denote aba^{-1}. This latter notation is too standard for us to alter, but it

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0021-8693/97 $25.00

gives rise to ambiguities, so we make the convention that juxtapositions are to be performed first, followed by the bracketing operation, followed by the dot multiplications; thus $a \cdot b(c) = abcb^{-1}$ and $ab(c) = abcb^{-1}a^{-1}$.

We write $[a]$ to denote the conjugacy class of a in G, so if G is a free group with a specified basis, $[a]$ is viewed as a cyclic word in the basis.

We use the same symbol a to denote both the group element and the inner automorphism $a(—)$ of G; the correct interpretation should be clear from the context.

The group of automorphisms of G will be denoted $\mathrm{Aut}(G)$, and the subgroup of inner automorphisms will be denoted $\mathrm{Inn}(G)$. The latter is isomorphic to G modulo its center, which we denote G/Ctr. When the center of G is trivial, we will tend to identify the foregoing groups with G, and, again, the interpretation will be clear from the context. In all our applications, G/Ctr will be either G or trivial.

Throughout this article, g and n will be nonnegative integers, and $\chi_{g,n}$ will denote the integer $2 - 2g - n$.

1.2. DEFINITIONS. We write

$$\Sigma_{g,n} = \left\langle x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n \,\middle|\, [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_n = 1 \right\rangle,$$

the fundamental group of an orientable closed surface of genus g with n punctures, which we call a (g, n)-*surface*. This surface has Euler characteristic $\chi_{g,n}$ and if $n \geq 1$, then $\Sigma_{g,n}$ is a free group of rank $2g + n - 1 = 1 - \chi_{g,n}$.

If $S_1, \ldots, S_m$ are elements, subsets, or sets of subsets, of $\Sigma_{g,n}$ we shall write

$$\mathrm{Aut}(\Sigma_{g,n}, S_1, \ldots, S_m)$$

to denote the subgroup of $\mathrm{Aut}(\Sigma_{g,n})$ consisting of those elements α such that $\alpha(S_i) = S_i$, $i = 1, \ldots, m$, with the natural interpretation of $\alpha(S_i)$.

Thus

$$\mathrm{Aut}\left(\Sigma_{g,n}, \left\{[z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right\}\right)$$

is the group of automorphisms that permute the conjugacy classes of the z_i. The (*algebraic*) *mapping class group* is defined as the quotient

$$\mathscr{MC}_{g,n} = \mathrm{Aut}\left(\Sigma_{g,n}, \left\{[z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right\}\right) / \mathrm{Inn}(\Sigma_{g,n}).$$

In a natural way $\mathscr{MC}_{g,n}$ maps onto the symmetric group on n letters, and the kernel is a normal subgroup of index $n!$ called the *pure* (or *unpermuted*) mapping class group, denoted

$$\mathscr{PMC}_{g,n} = \mathrm{Aut}\left(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right) / \mathrm{Inn}(\Sigma_{g,n}).$$

Our aim is to prove that, except in the degenerate case $(g, n) = (0, 1)$,

$$\begin{aligned}\mathscr{MC}_{g,n+1} &\geq \operatorname{Aut}\Bigl(\Sigma_{g,n+1}, \bigl\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\bigr\}, [z_{n+1}]^{\pm 1}\Bigr)/\operatorname{Inn}(\Sigma_{g,n+1}) \\ &\approx \operatorname{Aut}\Bigl(\Sigma_{g,n}, \bigl\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\bigr\}\Bigr), \end{aligned} \tag{1}$$

that is, a subgroup of index $n+1$ in $\mathscr{MC}_{g,n+1}$ is isomorphic to a group of automorphisms. Restricting to subgroups of index $n!$ gives

$$\begin{aligned}\mathscr{PMC}_{g,n+1} &= \operatorname{Aut}\Bigl(\Sigma_{g,n+1}, [z_1]^{\pm 1}, \dots, [z_{n+1}]^{\pm 1}\Bigr)/\operatorname{Inn}(\Sigma_{g,n+1}) \\ &\approx \operatorname{Aut}\Bigl(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Bigr). \end{aligned} \tag{2}$$

The case $g = 0$ of (1) is due to Magnus [13], and the case $n = 0$ is due to Dehn and Mangler [14]. Topological and analytic analogues of (1) were proved by Birman [3] and Bers [2]. Invoking the subsequent identification by Maclachlan and Harvey [12] of the algebraic and topological mapping class groups, we can express Birman's homotopy-fibration exact sequence [3] in the form

$$1 \to \Sigma_{g,n}/\mathrm{Ctr} \to \mathscr{PMC}_{g,n+1} \to \mathscr{PMC}_{g,n} \to 1, \tag{3}$$

which can be viewed as another way of stating (2) (see [3, Section 3]), and we can express Bers' result [2, Theorem 10] in the form (1). The form (1) was first explicitly stated by Maclachlan [11, Corollary 8]. Our main claim is to give a purely algebraic proof of (1). We do this by constructing and analyzing the kernel that appears in (3), a normal subgroup which has played a part in the calculation of presentations of mapping class groups in the work of McCool [16] and Wajnryb [20].

In Section 2, we briefly discuss the isomorphism (1) from a topological viewpoint. In Sections 3 and 4, we give the algebraic proof. Notice that

$$\begin{aligned}&\operatorname{Aut}\Bigl(\Sigma_{g,n+1}, \bigl\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\bigr\}, \{z_{n+1}\}^{\pm 1}\Bigr) \cap \operatorname{Inn}(\Sigma_{g,n+1}) \\ &\qquad = \operatorname{Inn}(\langle z_{n+1}\rangle),\end{aligned}$$

since, in a nonabelian free group, a basis element generates its own centralizer; also

$$\begin{aligned}&\operatorname{Aut}\Bigl(\Sigma_{g,n+1}, \bigl\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\bigr\}, \{z_{n+1}\}^{\pm 1}\Bigr) \cdot \operatorname{Inn}(\Sigma_{g,n+1}) \\ &\qquad = \operatorname{Aut}\Bigl(\Sigma_{g,n+1}, \bigl\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\bigr\}, [z_{n+1}]^{\pm 1}\Bigr).\end{aligned}$$

Thus we have an isomorphism

$$\begin{aligned}&\operatorname{Aut}\Big(\Sigma_{g,n+1},\big\{[z_1]^{\pm1},\dots,[z_n]^{\pm1}\big\},[z_{n+1}]^{\pm1}\Big)/\operatorname{Inn}\big(\Sigma_{g,n+1}\big)\\ &\qquad\approx \operatorname{Aut}\Big(\Sigma_{g,n+1},\big\{[z_1]^{\pm1},\dots,[z_n]^{\pm1}\big\},\{z_{n+1}\}^{\pm1}\Big)/\operatorname{Inn}\big(\langle z_{n+1}\rangle\big).\end{aligned}$$

There is also a natural map

$$\begin{aligned}&\operatorname{Aut}\Big(\Sigma_{g,n+1},\big\{[z_1]^{\pm1},\dots,[z_n]^{\pm1}\big\},\{z_{n+1}\}^{\pm1}\Big)\\ &\qquad\to \operatorname{Aut}\Big(\Sigma_{g,n},\big\{[z_1]^{\pm1},\dots,[z_n]^{\pm1}\big\}\Big),\end{aligned}$$

denoted collapse (z_{n+1}), and our task is reduced to showing that collapse(z_{n+1}) is surjective and has kernel $\operatorname{Inn}(\langle z_{n+1}\rangle)$. Surjectivity is not difficult, and it is the calculation of the kernel that occupies Sections 3 and 4. We conclude Section 4 with a brief proof of Zieschang's result that any endomorphism of $\Sigma_{g,n+1}$ that sends each of the $n+1$ z_is to a conjugate of itself is an automorphism.

In Section 5, we consider the embedding of $\mathscr{PMC}_{g,n+1}$ in $\mathscr{MC}_{g,n+1}$, which produces an image of the symmetric group S_{n+1} in the outer automorphism group of the groups in (2). Results of Ivanov [8] show that if $\chi_{g,n} \le -2$, then this image is the whole outer automorphism group. We study the way in which a specific transposition of S_{n+1} acts on $\operatorname{Aut}(\Sigma_{g,n},[z_1]^{\pm1},\dots,[z_n]^{+1})$. In consequence, we obtain an explicit free generating set of the kernel of the surjective map

$$\operatorname{collapse}(z_n)\colon \operatorname{Aut}\big(\Sigma_{g,n},[z_1],\dots,[z_n]\big) \to \operatorname{Aut}\big(\Sigma_{g,n-1},[z_1],\dots,[z_{n-1}]\big),$$

for each $n \ge 1$. For $n = 1$, the action of the transposition completely describes the outer automorphism group of $\operatorname{Aut}(\Sigma_{g,1},[z_1]^{\pm1})$, the group consisting of those automorphisms of the free group $\Sigma_{g,1}$ that stabilize the normal subgroup generated by the oriented surface relator z_1. The outer automorphism group is trivial for $g \le 1$ and has order 2 for $g \ge 2$.

2. THE TOPOLOGICAL VIEWPOINT

In this section we give a vague idea of how the topologists view the group isomorphism that we shall prove algebraically over the two subsequent sections.

Recall that the *topological mapping class group* $\mathscr{TMC}_{g,n}$ is the group of isotopy classes of homeomorphisms of the (g,n)-surface. It is convenient to think of the (g,n)-surface as a surface of genus g with n distinguished points, so that we can refer to homeomorphisms "permuting the punctures."

Let us choose a distinguished point p on a (g,n)-surface, and take $\Sigma_{g,n}$ to be the fundamental group with respect to this base point. By a p-homeomorphism of the (g,n)-surface we shall mean a homeomorphism which fixes p. In a natural way, a p-homeomorphism determines an element of

$$\mathrm{Aut}\left(\Sigma_{g,n}, \left\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\right\}\right),$$

and hence an element of $\mathscr{MC}_{g,n}$.

Dehn showed that every isotopy class contains a p-homeomorphism, and that two p-homeomorphisms lie in the same isotopy class if and only if they have the same image in $\mathscr{MC}_{g,n}$. Hence there is an injective homomorphism $\mathscr{TMC}_{g,n} \to \mathscr{MC}_{g,n}$. Combined work of Dehn, Nielsen, Magnus, Harvey, Maclachlan, and others culminated in a proof that this map is surjective; see [12]. Thus, the algebraic and topological mapping class groups can be identified with each other.

Two p-homeomorphisms are said to be *p-isotopic* if there is an isotopy through p-homeomorphisms between them. Epstein [7] showed that two p-homeomorphisms are p-isotopic if and only if they determine the same automorphism of the fundamental group. Together with the results described in the previous paragraph, this implies that $\mathrm{Aut}(\Sigma_{g,n}, \{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\})$ can be identified with the group of p-isotopy classes of p-homeomorphisms of the (g,n)-surface.

If we now delete p from the (g,n)-surface, we get a $(g, n+1)$-surface with a distinguished puncture, and there is a natural bijective correspondence between p-homeomorphisms of the (g,n)-surface and homeomorphisms of the $(g, n+1)$-surface that fix the distinguished puncture. Moreover, two p-homeomorphisms are p-isotopic if and only if they determine isotopic homeomorphisms of the $(g, n+1)$-surface.

This gives the topological interpretation of the isomorphism (1). Modulo identifying the algebraic and topological mapping class groups, Bers' article [2] gives two interpretations of the isomorphism—one topological and one analytic.

3. BIRMAN'S NORMAL SUBGROUP

Recall that n is a nonnegative integer.

3.1. Definitions. Throughout this section, Σ will denote a finitely generated free group with a specified (finite) basis X. Thus the elements of Σ are thought of as reduced words in $X^{\pm 1}$, and we use terms such as "subword," "cyclically reduced," etc., without explicitly mentioning X.

Let $t_1, \ldots, t_n, t$ be nontrivial elements of Σ and let $T = ([t_1], \ldots, [t_n], t)$. An element w of Σ is said to *occur* in T if w occurs as a subword of t (reduced) or of some cyclically written t_i, $1 \leq i \leq n$ (cyclically reduced). We denote by $\prec_T$ the relation on $X^{\pm 1}$ given by $y \prec_T z$ if and only if yz^{-1} occurs in T, for all y, z in $X^{\pm 1}$.

