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# Stiffness Matrix of Nonlinear FEM Equilibrium Equation

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**Abstract**

For structural stability analysis, either tangent stiffness matrix or secant stiffness matrix is used depending on different situations. In this study, the general mathematic relationship between structural secant and tangent stiffness matrices is developed in detail based on Taylor series expression of the total potential energy. The result is important to the analysis of structural nonlinear stability. Moreover, it can be used not only in finite element method but also Rayleigh-Ritz method, Galerkin method, etc.



*Key words*: FEM equilibrium equation; geometric nonlinearity; tangent stiffness matrix; secant stiffness matrix

## 1. Introduction

Structural geometric nonlinear FEM equilibrium equation can be solved with total amount or incremental method. Total amount method is used to study structural characteristic in some specific condition, also directly strives for structural total deformation and stress under known load and restraint (Zienkiewicz and Taylor 1989). When solving total amount form FEM equilibrium equations by direct iteration method, coefficient matrix of equations is secant stiffness matrix. But in practical applications, in order to get static and kinematics parameters in a series of discrete time points during the whole process of loading, such as deformation and stress, and ensure good convergence of the solution, it usually need to adopt incremental method(Zienkiewicz and Taylor 1991; Clough 2004). In order to get the finite element equations with incremental method, it need linearize nonlinear equations, the coefficient matrix for linear balance equations is tangent stiffness matrix (Belytschko et al. 2000). In this study, based on the finite element method and Taylor series expression of the total potential energy, the general mathematic

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relationship between structural secant and tangent stiffness matrices is developed for the structure, which is specified to any freedom and any parameter variables (such as arbitrary load or initial defects).

## 2. The relationship between structural secant and tangent stiffness matrices

Suppose that $d_i^0\left(i=1,2,\cdots,n\right)$ is the displacement of a certain equilibrium node O, $d_i\left(i=1,2,\cdots,n\right)$ is the displacement of another arbitrary point nearby the node O and on the equilibrium path, and $q_i=d_i-d_i^0\ \left(i=1,2,\cdots,n\right)$ is the displacement increment; $v_p\left(p=1,2,\cdots n\right)$ is the loading parameter or initial flaw parameter. The total potential energy $\Pi\left(q_i,v_p\right)$ of conservative system on equilibrium node O will be spread out into a Taylor series:

$$\Pi\left(q_i,v_p\right)=\Pi^0+\Pi_i^0q_i+\frac{1}{2!}\Pi_{ij}^0q_iq_j+\frac{1}{3!}\Pi_{ijk}^0q_iq_jq_k+\frac{1}{4!}\Pi_{ijkl}^0q_iq_jq_kq_l+h.o.t \tag{1}$$

or in a compact notation, $\Pi_i^0=\Pi_i^0\left(v_p\right)=\left.\frac{\partial\Pi}{\partial q_i}\right|_{q=0}$ , $\cdots$ .where， $j,k,l=1,2,\cdots,n$ ， $h.o.t.$ is the neglected terms which are higher than second order.

According to the principle of resident potential energy, the equilibrium equation for the general expression is $\Pi_i=\frac{\partial\Pi}{\partial d_i}=\frac{\partial\Pi}{\partial q_i}=0$ ， $\left(i=1,2,\cdots,n\right)$， and conservative system of total potential energy with partial derivative provides interchangeability (Chessa et al. 2002), therefore

$$\begin{aligned}\Pi_i=\frac{\partial\Pi}{\partial q_i}=\Pi_i^0+\Pi_{ij}^0q_j+\frac{1}{2!}\Pi_{ijk}^0q_jq_k+\frac{1}{3!}\Pi_{ijkl}^0q_jq_kq_l\quad\left(i=1,2,\cdots,n\right)\\+\frac{1}{4!}\Pi_{ijklm}^0q_jq_kq_lq_m+h.o.t.=0\end{aligned} \tag{2}$$

A total content balance equation (omitting higher order items) is written as

$$(\Pi_{ij}^0+\frac{1}{2!}\Pi_{ijk}^0q_k+\frac{1}{3!}\Pi_{ijkl}^0q_kq_l+\frac{1}{4!}\Pi_{ijklm}^0q_kq_lq_m)q_j=-\Pi_i^0\ \left(i=1,2,\cdots,n\right) \tag{3}$$

so

$$\begin{aligned}K_{s,ij}&=\Pi_{ij}^0+\frac{1}{2!}\Pi_{ijk}^0q_k+\frac{1}{3!}\Pi_{ijkl}^0q_kq_l+\frac{1}{4!}\Pi_{ijklm}^0q_kq_lq_m\ \left(i,j=1,2,\cdots,n\right)\\&=K_{0,ij}+K_{1,ij}+K_{2,ij}+K_{3,ij}\end{aligned} \tag{4}$$