Throughout this section, $\mathscr{T}$ will denote the set of sequences of the form

$$T = ([t_1], \ldots, [t_n], t),$$

where t and all the t_i are nontrivial elements of Σ, such that every element of $X^{\pm 1}$ occurs exactly once in T and such that $\prec_T$ uniquely determines a total order $<_T$, called the *T-ordering*, on $X^{\pm 1}$. For all y, z in $X^{\pm 1}$, z immediately follows y in the T-ordering if and only if $y \prec_T z$. A specific example is given in Definition 4.2, below. In the degenerate case where X is empty, we declare that the foregoing definition of $\mathscr{T}$ does not apply, and we take $\mathscr{T}$ to consist of the single element $T = (t)$, where t is the identity element.

In a natural way $\mathrm{Aut}(\Sigma)$ acts on $\mathscr{T}$.

For any $(T, U) \in \mathscr{T} \times \mathscr{T}$, we let

$$\mathscr{G}(T, U) = \{\alpha \in \mathrm{Aut}(\Sigma) \mid \alpha(T) = U\}$$

and we let $\mathscr{G}$ denote the disjoint union of the $\mathscr{G}(T, U)$, as (T, U) ranges over $\mathscr{T} \times \mathscr{T}$. Each element of $\mathscr{G}$ has, associated with it, an element of $\mathrm{Aut}(\Sigma)$, called the *underlying automorphism*, and we write $\alpha\colon T \to U$ to denote an element of $\mathscr{G}(T, U)$ that has underlying automorphism α. In a natural way, the group structure of $\mathrm{Aut}(\Sigma)$ induces on $\mathscr{G}$ a groupoid structure, with vertex set $\mathscr{T}$.

Notice that for any $T = ([t_1], \ldots, [t_n], t) \in \mathscr{T}$,

$$\mathscr{G}(T, T) = \mathrm{Aut}(\Sigma, [t_1], \ldots, [t_n], t).$$

An element $\alpha\colon T \to U$ of $\mathscr{G}$ is said to be a *signed permutation* if α permutes the elements of $X^{\pm 1}$. Notice that α commutes with the inverting operation.

An element $\alpha\colon T \to U$ of $\mathscr{G}$ is said to be a *Nielsen shift* if there are elements y, z of $X^{\pm 1}$ such that α acts trivially on $X^{\pm 1} - \{y\}^{\pm 1}$ and $\alpha(y) = yz$, where $(yz^{-1})^{\pm 1}$ occurs in T, that is, either $y \prec_T z$ or $z \prec_T y$. In this event we say that $\alpha\colon T \to U$ is a *right*, resp. *left*, Nielsen shift.

Since α is an automorphism, $z \neq y^{-1}$. Conversely, if we have $T \in \mathscr{T}$ and $y, z \in X^{\pm 1}$ such that either $y \prec_T z \neq y^{-1}$ or $y^{-1} \neq z \prec_T y$, then there exists an automorphism α that acts trivially on $X^{\pm 1} - \{y\}^{\pm 1}$ such that $\alpha(y) = yz$, and then $\alpha(T) \in \mathscr{T}$.

3.2. LEMMA (McCool). *With respect to the above, the groupoid $\mathscr{G}$ is generated by the signed permutations together with the Nielsen shifts.*

Proof. Recall that there are two types of Whitehead automorphism (with respect to X) of Σ. Those *of the first type* permute $X^{\pm 1}$. Those *of the second type* fix some $y \in X^{\pm 1}$ and send each $x \in X^{\pm 1} - \{y\}^{\pm 1}$ to $y^{(-1 \text{ or } 0)}xy^{(0 \text{ or } 1)}$.

Now let us pass to a free group of larger rank by adding one more free generator u and associate with each element $([t_1], \dots, [t_n], t)$ of $\mathscr{T}$, the sequence $([t_1], \dots, [t_n], [ut])$ of cyclic words in the new, larger, free group. As in the proof of [15, Lemma 2], by applying [15, Lemma 1] to the action on such sequences of cyclic words of automorphisms that fix u, we see that $\mathscr{G}$ is generated by those elements $\alpha: T \to U$ whose underlying automorphisms are Whitehead automorphisms. If α is of the first type, then $\alpha: T \to U$ is a signed permutation. If α is of the second type, then $\alpha: T \to U$ can be written as a product of Nielsen shifts by a simple inductive procedure; see [17, Corollary 3]. ∎

3.3. DEFINITIONS. Let $T = ([t_1], \dots, [t_n], t) \in \mathscr{T}$ and let $x, y \in X^{\pm 1}$.

In this section we will use angle brackets and a semicolon to denote a pairing, which we hope will not be confused with the usage in the other sections where angle brackets with no semicolons are used to denote groups generated by presentations, and subgroups generated by sets.

We define

$$(x <_T y) = \begin{cases} t, & \text{if } x <_T y, \\ 1, & \text{if } x \geq_T y. \end{cases}$$

Notice that T plays two roles here, since $(x <_T y)$ is a power of the last entry of T. (We would have preferred to use the notation $t^{(x <_T y)}$, but this would have complicated our formulas even more.) We also define

$$\frac{(x <_T y)}{(w <_T z)} = (x <_T y)(w <_T z)^{-1} = (w <_T z)^{-1}(x <_T y).$$

For each $w \in \Sigma$, we will eventually construct an element

$$\langle w; T \rangle \in \operatorname{Aut}(\Sigma, [t_1], \dots, [t_n], t).$$

To begin, for each $x \in X^{\pm 1}$, we define $\langle x; T\rangle$ to be the endomorphism of Σ such that

$$\langle x; T\rangle(y) = \begin{cases} y, & \text{if } y = x^{\pm 1}, \\ \dfrac{(y^{-1} <_T x)}{(x^{-1} <_T x)} x \dfrac{(x <_T x^{-1})}{(y^{-1} <_T x^{-1})} y \dfrac{(y <_T x^{-1})}{(x <_T x^{-1})} x^{-1} \dfrac{(x^{-1} <_T x)}{(y <_T x)}, & \\ & \text{if } y \in X^{\pm 1} - \{x^{\pm 1}\}. \end{cases}$$

Notice that $\langle x; T\rangle(y^{-1}) = (\langle x; T\rangle(y))^{-1}$.

3.4. LEMMA. *If $T = ([t_1], \dots, [t_n], t) \in \mathscr{T}$ and $x \in X^{\pm 1}$, then*

$$\langle x; T\rangle \in \operatorname{Aut}(\Sigma, [t_1], \dots, [t_n], t).$$

Proof. Consider any $y, z \in X^{\pm 1} - \{x^{\pm 1}\}$ such that yz or yxz or $yx^{-1}z$ occurs in T.

By construction, $\langle x; T\rangle$ sends y to an expression of the form ayb, where a, b are expressions in x, t. Hence $\langle x; T\rangle$ acts on a word or cyclic word by inserting, between each pair of adjacent letters, an expression in x and t.

If yz occurs in T, then $y \prec_T z^{-1}$ and $\langle x; T\rangle$ inserts between y and z the expression

$$\begin{aligned} &\frac{(y <_T x^{-1})}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{(y <_T x)} \frac{(z^{-1} <_T x)}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{(z^{-1} <_T x^{-1})} \\ &\qquad = \frac{(z^{-1} <_T x^{-1})}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{(z^{-1} <_T x)} \frac{(z^{-1} <_T x)}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{(z^{-1} <_T x^{-1})} = 1. \end{aligned}$$

If yxz occurs in T, then $y \prec_T x^{-1}$ and $x \prec_T z^{-1}$, and between y and z, $\langle x; T\rangle$ replaces x with

$$\begin{aligned} &\frac{(y <_T x^{-1})}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{(y <_T x)} x \frac{(z^{-1} <_T x)}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{(z^{-1} <_T x^{-1})} \\ &\qquad = \frac{t}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{(x^{-1} <_T x)} x \frac{1}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{(s <_T x^{-1})} \\ &\qquad = \frac{t}{(x <_T x^{-1})(x^{-1} <_T x)} x = x. \end{aligned}$$

The argument is similar if $yx^{-1}z$ occurs in T. Thus the n cyclic words of T are fixed by $\langle x; T\rangle$.

If z is the first letter in t and y is the last letter, then z^{-1} is the least element and y is the greatest element in the $<_T$ ordering. The foregoing shows that

$$\begin{aligned}\langle x;T\rangle(t) &= \frac{(z^{-1} <_T x)}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{(z^{-1} <_T x^{-1})} t \frac{(y <_T x^{-1})}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{(y <_T x)} \\ &= \frac{t}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{t} t \frac{1}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{1} \\ &= t.\end{aligned}$$

The same result holds if $x^{\pm 1}$ is the first or last letter of t, since x is fixed. ∎

3.5. DEFINITION. If $T = ([t_1], \ldots, [t_n], t) \in \mathscr{T}$ and $x \in X^{\pm 1}$, then, since $\langle x;T\rangle$ fixes x and t, it is not difficult to see that $\langle x;T\rangle$ is bijective and that $\langle x^{-1};T\rangle$ is its inverse. Thus $\langle —;T\rangle$ can be extended to a homomorphism

$$\Sigma \to \mathrm{Aut}(\Sigma, [t_1], \ldots, [t_n], t), \qquad w \mapsto \langle w;T\rangle.$$

3.6. LEMMA. *With the above notation, let* $\alpha\colon T \to U$ *be a right Nielsen shift in* $\mathscr{G}$, *with* $\alpha(y) = yz$, $y \prec_T z$ *in* $X^{\pm 1}$.

(i) $\alpha(\langle y;T\rangle) = u^m \langle y;U\rangle\langle z;U\rangle = u^m\langle \alpha(y);U\rangle$, *where* u *denotes the last entry in* U *and* $u^m = (z^{-1} <_U z)/((z^{-1} <_U y^{-1})(y^{-1} <_U z))$.

(ii) $\alpha(\langle z;T\rangle) = \langle z;U\rangle = \langle \alpha(z);U\rangle$.

(iii) *If* $x \in X^{\pm 1} - \{y,z\}^{\pm 1}$, *then* $\alpha(\langle x;T\rangle) = \langle x;U\rangle = \langle \alpha(x);U\rangle$.

[In (i), we can take $m = -1$ or 0, depending as the permutation that carries the sequence z^{-1}, y^{-1}, z into its correct T-ordering is even or odd, respectively.

Proof. Here $\alpha(t) = u$ and $y \prec_T z$, $z^{-1} \prec_U y$. Moreover the T-ordering and the U-ordering of $X^{\pm 1} - \{y\}$ are the same ordering.

To prove (i) we first recall the actions of $\langle y;T\rangle$ and $\langle y;U\rangle$.

If $w \in X^{\pm 1} - \{y\}^{\pm 1}$, then

$$\begin{aligned}&\langle y;T\rangle(w) \\ &\quad = \frac{(w^{-1} <_T y)}{(y^{-1} <_T y)} y \frac{(y <_T y^{-1})}{(w^{-1} <_T y^{-1})} w \frac{(w <_T y^{-1})}{(y <_T y^{-1})} y^{-1} \frac{(y^{-1} <_T y)}{(w <_T y)}.\end{aligned}$$

We have $y \prec_T z$, so if $w \in X^{\pm 1} - \{y, z\}^{\pm 1}$, then

$$\langle y;T\rangle(w) = \frac{(w^{-1} <_T z)}{(y^{-1} <_T z)} y \frac{(z <_T y^{-1})}{(w^{-1} <_T y^{-1})} w \frac{(w <_T y^{-1})}{(z <_T y^{-1})} y^{-1} \frac{(y^{-1} <_T z)}{(w <_T z)},$$

while

$$\langle y;T\rangle(z) = \frac{(z^{-1} <_T z)}{(y^{-1} <_T z)} y \frac{(z <_T y^{-1})}{(z^{-1} <_T y^{-1})} zy^{-1} (y^{-1} <_T z)$$

and

$$\langle y;T\rangle(z^{-1}) = \frac{1}{(y^{-1} <_T z)} yz^{-1} \frac{(z^{-1} <_T y^{-1})}{(z <_T y^{-1})} y^{-1} \frac{(y^{-1} <_T z)}{(z^{-1} <_T z)}.$$

If $w \in X^{\pm 1} - \{y\}^{\pm 1}$, then

$$\langle y;U\rangle(w) = \frac{(w^{-1} <_U y)}{(y^{-1} <_U y)} y \frac{(z^{-1} <_U y^{-1})}{(w^{-1} <_U y^{-1})} w \frac{(w <_U y^{-1})}{(z^{-1} <_U y^{-1})} y^{-1} \frac{(y^{-1} <_U y)}{(w <_U y)}.$$

Here $z^{-1} \prec_U y$, so if $w \in X^{\pm 1} - \{y, z\}^{\pm 1}$, then

$$\langle y;U\rangle(w) = \frac{(w^{-1} <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{(z^{-1} <_U y^{-1})}{(w^{-1} <_U y^{-1})} w \frac{(w <_U y^{-1})}{(z^{-1} <_U y^{-1})} y^{-1} \frac{(y^{-1} <_U z^{-1})}{(w <_U z^{-1})},$$

while

$$\begin{aligned}\langle y;U\rangle(z) &= \frac{u}{(y^{-1} <_U z^{-1})} yz \frac{(z <_U y^{-1})}{(z^{-1} <_U y^{-1})} y^{-1} \frac{(y^{-1} <_U z^{-1})}{(z <_U z^{-1})} \\ &= (z^{-1} <_U y^{-1}) yz \frac{(z <_U y^{-1})}{(z^{-1} <_U y^{-1})} y^{-1} \frac{(y^{-1} <_U z^{-1})}{(z <_U z^{-1})},\end{aligned}$$

and

$$\langle y;U\rangle(z^{-1}) = \frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{(z^{-1} <_U y^{-1})}{(z <_U y^{-1})} z^{-1} y^{-1} \frac{1}{(z^{-1} <_U y^{-1})}.$$

Notice also that

$$\langle z;U\rangle(y) = \frac{(y^{-1} <_U z)}{(z^{-1} <_U z)} z \frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{(y <_U z^{-1})}{(z <_U z^{-1})} z^{-1} \frac{(z^{-1} <_U z)}{(y <_U z)}$$

$$= \frac{(y^{-1} <_U z)}{(z^{-1} <_U z)} z \frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{1}{(z <_U z^{-1})} z^{-1}.$$

We can now verify (i) by considering three cases:

$$\begin{aligned}
&u^m\langle y;U\rangle\langle z;U\rangle(y)\\
&\quad = u^m\langle y;U\rangle(\langle z;U\rangle(y))\\
&\quad = u^m\langle y;U\rangle\left(\frac{(y^{-1} <_U z)}{(z^{-1} <_U z)} z \frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{1}{(z <_U z^{-1})} z^{-1}\right)\\
&\quad = u^m\left(\frac{(y^{-1} <_U z)}{(z^{-1} <_U z)} \cdot (z^{-1} <_U y^{-1}) yz \frac{(z <_U y^{-1})}{(z^{-1} <_U y^{-1})}\right.\\
&\qquad\qquad y^{-1} \frac{(y^{-1} <_U z^{-1})}{(z <_U z^{-1})} \cdot \frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{1}{(z <_U z^{-1})}\\
&\qquad\qquad \left.\cdot \frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})} y \frac{(z^{-1} <_U y^{-1})}{(z <_U y^{-1})} z^{-1} y^{-1} \frac{1}{(z^{-1} <_U y^{-1})}\right)\\
&\quad = u^m\left(\frac{(y^{-1} <_U z)(z^{-1} <_U y^{-1})}{(z^{-1} <_U z)} yz \frac{1}{(y^{-1} <_U z)} y \frac{(z^{-1} <_U y^{-1})}{(z <_U y^{-1})}\right.\\
&\qquad\qquad \left. z^{-1} y^{-1} \frac{1}{(z^{-1} <_U y^{-1})}\right)\\
&\quad = yz \frac{1}{(y^{-1} <_U z)} y \frac{(z^{-1} <_U y^{-1})}{(z <_U y^{-1})} z^{-1} y^{-1} \frac{(y^{-1} <_U z)}{(z^{-1} <_U z)}\\
&\quad = \alpha\left(y \frac{1}{(y^{-1} <_T z)} yz^{-1} \frac{(z^{-1} <_T y^{-1})}{(z <_T y^{-1})} y^{-1} \frac{(y^{-1} <_T z)}{(z^{-1} <_T z)}\right)\\
&\quad = \alpha\langle y;T\rangle(yz^{-1}) = \alpha\langle y;T\rangle\alpha^{-1}(y) = \alpha(\langle y;T\rangle)(y).
\end{aligned}$$

Also,

$$\begin{aligned}
\alpha\langle y;T\rangle(z) &= \alpha\langle y;T\rangle\alpha^{-1}(z) = \alpha\langle y;T\rangle(z)\\
&= \alpha\left(\frac{(z^{-1} <_T z)}{(y^{-1} <_T z)}y\frac{(z <_T y^{-1})}{(z^{-1} <_T y^{-1})}zy^{-1}(y^{-1} <_T z)\right)\\
&= \frac{(z^{-1} <_U z)}{(y^{-1} <_U z)}yz\frac{(z <_U y^{-1})}{(z^{-1} <_U y^{-1})}y^{-1}(y^{-1} <_U z)\\
&= u^m\left((z^{-1} <_U y^{-1})yz\frac{(z <_U y^{-1})}{(z^{-1} <_U y^{-1})}y^{-1}\frac{(y^{-1} <_U z^{-1})}{(z <_U z^{-1})}\right)\\
&= u^m\langle y;U\rangle(z) = u^m\langle y;U\rangle(\langle z;U\rangle(z)) = u^m\langle y;U\rangle\langle z;U\rangle(z)
\end{aligned}$$

Moreover, for $w \in X^{\pm 1} - \{y, z\}^{\pm 1}$,

$$\begin{aligned}
&u^m\langle y;U\rangle\langle z;U\rangle(w)\\
&\quad= \langle y;U\rangle(\langle z;U\rangle(w))\\
&\quad= u^m\langle y;U\rangle\left(\frac{(w^{-1} <_U z)}{(z^{-1} <_U z)}z\frac{(z <_U z^{-1})}{(w^{-1} <_U z^{-1})}w\frac{(w <_U z^{-1})}{(z <_U z^{-1})}\right.\\
&\qquad\qquad\qquad\left. z^{-1}\frac{(z^{-1} <_U z)}{(w <_U z)}\right)\\
&\quad= u^m\left(\frac{(w^{-1} <_U z)}{(z^{-1} <_U z)}\cdot(z^{-1} <_U y^{-1})yz\frac{(z <_U y^{-1})}{(z^{-1} <_U y^{-1})}\right.\\
&\qquad\qquad y^{-1}\frac{(y^{-1} <_U z^{-1})}{(z <_U z^{-1})}\cdot\frac{(z <_U z^{-1})}{(w^{-1} <_U z^{-1})}\cdot\frac{(w^{-1} <_U z^{-1})}{(y^{-1} <_U z^{-1})}\\
&\qquad\qquad y\frac{(z^{-1} <_U y^{-1})}{(w^{-1} <_U y^{-1})}w\frac{(w <_U y^{-1})}{(z^{-1} <_U y^{-1})}y^{-1}\frac{(y^{-1} <_U z^{-1})}{(w <_U z^{-1})}\\
&\qquad\qquad \cdot\frac{(w <_U z^{-1})}{(z <_U z^{-1})}\cdot\frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})}y\frac{(z^{-1} <_U y^{-1})}{(z <_U y^{-1})}\\
&\qquad\qquad\left. z^{-1}y^{-1}\frac{1}{(z^{-1} <_U y^{-1})}\cdot\frac{(z^{-1} <_U z)}{(w <_U z)}\right)
\end{aligned}$$

$$= u^m\left(\frac{(w^{-1} <_U z)(z^{-1} <_U y^{-1})}{(z^{-1} <_U z)}yz\frac{(z <_U y^{-1})}{(w^{-1} <_U y^{-1})}\right.$$

$$\left. w\frac{(w <_U y^{-1})}{(z <_U y^{-1})}z^{-1}y^{-1}\frac{(z^{-1} <_U z)}{(z^{-1} <_U y^{-1})(w <_U z)}\right)$$

$$= \frac{(w^{-1} <_U z)}{(y^{-1} <_U z)}yz\frac{(z <_U y^{-1})}{(w^{-1} <_U y^{-1})}w\frac{(w <_U y^{-1})}{(z <_U y^{-1})}z^{-1}y^{-1}\frac{(z^{-1} <_U z)}{(w <_U z)}$$

$$= \alpha\left(\frac{(w^{-1} <_T z)}{(y^{-1} <_T z)}y\frac{(z <_T y^{-1})}{(w^{-1} <_T y^{-1})}w\frac{(w <_T y^{-1})}{(z <_T y^{-1})}y^{-1}\frac{(y^{-1} <_T z)}{(w <_T z)}\right)$$

$$= \alpha\left(\frac{(w^{-1} <_T y)}{(y^{-1} <_T y)}y\frac{(y <_T y^{-1})}{(w^{-1} <_T y^{-1})}w\frac{(w <_T y^{-1})}{(y <_T y^{-1})}y^{-1}\frac{(y^{-1} <_T y)}{(w <_T y)}\right)$$

$$= \alpha(\langle y;T\rangle(w)) = \alpha(\langle y;T\rangle(\alpha^{-1}(w)) = \alpha(\langle y;T\rangle)(w).$$

Thus $\alpha(\langle y;T\rangle)$ and $u^m\langle y;U\rangle\langle z;U\rangle$ agree on y, on z, and on $X^{\pm 1} - \{y,z\}^{\pm 1}$, so are equal. This proves (i).

We now prove (ii) in the same way:

$$\alpha(\langle z;T\rangle)(y)$$

$$= \alpha\langle z;T\rangle\alpha^{-1}(y) = \alpha\langle z;T\rangle(yz^{-1})$$

$$= \alpha\left(\frac{(y^{-1} <_T z)}{(z^{-1} <_T z)}z\frac{(z <_T z^{-1})}{(y^{-1} <_T z^{-1})}y\frac{(y <_T z^{-1})}{(z <_T z^{-1})}z^{-1}\frac{(z^{-1} <_T z)}{(y <_T z)}z^{-1}\right)$$

$$= \alpha\left(\frac{(y^{-1} <_T z)}{(z^{-1} <_T z)}z\frac{(z <_T z^{-1})}{(y^{-1} <_T z^{-1})}yz^{-1}\frac{(z^{-1} <_T z)}{t}z^{-1}\right)$$

$$= \alpha\left(\frac{(y^{-1} <_T z)}{(z^{-1} <_T z)}z\frac{(z <_T z^{-1})}{(y^{-1} <_T z^{-1})}yz^{-1}\frac{1}{(z <_T z^{-1})}z^{-1}\right)$$

$$= \frac{(y^{-1} <_U z)}{(z^{-1} <_U z)}z\frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})}y\frac{1}{(z <_U z^{-1})}z^{-1}$$

$$= \frac{(y^{-1} <_U z)}{(z^{-1} <_U z)}z\frac{(z <_U z^{-1})}{(y^{-1} <_U z^{-1})}y\frac{(y <_U z^{-1})}{(z <_U z^{-1})}z^{-1}\frac{(z^{-1} <_U z)}{(y <_U z)}$$

$$= \langle z;U\rangle(y).$$

Also $\alpha(\langle z;T\rangle)(z)=\alpha\langle z;T\rangle\alpha^{-1}(z)=\alpha\langle z;T\rangle(z)=\alpha(z)=z=\langle z;U\rangle(z)$. In addition, for $w\in X^{\pm 1}-\{y,z\}^{\pm 1}$,

$$\begin{aligned}
&\alpha(\langle z;T\rangle)(w)\\
&\quad=\alpha\langle z;T\rangle\alpha^{-1}(w)=\alpha\langle z;T\rangle(w)\\
&\quad=\alpha\left(\frac{(w^{-1}<_T z)}{(z^{-1}<_T z)}z\frac{(z<_T z^{-1})}{(w^{-1}<_T z^{-1})}w\frac{(w<_T z^{-1})}{(z<_T z^{-1})}z^{-1}\frac{(z^{-1}<_T z)}{(w<_T z)}\right)\\
&\quad=\frac{(w^{-1}<_U z)}{(z^{-1}<_U z)}z\frac{(z<_U z^{-1})}{(w^{-1}<_U z^{-1})}w\frac{(w<_U z^{-1})}{(z<_U z^{-1})}z^{-1}\frac{(z^{-1}<_U z)}{(w<_U z)}\\
&\quad=\langle z;U\rangle(w).
\end{aligned}$$

Thus $\alpha(\langle z;T\rangle)$ and $\langle z;U\rangle$ agree on y, on z, and on $X^{\pm 1}-\{y,z\}^{\pm 1}$, so are equal. This proves (ii).

Finally, we prove (iii) in the same way:

$$\begin{aligned}
&\alpha(\langle x;T\rangle)(y)\\
&\quad=\alpha\langle x;T\rangle\alpha^{-1}(y)=\alpha\langle x;T\rangle(yz^{-1})\\
&\quad=\alpha\left(\frac{(y^{-1}<_T x)}{(x^{-1}<_T x)}x\frac{(x<_T x^{-1})}{(y^{-1}<_T x^{-1})}y\frac{(y<_T x^{-1})}{(x<_T x^{-1})}x^{-1}\frac{(x^{-1}<_T x)}{(y<_T x)}\right.\\
&\qquad\left.\frac{(z<_T x)}{(x^{-1}<_T x)}x\frac{(x<_T x^{-1})}{(z<_T x^{-1})}z^{-1}\frac{(z^{-1}<_T x^{-1})}{(x<_T x^{-1})}x^{-1}\frac{(x^{-1}<_T x)}{(z^{-1}<_T x)}\right)\\
&\quad=\alpha\left(\frac{(y^{-1}<_T x)}{(x^{-1}<_T x)}x\frac{(x<_T x^{-1})}{(y^{-1}<_T x^{-1})}y\frac{(z<_T x^{-1})}{(x<_T x^{-1})}x^{-1}\frac{(x^{-1}<_T x)}{(z<_T x)}\right.\\
&\qquad\left.\frac{(z<_T x)}{(x^{-1}<_T x)}x\frac{(x<_T x^{-1})}{(z<_T x^{-1})}z^{-1}\frac{(z^{-1}<_T x^{-1})}{(x<_T x^{-1})}x^{-1}\frac{(x^{-1}<_T x)}{(z^{-1}<_T x)}\right)\\
&=\alpha\left(\frac{(y^{-1}<_T x)}{(x^{-1}<_T x)}x\frac{(x<_T x^{-1})}{(y^{-1}<_T x^{-1})}yz^{-1}\frac{(z^{-1}<_T x^{-1})}{(x<_T x^{-1})}x^{-1}\frac{(x^{-1}<_T x)}{(z^{-1}<_T x)}\right)\\
&=\frac{(y^{-1}<_U x)}{(x^{-1}<_U x)}x\frac{(x<_U x^{-1})}{(y^{-1}<_U x^{-1})}y\frac{(z^{-1}<_U x^{-1})}{(x<_U x^{-1})}x^{-1}\frac{(x^{-1}<_U x)}{(z^{-1}<_U x)}\\
&\quad=\frac{(y^{-1}<_U x)}{(x^{-1}<_U x)}x\frac{(x<_U x^{-1})}{(y^{-1}<_U x^{-1})}y\frac{(y<_U x^{-1})}{(x<_U x^{-1})}x^{-1}\frac{(x^{-1}<_U x)}{(y<_U x)}\\
&\quad=\langle x;U\rangle(y).
\end{aligned}$$

Also $\alpha(\langle x;T\rangle)(x)=\alpha\langle x;T\rangle\alpha^{-1}(x)=\alpha\langle x;T\rangle(x)=x=\langle x;U\rangle(x)$.

Moreover, for $w \in X^{\pm 1} - \{x, y\}^{\pm 1}$,

$$
\begin{aligned}
&\alpha(\langle x; T\rangle)(w) \\
&\quad = \alpha\langle x; T\rangle\alpha^{-1}(w) = \alpha\langle x; T\rangle(w) \\
&\quad = \alpha\left(\frac{(w^{-1} <_T x)}{(x^{-1} <_T x)} x \frac{(x <_T x^{-1})}{(w^{-1} <_T x^{-1})} w \frac{(w <_T x^{-1})}{(x <_T x^{-1})} x^{-1} \frac{(x^{-1} <_T x)}{(w <_T x)}\right) \\
&\quad = \frac{(w^{-1} <_U x)}{(x^{-1} <_U x)} x \frac{(x <_U x^{-1})}{(w^{-1} <_U x^{-1})} w \frac{(w <_U x^{-1})}{(x <_U x^{-1})} x^{-1} \frac{(x^{-1} <_U x)}{(w <_U x)} \\
&\quad = \langle x; U\rangle(w).
\end{aligned}
$$

Thus $\alpha(\langle x; T\rangle)$ and $\langle x; U\rangle$ agree on y, on x, and on $X^{\pm 1} - \{y, x\}^{\pm 1}$, so are equal. This proves (iii). ∎

3.7. THEOREM. *With the above notation, for each* $\alpha\colon T \to U$ *in* $\mathscr{G}$ *and each* $x \in \Sigma$, *there exists an integer* m *such that* $\alpha(\langle x; T\rangle) = u^m\langle \alpha(x); U\rangle$, *where* u *denotes the last entry in* U.

(If Σ is not abelian, the integer m is uniquely determined by $\alpha\colon T \to U$ and x, since no positive power of u is central.)

Proof. It follows from McCool's Lemma 3.2, that it suffices to consider the case where $\alpha\colon T \to U$ is a signed permutation or a Nielsen shift.

The result is clear for a signed permutation, and here $m = 0$.

Thus we may assume that $\alpha\colon T \to U$ is a right or left Nielsen shift. The case of a right Nielsen shift is covered by Lemma 3.6, and here $m = 0$ or -1. The case of a left Nielsen shift is quite similar, and is converted to a right Nielsen shift if we invert the elements of T, U. This completes the proof. ∎

3.8. *Open Problem.* What is the topological significance of the integer m occurring in Theorem 3.7? We have not been able to extract any useful information from this invariant, but we suspect it must have applications.

We have now proved the main result of this section, which originally came from topological results of Birman [3], and Maclachlan and Harvey [12], which in turn depended on results of Dehn, Nielsen, and others. Our proof is algebraic, since it is based on the above result of McCool.

3.9. COROLLARY. *With the above notation, for any* $([t_1], \ldots, [t_n], t) \in \mathscr{T}$ *any* $\alpha \in \mathrm{Aut}(\Sigma, \{[t_1], \ldots, [t_n]\}, t)$, *and any* $x \in \Sigma$, *there exists an integer* m *such that* $\alpha(\langle x; T\rangle) = t^m\langle \alpha(x); \alpha(T)\rangle = t^m\langle \alpha(x); T\rangle$.

Proof. This is immediate from Theorem 3.7, since $\langle \alpha(x); T\rangle$ does not depend on the order of the cyclic words in the sequence T. ∎

4. THE ISOMORPHISMS

4.1. *Notation.* Throughout this section, $\Sigma_{g,n+1}$ denotes the free group with the specified basis $X = \{x_i, y_i, z_j \mid 1 \le i \le g,\ 1 \le j \le n\}$ and a specified element

$$z_{n+1} = z_n^{-1} \cdots z_1^{-1}[y_g, x_g] \cdots [y_1, x_1].$$

For $1 \le i \le n$, we write $p_i = [x_i, y_i]$, so $z_{n+1}^{-1} = p_1 \cdots p_g z_1 \cdots z_n$.

Let N denote the normal subgroup of $\Sigma_{g,n+1}$ generated by z_{n+1}, and identify $\Sigma_{g,n} = \Sigma_{g,n+1}/N$.

4.2. Definitions. Let $T = ([z_1], \dots, [z_n], z_{n+1})$. It is readily verified that T lies in the set $\mathscr{T}$ described in Definition 3.1. The T-ordering on $X^{\pm 1}$ of Definition 3.1 will be denoted $<$ for the purposes of the current definition. Thus

$$z_n < z_n^{-1} < \cdots < z_1 < z_1^{-1} < y_g^{-1} < x_g < y_g < x_g^{-1} < y_{g-1}^{-1} < \cdots < x_1^{-1}.$$

For each $w \in \Sigma_{n+1,g}$, we denote by $\hat{w}$ the automorphism $\langle w; T\rangle$ of Definition 3.5, so we have a homomorphism

$$\Sigma_{g,n+1} \to \mathrm{Aut}\Big(\Sigma_{g,n+1}, \big\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\big\}, \{z_{n+1}\}^{\pm 1}\Big), \qquad w \mapsto \hat{w}.$$

Let us record the following, after which we will no longer need T nor the T-ordering:

$$\hat{x}_i(w) = \begin{cases} x_i z_{n+1}(w), & \text{for } w = x_1, y_1, \dots, x_{i-1}, y_{i-1}, \\ x_i, & \text{for } w = x_i, \\ z_{n+1} x_i y_i x_i^{-1}, & \text{for } w = y_i, \\ z_{n+1} x_i(w), & \text{for } w = x_{i+1}, y_{i+1}, \dots, y_g, z_1, \dots, z_n, \end{cases}$$

$$\hat{y}_i(w) = \begin{cases} z_{n+1}^{-1} y_i(w), & \text{for } w = x_1, y_1, \dots, x_{i-1}, y_{i-1}, \\ z_{n+1}^{-1} y_i x_i y_i^{-1}, & \text{for } w = x_i, \\ y_i, & \text{for } w = y_i, \\ y_i z_{n+1}^{-1}(w), & \text{for } w = x_{i+1}, y_{i+1}, \dots, y_g, z_1, \dots, z_n, \end{cases}$$

$$\hat{z}_j(w) = \begin{cases} z_j z_{n+1}(w), & \text{for } w = x_1, y_1, \dots, y_g, z_1, \dots, z_{j-1}, \\ z_j, & \text{for } w = z_j, \\ z_{n+1} z_j(w), & \text{for } w = z_{j+1}, \dots, z_n. \end{cases}$$

A straightforward but tedious calculation shows that

$$\hat{p}_i(w) = \begin{cases} z_{n+1} p_i z_{n+1}(w), & \text{for } w = x_1, y_1, \ldots, x_{i-1}, y_{i-1}, \\ z_{n+1} p_i(w), & \text{for } w = x_i, y_i, \\ z_{n+1}^2 p_i(w), & \text{for } w = x_{i+1}, y_{i+1}, \ldots, y_g, z_1, \ldots, z_n. \end{cases}$$

Let us also record the only case of Corollary 3.9 that we shall use.

4.3. COROLLARY. *For any* $\alpha \in \operatorname{Aut}(\Sigma_{g,n+1}, \{[z_1], \ldots, [z_n]\}, z_{n+1})$ *and any* $w \in \Sigma_{g,n+1}$, *there exists an integer m such that* $\alpha(\hat{w}) = z_{n+1}^m \widehat{\alpha(w)}$.

In the following statement we use Fix to denote the fixed subgroup of an automorphism and δ to denote Kronecker's delta.

4.4. LEMMA. $\operatorname{Fix}(z_{n+1}^k \hat{p}_i) = \langle p_i^{\delta_{k,-1}}, z_{n+1} \rangle$ *and* $\operatorname{Fix}(z_{n+1}^k \hat{z}_j) = \langle z_j^{\delta_{k,0}}, z_{n+1} \rangle$, *for any integers i, j, k, with* $1 \le i \le g$, $1 \le j \le n$.

Proof. It follows from the formulas in Definition 4.2 that

$$\hat{z}_j(w) = \begin{cases} z_j z_{n+1}(w), & \text{for } w = x_1, y_1, \ldots, y_g, z_1, \ldots, z_{j-1}, \\ z_j(w), & \text{for } w = z_j, \\ z_j z_{n+1}(w), & \text{for } w = z_j(z_{j+1}), \ldots, z_j(z_n). \end{cases}$$

Let $G_1 = \langle z_j \rangle$ and $G_2 = \langle x_1, y_1, \ldots, y_g, z_1, \ldots, z_{j-1}, z_j(z_{j+1}), \ldots, z_j(z_n) \rangle$. Thus we have a free product decomposition $\Sigma_{g,n+1} = G_1 * G_2$ and we have elements $g_1 = z_j \in G_1$, $g_2 = z_j z_{n+1} \in G_2$, since

$$p_1 \cdots p_g \cdot z_1 \cdots z_{j-1} \cdot z_j(z_{j+1}) \cdots z_j(z_n) = z_{n+1}^{-1} z_j^{-1}.$$

Now $\hat{z}_j$ acts as conjugation by g_i on G_i, for $i = 1, 2$, so $\operatorname{Fix}(\hat{z}_j) = \langle g_1, g_2 \rangle$ and this is $\langle z_j, z_{n+1} \rangle$.

Since $z_{n+1} = g_1^{-1} g_2$ has length exactly 2 with respect to the free product decomposition, conjugation by z_{n+1}^k increases this length for any element whose free product normal form does not begin or end with $z_{n+1}^{\pm 1}$. Since $\hat{z}_j$ preserves the length, we see that if $k \neq 0$, then $\operatorname{Fix}(z_{n+1}^k \hat{z}_j)$ lies entirely in $\langle z_{n+1} \rangle$.

Similarly, it follows from Definition 4.2 that

$$z_{n+1}^{-1} \hat{p}_i(w) = \begin{cases} p_i z_{n+1}(w), & \text{for } w = x_1, y_1, \ldots, x_{i-1}, y_{i-1}, \\ p_i(w), & \text{for } w = x_i, y_i, \\ p_i z_{n+1}(w), & \text{for } w = p_i(x_{i+1}), \ldots, p_i(y_g), p_i(z_1), \ldots, p_i(z_n). \end{cases}$$

Now let $G_1 = \langle x_i, y_i \rangle$ and

$$G_2 = \Big\langle x_1, y_1, \ldots, x_{i-1}, y_{i-1}, p_i(x_{i+1}), p_i(y_{i+1}), \ldots, p_i(y_g), p_i(z_1), \\ \ldots, p_i(z_n) \Big\rangle .$$

Again $\Sigma_{g,n+1} = G_1 * G_2$ and we have elements $g_1 = p_i \in G_1$, $g_2 = p_i z_{n+1} \in G_2$, since $p_1 \cdots p_{i-1} \cdot p_i(p_{i+1}) \cdots p_i(p_n) \cdot p_i(z_1) \cdots p_i(z_n) = z_{n+1}^{-1} p_i^{-1}$. Now $z_{n+1}^{-1} \hat{p}_i$ acts as conjugation by g_i on G_i, for $i = 1, 2$, so $\mathrm{Fix}(t^{-1}\hat{p}_i) = \langle g_1, g_2 \rangle$, and this is $\langle p_i, z_{n+1} \rangle$.

As before, $\mathrm{Fix}(z_{n+1}^k \hat{p}_i) = \langle z_{n+1} \rangle$, if $k \neq -1$. We will not be using this particular fact later, but have included it for completeness. ∎

Recall that N denotes the normal closure of z_{n+1} in $\Sigma_{g,n+1}$ and that $\chi_{g,n} = 2 - 2g - n$.

4.5. Lemma. *$\hat{z}_{n+1} = z_{n+1}^{\chi_{g,n}}$ and $\hat{N} = \langle \hat{z}_{n+1} \rangle = \langle z_{n+1}^{\chi_{g,n}} \rangle$.*

Proof. We have

$$\begin{aligned} z_{n+1}^{-g} \hat{z}_{n+1}^{-1} &= z_{n+1}^{-g} \hat{p}_1 \hat{p}_2 \cdots \hat{p}_g \hat{z}_1 \cdots \hat{z}_n \\ &= z_{n+1}^{-1} \hat{p}_1 z_{n+1}^{-1} \hat{p}_2 \cdots z_{n+1}^{-1} \hat{p}_g \hat{z}_1 \cdots \hat{z}_n . \end{aligned}$$

Using the descriptions

$$\hat{z}_j(w) = \begin{cases} z_j z_{n+1}(w), & \text{for } w = x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_{j-1}, \\ z_j, & \text{for } w = z_j, \\ z_{n+1} z_j(w), & \text{for } w = z_{j+1}, \ldots, z_n, \end{cases}$$

$$z_{n+1}^{-1} \hat{p}_i(w) = \begin{cases} p_i z_{n+1}(w), & \text{for } w = x_1, y_1, \ldots, x_{i-1}, y_{i-1}, \\ p_i(w), & \text{for } w = x_i, y_i, \\ z_{n+1} p_i(w), & \text{for } w = x_{i+1}, y_{i+1}, \ldots, y_g, z_1, \ldots, z_n, \end{cases}$$

it is straightforward to calculate that $z_{n+1}^{-1} \hat{p}_1 z_{n+1}^{-1} \hat{p}_2 \cdots z_{n+1}^{-1} \hat{p}_g \hat{z}_1 \cdots \hat{z}_n$ acts as z_{n+1}^{g+n-2} on each generator, so $\hat{z}_{n+1} = z_{n+1}^{2-2g-n} = z_{n+1}^{\chi_{g,n}}$.

Now $\hat{N}$ is the normal subgroup of $\hat{\Sigma}_{g,n+1}$ generated by the (central) element $\hat{z}_{n+1} = z_{n+1}^{\chi_{g,n}}$, so $\hat{N} = \langle z_{n+1}^{\chi_{g,n}} \rangle$. ∎

4.6. Definitions. We have a natural homomorphism

$$\Sigma_{g,n+1} \to \Sigma_{g,n+1}/N = \Sigma_{g,n},$$

and hence an induced homomorphism

$$\operatorname{collapse}(z_{n+1})\colon \operatorname{Aut}\Big(\Sigma_{g,n+1}, \big\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\big\}, \{z_{n+1}\}^{\pm 1}\Big) \\ \to \operatorname{Aut}\Big(\Sigma_{g,n}, \big\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\big\}\Big),$$

since N is stabilized by maps which fix or invert z_{n+1}.

Let us consider the composite of the latter map, with the map

$$\Sigma_{g,n+1} \to \operatorname{Aut}\Big(\Sigma_{g,n+1}, \big\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\big\}, \{z_{n+1}\}^{\pm 1}\Big), \qquad w \mapsto \hat{w}.$$

It is immediate from the formulas in Definition 4.2 that this composite carries each element of X to the corresponding element of $\operatorname{Inn}(\Sigma_{g,n})$. Hence the image of $\hat{\Sigma}_{g,n+1}$ under $\operatorname{collapse}(z_{n+1})$ is precisely $\operatorname{Inn}(\Sigma_{g,n})$, and we have a surjective map

$$\hat{\Sigma}_{g,n+1} \to \Sigma_{g,n}/\mathrm{Ctr}.$$

4.7. PROPOSITION. *The map* $\hat{\Sigma}_{g,n+1} \to \Sigma_{g,n}/\mathrm{Ctr}$ *gives a universal central extension.*

Proof. If $\chi_{g,n} \geq 0$, then $(g,n) \in \{(0,0),\ (0,1),\ (0,2),\ (1,0)\}$, and it follows from the formulas in Definition 4.2 that $\hat{\Sigma}_{g,n+1}$ is trivial, as is $\Sigma_{g,n}/\mathrm{Ctr}$.

Thus we may assume that $\chi_{g,n} < 0$, so $\Sigma_{g,n}/\mathrm{Ctr} = \Sigma_{g,n}$. The kernel of the map $\hat{\Sigma}_{g,n+1} \to \Sigma_{g,n}$ is clearly $\hat{N}$ and, by Lemma 4.5, $\hat{N}$ is central and infinite cyclic. Thus we have a presentation

$$\hat{\Sigma}_{g,n+1} \\ \approx \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_n \text{ is central} \rangle,$$

which is the universal central extension of $\Sigma_{g,n}$. ∎

4.8. REMARK. Here we have a phenomenon similar to one encountered by Milnor [18]. In lifting representations of surface groups in $SL_2(\mathbb{R})$ to representations of central extensions in $\widetilde{SL_2(\mathbb{R})}$, he found that the power of the generator of the center of $\widetilde{SL_2(\mathbb{R})}$, to which the "relator" gets mapped, is a nonzero multiple of the Euler characteristic of the surface.

We now come to the delicate algebraic proof of a theorem, which Birman [3], Bers [2], and Maclachlan [11] obtained using topology and analysis.

4.9. THEOREM. *The homomorphism*

$$\operatorname{collapse}(z_{n+1})\colon \operatorname{Aut}\Big(\Sigma_{g,n+1},\big\{[z_1]^{\pm 1},\dots,[z_n]^{\pm 1}\big\},\{z_{n+1}\}^{\pm 1}\Big)$$
$$\to \operatorname{Aut}\Big(\Sigma_{g,n},\big\{[z_1]^{\pm 1},\dots,[z_n]^{\pm 1}\big\}\Big)$$

induced by the homomorphism $\Sigma_{g,n+1}\to\Sigma_{g,n+1}/N=\Sigma_{g,n}$, *is surjective and, except in the case* $(g,n)=(0,1)$, *has kernel* $\operatorname{Inn}\langle z_{n+1}\rangle$.

Proof. Let the map $\Sigma_{g,N+1}\to\Sigma_{g,n+1}/N=\Sigma_{g,n}$ be denoted $x\mapsto\bar{x}$.

We begin by proving surjectivity. Notice that, in Definitions 4.6, we saw that $\operatorname{Inn}(\Sigma_{g,n})$ lies in the image of $\operatorname{collapse}(z_{n+1})$. Let $\beta\in\operatorname{Aut}(\Sigma_{g,n},\{[z_1]^{\pm 1},\dots,[z_n]^{\pm 1}\})$.

Consider first the case $n=0$. Here there exists $\alpha\in\operatorname{Aut}(\Sigma_{g,1},[z_1]^{\pm 1})$ that induces β, by a topological result of Nielsen; for algebraic proofs, see [22, Corollary 5.4.3] or [5, Theorem 4.9]. Hence there exists $x\in\Sigma_{g,1}$ such that $x\alpha\in\operatorname{Aut}(\Sigma_{g,1},\{z_1\}^{\pm 1})$. Thus $\operatorname{collapse}(z_{n+1})$ sends $x\alpha$ to $y\beta$ for some inner automorphism y. Since y and $y\beta$ lie in the image of $\operatorname{collapse}(z_{n+1})$, we see that β does also.

Thus, in proving surjectivity, we may assume $n\ge 1$. Since the inner automorphisms lies in the image, we may assume that β fixes or inverts $\bar{z}_n$. Let us identify $\Sigma_{g,n}$ with the free factor of $\Sigma_{g,n+1}$ generated by $X-\{z_n\}$. Let $\alpha\in\operatorname{Aut}(\Sigma_{g,n+1})=\operatorname{Aut}(\Sigma_{g,n}*\langle z_n\rangle)$ be the automorphism that acts on the free factor $\Sigma_{g,n}$ as β and sends z_n to z_n or $\bar{z}_n^{-1}z_n^{-1}\bar{z}_n$, depending as β fixes or inverts $\bar{z}_n$, respectively. This fixes or inverts z_{n+1}, so $\operatorname{collapse}(z_{n+1})$ sends α to β, and this completes the proof of surjectivity.

Now it remains to show that the inner automorphism z_{n+1} generates the kernel of $\operatorname{collapse}(z_{n+1})$. Let α be an arbitrary element of the kernel of $\operatorname{collapse}(z_{n+1})$. Since $(g,n)\ne(0,1)$, it can be seen that α lies in

$$\operatorname{Aut}\big(\Sigma_{g,n+1},[z_1],\dots,[z_n],\{z_{n+1}\}\big).$$

Consider any $w\in\Sigma_{g,n+1}$. Then $\overline{\alpha(w)}=\overline{w}$, so $\alpha(w)\in wN$ and hence $\widehat{\alpha(w)}\in\widehat{wN}=\hat{w}\langle z_{n+1}^{\chi_{g,n}}\rangle$ by Lemma 4.5. However, $\widehat{\alpha(w)}\in\alpha(\hat{w})\langle z_{n+1}\rangle$ by Corollary 4.3; thus $\alpha(\hat{w})\in\hat{w}\langle z_{n+1}\rangle$.

Consider any $i\in\{1,\dots,g\}$. We have

$$\alpha(\hat{p}_i)=\big[\alpha(\hat{x}_i),\alpha(\hat{y}_i)\big]\in\big[\hat{x}_i\langle z_{n+1}\rangle,\hat{y}_i\langle z_{n+1}\rangle\big]=\big\{[\hat{x}_i,\hat{y}_i]\big\}=\{\hat{p}_i\}.$$

That is, $\alpha(\hat{p}_i)=\hat{p}_i$, so

$$\alpha\hat{p}_i\alpha^{-1}(p_i)=\alpha(\hat{p}_i)(p_i)=\hat{p}_i(p_i)=z_{n+1}(p_i).$$

Applying α^{-1} we get

$$\hat{p}_i\big(\alpha^{-1}(p_i)\big) = \alpha^{-1}\big(z_{n+1}(p_i)\big) = z_{n+1}\big(\alpha^{-1}(p_i)\big).$$

Hence $\alpha^{-1}(p_i) \in \mathrm{Fix}(z_{n+1}^{-1}\hat{p}_i) = \langle z_{n+1}, p_i\rangle$ by Lemma 4.4. By symmetry, we then have $\alpha(p_i), \alpha^{-1}(p_i) \in \langle z_{n+1}, p_i\rangle$. Hence α induces an automorphism of $\langle z_{n+1}, p_i\rangle$ fixing z_{n+1}, so $\langle z_{n+1}, \alpha(p_i)\rangle = \langle z_{n+1}, p_i\rangle$. Since $\langle z_{n+1}, p_i\rangle$ is free of rank 2, or $p_i = z_{n+1}$, a normal form argument shows that there exist integers a_i, b_i such that $\alpha(p_i) = z_{n+1}^{a_i} p_i z_{n+1}^{b_i}$.

Let $\Sigma_{g,n+1}/M$ be the quotient group in which z_{n+1} is made central. Then

$$x_i^{-1}\alpha(x_i), y_i^{-1}\alpha(y_i) \in N \subseteq NM = \langle z_{n+1}\rangle M,$$

so

$$\begin{aligned} p_i z_{n+1}^{a_i+b_i} M = z_{n+1}^{a_i} p_i z_{n+1}^{b_i} M = \alpha(p_i) M = \big[\alpha(x_i), \alpha(y_i)\big] M \\ \in \big[x_i\langle z_{n+1}\rangle, y_i\langle z_{n+1}\rangle\big] M = \{[x_i, y_i]\} M = \{p_i\} M. \end{aligned}$$

Hence $z_{n+1}^{a_i+b_i} \in M$. Since making z_{n+1} central in the free group does not make it have finite order, we see that $b_i = -a_i$. Hence we have $\alpha(p_i) = z_{n+1}^{a_i}(p_i)$, $i = 1, \dots, g$.

Now consider any $j \in \{1, \dots, n\}$. We have seen that $\alpha(\hat{z}_j) = z_{n+1}^{k_j}\hat{z}_j$ for some k_j. Hence

$$\alpha\hat{z}_j\alpha^{-1}(z_j) = \alpha(\hat{z}_j)(z_j) = z_{n+1}^{k_j}\hat{z}_j(z_j) = z_{n+1}^{k_j}(z_j).$$

Applying α^{-1} we get

$$\hat{z}_j\big(\alpha^{-1}(z_j)\big) = \alpha^{-1}\big(z_{n+1}^{k_j}(z_j)\big) = z_{n+1}^{k_j}\big(\alpha^{-1}(z_j)\big).$$

Hence $\alpha^{-1}(z_j) \in \mathrm{Fix}(z_{n+1}^{-k_j}\hat{z}_j) = \langle z_{n+1}, z_j^{\delta_{k_j,0}}\rangle$ by Lemma 4.4. However, $\alpha^{-1}(z_j)$ is a free generator of $\Sigma_{g,n+1}$, so $k_j = 0$ and, moreover, by symmetry, we then have $\alpha(z_j), \alpha^{-1}(z_j) \in \langle z_{n+1}, z_j\rangle$. As before, there exist integers c_j, d_j such that

$$\alpha(z_j) = z_{n+1}^{c_j} z_j z_{n+1}^{d_j}.$$

Since $\alpha(z_j)$ has to be a conjugate of some $z_k^{\pm 1}$, which becomes z_j modulo N, we see that $\alpha(z_j)$ has to be a conjugate of z_j, so making z_j central makes $z_{n+1}^{c_j+d_j}$ vanish. This means that $d_j = -c_j$. Hence we have $\alpha(z_j) = z_{n+1}^{c_j}(z_j)$, $j = 1, \dots, n$.

Now $\alpha(z_{n+1}) = z_{n+1}$ implies that

$$z_{n+1}^{a_1} p_1 z_{n+1}^{-a_1+a_2} p_2 \cdots z_{n+1}^{-a_{g-1}+a_g} p_g z_{n+1}^{-a_g+c_1} z_1 z_{n+1}^{-c_1+c_2} z_2 \cdots z_{n-1} z_{n+1}^{-c_{n-1}+c_n} z_n z_{n+1}^{-c_n}$$
$$= p_1 \cdots p_g z_1 \cdots z_n.$$

This can be viewed as an equation in the free group on the p_i, z_j, and it is straightforward to show that all the "internal" powers of z_{n+1} must be 0, since the only place any cancellation can occur is at $z_n z_{n+1}$. Hence

$$a_1 = a_2 = \cdots = a_g = c_1 = \cdots = c_n.$$

If b denotes the common value, $z_{n+1}^{-b}\alpha$ lies in the kernel of collapse(z_{n+1}) and fixes the p_i and the z_j. Thus we may assume that α fixes the p_i and the z_j.

Fix an $i \in \{1, \ldots, g\}$. We have $[\alpha(x_i), \alpha(y_i)] = \alpha([x_i, y_i]) = [x_i, y_i]$. A well-known algebraic result of Nielsen [19] (who attributes it to Dehn) shows that

$$\langle x_i, y_i \rangle \subseteq \langle \alpha(x_i), \alpha(y_i) \rangle = \alpha(\langle x_i, y_i \rangle);$$

see the proof of Proposition 4.12 below for a simple argument using Fox derivatives. Replacing α with α^{-1} shows that $\langle x_i, y_i \rangle = \langle \alpha(x_i), \alpha(y_i) \rangle$, and α induces an automorphism α_i of $\langle x, y \rangle$. Since α becomes trivial modulo the normal closure N of z_{n+1} in $\Sigma_{n+1,g}$, α_i becomes trivial modulo $N \cap \langle x_i, y_i \rangle$. We now consider two cases.

Case 1. $n + g = 1$. Here $n = 0$, $g = i = 1$, $\Sigma_{g,n+1} = \langle x_1, y_1 \mid \ \rangle$, $z_1 = [x_1, y_1]$ has normal closure N, and α becomes trivial modulo N and fixes z_1. Thus α induces the trivial automorphism of $\Sigma_{g,n+1}/N = \langle x_1, y_1 \mid [x_1, y_1] \rangle$. Another well-known result from the same paper of Nielsen [19] says that α must then be an inner automorphism, and since it fixes z_1, it must be a power of z_1, as desired. This result of Nielsen is quite simple to prove, once one knows yet another result of Nielsen, McCool's generalization of which was used in the preceding section, that the automorphism group of $F = \langle x, y \mid \ \rangle$ is generated by $(y, x), (x^{-1}, y)$, $(xy^{\pm 1}, y), (y^{\pm 1}x, y)$.

Since F/N is abelian, every inner automorphism of F acts trivially on F/N, and we want to show, conversely, that any automorphism which acts trivially on F/N is inner. We can work modulo the group F of inner automorphisms. Notice that

$$(x^{-1}, y) = (y, x)(y^{-1}x, y)(y, x)(xy, y)(y, x)(xy^{-1}, y),$$
$$(xy^{-1}, y) = (xy, y)^{-1},$$
$$(y^{-1}x, y) = (yx, y)^{-1},$$
$$(yx, y) = y(xy, y).$$

Thus, $(y, x), (xy, y), F$, together generate $\mathrm{Aut}(F)$, so it suffices to show that, in the subgroup G generated by (y, x) and (xy, y), any element which acts trivially on F/N is inner.

Since $(y, x)^2 = 1$, the subgroup H of G generated by the two elements

$$(xy, y) \quad \text{and} \quad (y, x)(xy, y)(y, x) = (x, yx)$$

has index at most 2 in G. Moreover, (y, x) acts with determinant -1 on F/N, while the elements of H act with determinant $+1$, so it suffices to show that any element of H that acts trivially on F/N is inner.

Now the elements

$$\beta = (xy, y)^{-1}(x, yx) = \left(xy^{-1}, yxy^{-1}\right) \text{ and } \gamma = (xy, y)\beta^{-1} = \left(xyx^{-1}, x^{-1}\right)$$

generate H and satisfy $\gamma^2 = xy\delta$, $\beta^3 = yx\delta$, where $\delta = (x^{-1}, y^{-1})$. Thus δ becomes central in $HF/F = H/(H \cap F)$ and $\delta^2 = 1$. Hence every nontrivial element of HF/F is represented by a nontrivial monoid word in

$$\gamma\beta = (xy, y) \quad \text{and} \quad \gamma\beta^2 = (x, yx),$$

or such a word followed by δ, and it is clear that such elements do not act trivially on F/N. Thus we have the desired result.

Case 2. $n + g \geq 2$. Here $N \cap \langle x_i, y_i \rangle = 1$, since $\Sigma_{g,n+1}/N$ can be expressed as a free product with amalgamation over the infinite cyclic group generated by $[x_i, y_i]$ in which one of the factors is the free group on x_i, y_i. Thus α_i is trivial. Since this holds for each i, α is trivial. Hence the kernel is precisely $\langle z_{n+1} \rangle$. ∎

4.10. *Remark.* It was shown at the beginning of the above proof that each automorphism f of the surface group $\Sigma_{g,0}$, lifts back to an automorphism $\tilde{f}$ of the free group $\Sigma_{g,1} = \langle x_1, y_1, \ldots, x_g, y_g \mid \rangle$ which fixes or inverts the relator $z_1^{-1} = [x_1, y_1] \cdots [x_g, y_g]$.

As remarked in the Introduction, we now have the following.

4.11. COROLLARY (Birman and Bers). *Suppose that* $(g, n) \neq (0, 1)$.

$$\text{(i)} \quad \mathrm{Aut}\left(\Sigma_{g,n}, \left\{[z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right\}\right)$$
$$\approx \mathrm{Aut}\left(\Sigma_{g,n+1}, \left\{[z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right\}, [z_{n+1}]^{\pm 1}\right)/\mathrm{Inn}\left(\Sigma_{g,n+1}\right)$$
$$\leq \mathscr{MC}_{g,n+1}.$$

(ii) $$\begin{aligned}\operatorname{Aut}\left(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right) &\approx \operatorname{Aut}\left(\Sigma_{g,n+1}, [z_1]^{\pm 1}, \ldots, [z_{n+1}]^{\pm 1}\right) / \operatorname{Inn}(\Sigma_{g,n+1}) \\ &= \mathscr{PMC}_{g,n+1}.\end{aligned}$$

We conclude this section with an amusing proof of a known result. The case $(g, n) = (2, 0)$ is due to Nielsen [19], the case $g = 0$ is due to Artin [1], and the general case is due to Zieschang [21, 4, Theorem V.4.11; 22, Corollary 5.2.13]. Let

$$\operatorname{End}\left(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right)$$

denote the monoid of endomorphisms of $\Sigma_{g,n}$ that carry each z_i into a conjugate of itself or its inverse.

4.12. PROPOSITION.

$$\operatorname{End}\left(\Sigma_{g,n+1}, [z_1]^{\pm 1}, \ldots, [z_{n+1}]^{\pm 1}\right) = \operatorname{Aut}\left(\Sigma_{g,n+1}, [z_1]^{\pm 1}, \ldots, [z_{n+1}]^{\pm 1}\right).$$

Proof. Since we can compose with automorphisms, we see that it suffices to show that

$$\operatorname{End}\left(\Sigma_{g,n+1}, [z_1], \ldots, [z_n], z_{n+1}\right) = \operatorname{Aut}\left(\Sigma_{g,n+1}, [z_1], \ldots, [z_n], z_{n+1}\right).$$

Let $\alpha \in \operatorname{End}(\Sigma_{g,n+1}, [z_1], \ldots, [z_n], z_{n+1})$. It suffices to show that α is an automorphism.

Let H denote the image $\alpha(\Sigma_{g,n+1})$. Nielsen showed that all finitely generated free groups are Hopfian; see, for example, [10, Proposition I.3.5] or [4, Theorem I.10.5]. Hence, it suffices to show that $H = \Sigma_{g,n+1}$.

There exist $c_1, \ldots, c_n \in \Sigma_{g,n+1}$ such that $\alpha(z_j) = c_j z_j c_j^{-1}$, $1 \le j \le n$. For $1 \le i \le g$, let X_i denote $\alpha(x_i)$ and Y_i denote $\alpha(y_i)$. Then H is generated by the X_i, the Y_i, and the $c_j z_j c_j^{-1}$.

Since α fixes z_{n+1}^{-1} we have

$$[X_1, Y_1] \cdots [X_g, Y_g] c_1 z_1 c_1^{-1} \cdots c_n z_n c_n^{-1} = [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_n. \tag{4}$$

For each w in the free generating set $\{x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_n\}$, the *left Fox derivative* $\partial/\partial w\colon \Sigma_{g,n+1} \to \mathbb{Z}[\Sigma_{g,n+1}]$ is the unique left derivation vanishing on all the free generators but w, and taking the value 1 on w; see, for example [10, Sect. I.10] or [4, Sect. V.4.7].

Suppose that $1 \le i \le g$ and that H contains $x_1, y_1, \dots, x_{i-1}, y_{i-1}$. On applying $\partial/\partial x_i$ to (4) we find that, in $\mathbb{Z}[\Sigma_{g,n+1}]$,

$$\sum_{p=1}^{g} [X_1, Y_1] \cdots [X_{p-1}, Y_{p-1}] \left(\left(1 - X_p Y_p X_p^{-1}\right) \frac{\partial X_p}{\partial x_i} \right.$$

$$\left. + X_p \left(1 - Y_p X_p^{-1} Y_p^{-1}\right) \frac{\partial Y_p}{\partial x_i} \right)$$

$$+ \sum_{q=1}^{n} [X_1, Y_1] \cdots [X_g, Y_g] c_1 z_1 c_1^{-1} \cdots c_{q-1} z_{q-1} c_{q-1}^{-1} \left(1 - c_q z_q c_q^{-1}\right) \frac{\partial c_q}{\partial x_i}$$

$$= [x_1, y_1] \cdots [x_{i-1}, y_{i-1}] \left(1 - x_i y_i x_i^{-1}\right).$$

Now let both sides of this last equation act on the element $H1$ in the right $\mathbb{Z}[\Sigma_{g,n+1}]$-module $\mathbb{Z}[H \setminus \Sigma_{g,n+1}]$, where $H \setminus \Sigma_{g,n+1}$ denotes the set of orbits for the left H-action on $\Sigma_{g,n+1}$. We then find that

$$0 = H[x_1, y_1] \cdots [x_{i-1}, y_{i-1}]\left(1 - x_i y_i x_i^{-1}\right) = H\left(1 - x_i y_i x_i^{-1}\right).$$

Hence $x_i y_i x_i^{-1} \in H$. A similar argument with $\partial/\partial y_i$ shows that

$$0 = Hx_i\left(1 - y_i x_i^{-1} y_i^{-1}\right).$$

Hence $x_i y_i x_i^{-1} y_i^{-1} x_i^{-1} \in H$. It follows that $x_i, y_i \in H$, so, by induction, H contains all the x_i and all the y_i.

Now suppose that $1 \le j \le n$, and that H contains $z_1, \dots, z_{j-1}$. On applying $\partial/\partial z_j$ to (4) we find that, in $\mathbb{Z}[\Sigma_{g,n+1}]$,

$$\sum_{p=1}^{g} [X_1, Y_1] \cdots [X_{p-1}, Y_{p-1}] \left(\left(1 - X_p Y_p X_p^{-1}\right) \frac{\partial X_p}{\partial z_j} \right.$$

$$\left. + X_p \left(1 - Y_p X_p^{-1} Y_p^{-1}\right) \frac{\partial Y_p}{\partial z_j} \right)$$

$$+ \sum_{q=1}^{n} [X_1, Y_1] \cdots [X_g, Y_g] c_1 z_1 c_1^{-1} \cdots c_{q-1} z_{q-1} c_{q-1}^{-1} \left(1 - c_q z_q c_q^{-1}\right) \frac{\partial c_q}{\partial z_j}$$

$$+ [X_1, Y_1] \cdots [X_g, Y_g] c_1 z_1 c_1^{-1} \cdots c_{j-1} z_{j-1} c_{j-1}^{-1} c_j$$

$$= [x_1, y_1] \cdots [x_n, y_n] z_1 \cdots z_{j-1}.$$

If we now let both sides of this last equation act on $H1 \in \mathbb{Z}[H \setminus \Sigma_{g,n+1}]$, we find that

$$Hc_j = H[x_1, y_1] \cdots [x_n, y_n] z_1 \cdots z_{j-1} = H.$$

Hence $c_j \in H$. Since $c_j z_j c_j^{-1} \in H$, we find that $z_j \in H$. So, by induction, H contains all the z_j, and this completes the proof. ∎

5. A DUALITY FOR THE NORMALIZER OF THE NORMAL CLOSURE OF A SURFACE RELATOR

Throughout this section we assume $(g, n) \neq (0, 1), (0, 2)$ and we study the group

$$\mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big) \approx \mathscr{P}\mathscr{M}\mathscr{C}_{g,n+1}. \tag{5}$$

The embedding of $\mathscr{P}\mathscr{M}\mathscr{C}_{g,n+1}$ in $\mathscr{M}\mathscr{C}_{g,n+1}$ gives us an image of the symmetric group S_{n+1} in the outer automorphism group of the groups in (5). Ivanov [8, Theorem 1] has shown that the outer automorphism group is S_{n+1} if $\chi_{g,n} \leq -2$. The action on $\mathscr{P}\mathscr{M}\mathscr{C}_{g,n+1}$ is quite clear, since we are just permuting the $n + 1$ punctures. However, when viewed on the isomorphic group given in (5), the action becomes rather unnatural.

Thus, for example, S_{n+1} permutes the kernels of the $n + 1$ natural maps

$$\mathscr{P}\mathscr{M}\mathscr{C}_{g,n+1} \to \mathscr{P}\mathscr{M}\mathscr{C}_{g,n},$$

each one obtained by filling in one of the $n + 1$ punctures. If we carry these maps over via the isomorphism in (5), we find that S_{n+1} permutes the kernels of $n + 1$ maps from

$$\mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big)$$

to the groups $\mathrm{Aut}(\Sigma_{g,n-1}, [z_1]^{\pm 1}, \dots, [z_{n-1}]^{\pm 1})$ and $\mathscr{P}\mathscr{M}\mathscr{C}_{g,n}$. The first n maps are the maps collapse(z_j), $j = 1, \dots, n$, and the $n + 1$st map is the quotient map corresponding to the normal subgroup $\mathrm{Inn}(\Sigma_{g,n}) \approx \Sigma_{g,n}/\mathrm{Ctr}$.

5.1. Definition. We assume $n \geq 1$ and $(g, n) \neq (0, 1), (0, 2)$. Here we can describe a particular automorphism Θ which interchanges $\mathrm{Inn}(\Sigma_{g,n})$ with the kernel of collapse(z_n).

Let

$$\Theta' \in \mathrm{Aut}\Big(\Sigma_{g,n+1}, \Big\{[z_1]^{\pm 1}, \dots, [z_{n+1}]^{\pm 1}\Big\}\Big)$$

be given by $\Theta'(z_n) = z_{n+1}$, $\Theta'(z_{n+1}) = z_{n+1}^{-1} z_n z_{n+1}$, with Θ' acting as the identity on $x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_{n-1}$. Topologically, Θ' arises from a Dehn twist along a (punctured) curve that passes through the nth and the $(n + 1)$st punctures. Then Θ' determines an element of $\mathscr{M}\mathscr{C}_{g,n+1}$ that acts on $\mathscr{P}\mathscr{M}\mathscr{C}_{g,n+1}$ by conjugation, and hence determines an automorphism Θ of

$$\mathrm{Aut}\Big(\Sigma_{g,n+1}, \Big\{[z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big\}\Big).$$

Thus Θ arises from a Dehn twist that interchanges the base point and a puncture.

Recall that the map

$$\mathscr{PMC}_{g,n+1} \to \operatorname{Aut}\left(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right)$$

acts on an element of $\mathscr{PMC}_{g,n+1}$ by first choosing a representative of the element, modulo inner automorphisms, in

$$\operatorname{Aut}\left(\Sigma_{g,n+1}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}, \{z_{n+1}\}^{\pm 1}\right)$$

and then collapsing z_{n+1}. Thus Θ acts on an

$$\alpha \in \operatorname{Aut}\left(\Sigma_{g,n+1}, \left\{[z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right\}\right)$$

by lifting it back to an

$$\alpha' \in \operatorname{Aut}\left(\Sigma_{g,n+1}, \left\{[z_1]^{\pm 1}, \ldots, [z_{n+1}]^{\pm 1}\right\}\right)$$

that fixes $\{z_{n+1}\}^{\pm 1}$, then applying Θ' to α', then composing $\Theta'(\alpha')$ with conjugation to get something that fixes $\{z_{n+1}\}^{\pm 1}$, and then collapsing z_{n+1}.

Since

$$\Theta'^2(w) = \begin{cases} w, & \text{for } w = x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_{n-1}, \\ z_{n+1}^{-1}(z_n), & \text{for } w = z_n, \\ z_{n+1}^{-1} z_n^{-1}(z_{n+1}), & \text{for } w = z_{n+1}, \end{cases}$$

we see that $z_n z_{n+1} \Theta'^2$ fixes both z_n and z_{n+1}, and acts as $z_n z_{n+1}$ on

$$x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_{n-1}.$$

Since this fixes z_{n+1}, we can collapse z_{n+1} and find that Θ^2 acts as z_n on $\Sigma_{g,n}$. That is, $\Theta^2 = z_n$.

Since $\operatorname{Aut}(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1})$ is the product of the two subgroups

$$\operatorname{Aut}\left(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_{n-1}]^{\pm 1}, \{z_n\}^{\pm 1}\right) \quad \text{and} \quad \operatorname{Inn}(\Sigma_{g,n}),$$

we can understand how Θ acts by examining what it does to each of these subgroups.

We have an embedding $\Sigma_{g,n} \to \Sigma_{g,n+1}$, $w \mapsto \tilde{w}$, that looks like the identity map on $x_1, y_1, \ldots, x_g, y_g, z_1, \ldots, z_{n-1}$. This notation will be useful when we wish to indicate where elements belong. Thus we can write

$$\tilde{z}_n = z_{n-1}^{-1} \cdots z_1^{-1} [y_g, x_g] \cdots [y_1, x_1] = z_n z_{n+1}.$$

Consider an $\alpha \in \operatorname{Aut}(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_{n-1}]^{\pm 1}, \{\tilde{z}_n\}^{\pm 1})$. In the case where α fixes $\tilde{z}_n$, α can be extended to an $\alpha' \in \operatorname{Aut}(\Sigma_{g,n+1}, \{[z_1]^{\pm 1}, \dots, [z_{n+1}]^{\pm 1}\})$, which acts in the same way as α on the free factor $\Sigma_{g,n}$ of $\Sigma_{g,n+1}$ and fixes z_n and hence z_{n+1}. Here Θ' commutes with α' so $\Theta(\alpha) = \alpha$. In the case where α inverts $\tilde{z}_n$, α can be extended to an $\alpha' \in \operatorname{Aut}(\Sigma_{g,n+1}, \{[z_1]^{\pm 1}, \dots, [z_{n+1}]^{\pm 1}\})$, which acts in the same way as α on the free factor $\Sigma_{g,n}$ of $\Sigma_{g,n+1}$ and sends z_n to $z_{n+1}^{-1} z_n^{-1} z_{n+1}$ and hence inverts z_{n+1}. Thus $z_n z_{n+1} \cdot \Theta'(\alpha')$ inverts z_{n+1}. On collapsing z_{n+1}, we see that $\Theta(\alpha) = z_n \alpha$. [This is reassuringly consistent with Θ^2 acting as z_n, since here $\Theta^2(\alpha) = z_n^2 \alpha = z_n \alpha z_n^{-1} = z_n(\alpha)$.] In summary then, Θ fixes those elements that fix z_n and left multiplies by z_n those elements that invert z_n.

We next examine the action of Θ on an inner automorphism u, where

$$u \in \left\{x_i, y_i^{-1}, z_j \mid 1 \le i \le g, 1 \le j \le n-1\right\}.$$

Thus $\tilde{u} \in \Sigma_{g,n+1}$ and Definition 4.2 shows that $\hat{\tilde{u}}$ lifts back u and fixes z_{n+1}. Notice that we are considering those u such that $\hat{\tilde{u}}$ sends z_n to $z_{n+1} u(z_n)$. Thus $\Theta'(\hat{\tilde{u}})$ sends z_{n+1} to $z_{n+1}^{-1} z_n z_{n+1} \tilde{u}(z_{n+1})$, so $\tilde{u}^{-1} z_{n+1}^{-1} z_n^{-1} z_{n+1} \cdot \Theta'(\hat{\tilde{u}})$ fixes z_{n+1}. On applying the formulas in Definitions 4.2 and the definition of Θ', and collapsing z_{n+1}, we find that $\Theta(u) = u^{-1} z_n^{-1} \hat{u}$, where $\hat{u}$ is given by Definitions 4.2 with $n-1$ in place of n. It follows that $\Theta(u^{-1}) = \hat{u}^{-1} z_n u = z_n u \hat{u}^{-1}$.

We record the action explicitly:

$$\Theta(x_i)(w) = \begin{cases} x_i^{-1} z_n^{-1} x_i z_n(w), & \text{for } w = x_1, y_1, \dots, x_{i-1}, y_{i-1}, \\ x_i^{-1} z_n^{-1}(x_i), & \text{for } w = x_i, \\ y_i x_i^{-1} z_n x_i, & \text{for } w = y_i, \\ w, & \text{for } w = x_{i+1}, y_{i+1}, \dots, y_g, z_1, \dots, z_{n-1}, \end{cases}$$

$$\Theta(y_i)(w) = \begin{cases} z_n y_i^{-1} z_n^{-1} y_i(w), & \text{for } w = x_1, y_1, \dots, x_{i-1}, y_{i-1}, \\ z_n y_i^{-1} z_n^{-1} y_i x_i z_n^{-1}, & \text{for } w = x_i, \\ z_n(y_i), & \text{for } w = y_i, \\ w, & \text{for } w = x_{i+1}, y_{i+1}, \dots, y_g, z_1, \dots, z_{n-1}, \end{cases}$$

$$\Theta(z_j)(w) = \begin{cases} z_j^{-1} z_n^{-1} z_j z_n(w), & \text{for } w = x_1, y_1, \dots, y_g, z_1, \dots, z_{j-1}, \\ z_j^{-1} z_n^{-1}(z_j), & \text{for } w = z_j, \\ w, & \text{for } w = z_{j+1}, \dots, z_{n-1}. \end{cases}$$

5.2. *Remarks*. Notice that the diagram

$$\begin{array}{ccc}
\mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big) & \overset{\Theta}{\to} & \mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big) \\
{\scriptstyle \text{mod inners}} \downarrow & & {\scriptstyle \text{collapse}(z_n)} \downarrow \\
\mathscr{PMC}_{g,n-1} & \overset{\approx}{\to} & \mathrm{Aut}\Big(\Sigma_{g,n-1}, [z_1]^{\pm 1}, \dots, [z_{n-1}]^{\pm 1}\Big)
\end{array}$$

commutes, since each inner automorphism ends up at the identity automorphism along both routes, while both the z_n-fixing and the z_n-inverting automorphisms end up at the induced map along both routes.

Let K_n denote the kernel of the map

$$\mathrm{collapse}(z_n)\colon \mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big) \to \mathrm{Aut}\Big(\Sigma_{g,n-1}, [z_1]^{\pm 1}, \dots, [z_{n-1}]^{\pm 1}\Big)$$

and let L_n denote the kernel of the natural map $\mathscr{PMC}_{g,n} \to \mathscr{PMC}_{g,n-1}$. Then Θ interchanges K_n and $\mathrm{Inn}(\Sigma_{g,n})$, and we have a commuting diagram that is reflected about the main diagonal by Θ:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathrm{Inn}(\Sigma_{g,n}) \cap K_n & \to & K_n & \to & L_n & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \Sigma_{g,n}/\mathrm{Ctr} & \to & \mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big) & \to & \mathscr{PMC}_{g,n} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \Sigma_{g,n-1}/\mathrm{Ctr} & \to & \mathrm{Aut}\Big(\Sigma_{g,n-1}, [z_1]^{\pm 1}, \dots, [z_{n-1}]^{\pm 1}\Big) & \to & \mathscr{PMC}_{g,n-1} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

5.3. COROLLARY. *Suppose that* $n \geq 1$ *and that* $\Sigma_{g,n}$ *is nonabelian. Then the kernel of the map*

$$\mathrm{collapse}(z_n)\colon \mathrm{Aut}\Big(\Sigma_{g,n}, [z_1]^{\pm 1}, \dots, [z_n]^{\pm 1}\Big) \to \mathrm{Aut}\Big(\Sigma_{g,n-1}, [z_1]^{\pm 1}, \dots, [z_{n-1}]^{\pm 1}\Big)$$

is freely generated by $\{\Theta(u) \mid u = x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_{n-1}\}$.

The elements of this free generating set were given explicitly in Definition 5.1.

5.4. PROPOSITION. *Suppose that* $n \geq 1$ *and that* $\Sigma_{g,n}$ *is nonabelian, that is,* $2g + n \geq 3$. *Let* N *denote the normal closure of* z_n *in* $\Sigma_{g,n}$.

(i) *If* $\Sigma_{g,n-1}$ *is abelian, that is,* $2g + n = 3$, *then* $\mathrm{Inn}(\Sigma_{g,n}) = K_n$.

(ii) *If* $\Sigma_{g,n-1}$ *is nonabelian, that is,* $2g + n \geq 4$, *then* $\mathrm{Inn}(\Sigma_{g,n}) \cap K_n = N$ *and* $\mathrm{Inn}(\Sigma_{g,n})K_n/N \approx \Sigma_{g,n-1} \times \Sigma_{g,n-1}$.

Proof. It is clear from the commutative diagram that $\Sigma_{g,n} \cap K_n$ is the kernel of the map from $\Sigma_{g,n}$ to $\Sigma_{g,n-1}/\mathrm{Ctr}$. Since Θ reflects the diagram about the diagonal, it induces Birman's isomorphism $\Sigma_{g,n-1}/\mathrm{Ctr} \approx L_n$.

Consider first the case where $2g + n = 3$. Here $\Sigma_{g,n-1}/\mathrm{Ctr}$ is trivial, so $\Sigma_{g,n} = K_n$. Thus we may assume that $2g + n \geq 4$. Here $\Sigma_{g,n-1}$ has trivial center, so $\Sigma_{g,n} \cap K_n = N$. ∎

5.5. *Remarks.* (1) If $n \geq 1$, $g \geq 1$, and $2g + n \geq 4$, then

$$\mathrm{Inn}(\Sigma_{g,n}) \cap K_n = \left[\mathrm{Inn}(\Sigma_{g,n}), K_n\right] = N.$$

To see this, notice that since we are dealing with normal subgroups, we have $[\mathrm{Inn}(\Sigma_{g,n}), K_n] \subseteq \mathrm{Inn}(\Sigma_{g,n}) \cap K_n = N$, and it remains to show the reverse inclusion. Since $[\mathrm{Inn}(\Sigma_{g,n}), K_n]$ is a normal subgroup of $\mathrm{Inn}(\Sigma_{g,n})$, it remains to show that z_n lies in $[\mathrm{Inn}(\Sigma_{g,n}), K_n\}$, and this holds since $[\mathrm{Inn}(\Sigma_{g,n}), K_n]$ contains

$$\begin{aligned} \left[y_1^{-1}, \Theta(x_1)\right] &= \left[y_1^{-1}, x_1^{-1} z_n^{-1} \hat{x}_1\right] \\ &= y_1^{-1} \cdot x_1^{-1} z_n^{-1} \hat{x}_1(y_1) \\ &= z_n. \end{aligned}$$

(2) There are many different maps that flip the diagram about the diagonal. We have found it convenient to leave the centralizer of z_n fixed, but this can be exchanged for other properties, such as having order 2, since, for any z_n-inverting α in

$$\mathrm{Aut}\left(\Sigma_{g,n}, [z_1]^{\pm 1}, \ldots, [z_n]^{\pm 1}\right)$$

such that $\alpha^2 = 1$, we have $(\Theta\alpha)^2 = 1$.

(3) In the above setting we are free to permute $z_1, \ldots, z_{n-1}$, so Θ extends to

$$\mathrm{Aut}\left(\Sigma_{g,n}, \left\{[z_1]^{\pm 1}, \ldots, [z_{n-1}]^{\pm 1}\right\}, \{z_n\}^{\pm 1}\right),$$

and, here, in place of pure mapping class groups we must consider subgroups of the mapping class group that permute $n - 1$ of the punctures.

(4) The case $n = 1$, $g \geq 1$, deserves special mention. Here $A = \mathrm{Aut}(\Sigma_{g,1}, [z_1]^{\pm 1})$ is the set of automorphisms of the free group of rank $2g$, which send the element $z_1^{-1} = [x_1, y_1] \cdots [x_g, y_g]$ to a conjugate of itself or its inverse, so A is the normalizer of the normal closure N of the surface relator z_1 in the free group $\Sigma_{g,1}$; see, for example, [10, Proposition II.5.8]. By Corollary 4.11, A is isomorphic to

$$\mathscr{PMC}_{g,2} = \mathrm{Aut}\Big(\Sigma_{g,2}, [z_1]^{\pm 1}, [z_2]^{\pm 1}\Big) / \mathrm{Inn}\big(\Sigma_{g,2}\big),$$

which is a (normal) subgroup of index 2 in

$$\mathscr{MC}_{g,2} = \mathrm{Aut}\Big(\Sigma_{g,2}, \big\{[z_1]^{\pm 1}, [z_2]^{\pm 1}\big\}\Big) / \mathrm{Inn}\big(\Sigma_{g,2}\big).$$

For $g = 1$, by [6], the outer automorphism group of A is trivial. For $g \geq 2$, by [8], the other automorphism group of A has order 2 and is generated by the image of Θ.

ACKNOWLEDGMENTS

We thank Jim McCool for providing us with his unpublished manuscripts, which were fundamental in the results of Section 3. We also thank Colin Maclachlan for his comments on an earlier version of this article. This work was carried out while the second author held a grant from the MEC (Spain) at the Centre de Recerca Matemàtica, of the Institut d'Estudis Catalans, and he is grateful for the hospitality he received.

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