where，

$$K_{0,ij}=\Pi_{ij}^0 \tag{5}$$

$$K_{1,ij}=\frac{1}{2!}\Pi_{ijk}^0q_k \tag{6}$$

$$K_{2,ij}=\frac{1}{3!}\Pi_{ijkl}^0q_kq_l \tag{7}$$

$$K_{3,ij}=\frac{1}{4!}\Pi_{ijklm}^0q_kq_lq_m \tag{8}$$

The structure secant stiffness matrix may be further decomposed as follows:

$$K_s=K_0+K_1+K_2+K_3 \tag{9}$$

where $K_0$ is linear tangent stiffness matrix , $K_1$， $K_2$， $K_3$ are nonlinear tangent stiffness matrices. Applying Eq. (2), one calculates

$$\mathrm{d}\Pi_i^0+\Pi_{ij}^0\mathrm{d}q_j+\mathrm{d}\Pi_{ij}^0q_j+\Pi_{ijk}^0q_k\mathrm{d}q_j+\frac{1}{2!}\mathrm{d}\Pi_{ijk}^0q_jq_k+\frac{1}{2!}\Pi_{ijk}^0q_kq_l\mathrm{d}q_j+\frac{1}{3!}\mathrm{d}\Pi_{ijkl}^0q_jq_kq_l+\frac{1}{3!}\Pi_{ijklm}^0q_kq_lq_m\mathrm{d}q_j+h.o.t.=0 \quad (i=1,2,\cdots,n) \tag{10}$$

Eq.(10) is the principle of resident potential energy, the higher order terms are neglected, increment equilibrium equation is written as

$$\left(\Pi_{ij}^0+\Pi_{ijk}^0q_k+\frac{1}{2!}\Pi_{ijk}^0q_kq_l+\frac{1}{3!}\Pi_{ijklm}^0q_kq_lq_m\right)\mathrm{d}q_j=-\mathrm{d}\Pi_i^0-\mathrm{d}\Pi_{ij}^0q_j-\frac{1}{2!}\mathrm{d}\Pi_{ijk}^0q_jq_k-\frac{1}{3!}\mathrm{d}\Pi_{ijkl}^0q_jq_kq_l \tag{11}$$

so

$$K_{t,ij}=\Pi_{ij}^0+\Pi_{ijk}^0q_k+\frac{1}{2!}\Pi_{ijkl}^0q_kq_l+\frac{1}{3!}\Pi_{ijklm}^0q_kq_lq_m \quad (i,j=1,2,\cdots,n) = K_{0,ij}+2K_{1,ij}+3K_{2,ij}+4K_{3,ij} \tag{12}$$

Similarly, it is easy to achieve the following representation of tangent stiffness matrix:

$$K_t=K_0+2K_1+3K_2+4K_3 \tag{13}$$

Eq.(9) and (13) indicate the general mathematic relationship between structural secant and tangent stiffness matrices. According to the above formula, secant and tangent stiffness matrices expressions are approximate when omitting higher order forms.

## 3. Numerical example

This example, Fig. 1, shows the behavior of a supported arch subjected to central loads. The analysis starts from the initial position. The initial defects $\varepsilon$ expresses node A off center position, BD and DC lengths are:

$$b_{02}=\sqrt{L_{02}^2-h_0^2}=b_0(1-\varepsilon) \tag{14}$$

$$b_{01}=\sqrt{L_{01}^2-h_0^2}=b_0(1+\varepsilon) \tag{15}$$

where, $b_0$ is the half of the length of BC. $u_1$ and $u_2$ are horizontal and vertical displacements of node A, respectively. The system's total potential energy can be obtained

$$\Pi(u_1,u_2,\varepsilon)=\frac{EA}{2}\cdot\frac{\left(L_{01}-\sqrt{\left[b_0(1+\varepsilon)+u_1\right]^2+(h_0-u_2)^2}\right)^2}{L_{01}}+\frac{EA}{2}\cdot\frac{\left(L_{02}-\sqrt{\left[b_0(1-\varepsilon)-u_1\right]^2+(h_0-u_2)^2}\right)^2}{L_{02}}-P_1u_1-P_2u_2 \tag{16}$$

Using symbols operation function of mathematical software MathCad, It is easy to achieve Taylor series forms of balance equation, the terms higher than second order are neglected, four elements of secant and tangent stiffness matrices are as follows:

$$K_{s,ij}=K_{0,ij}+K_{1,ij}+K_{2,ij} \quad (i,j=1,2) \tag{17}$$

$$K_{0,11} = EAb_0^2\left(\frac{(1+\varepsilon)^2}{L_{01}^3}+\frac{(1-\varepsilon)^2}{L_{02}^3}\right) \tag{18}$$

$$\begin{aligned}K_{1,\,11} &= \frac{EAb_0}{2}\left(\frac{3(1+\varepsilon)}{L_{01}^3}-\frac{3(1-\varepsilon)}{L_{02}^3}-\frac{3b_0^2(1+\varepsilon)^3}{L_{01}^5}+\frac{3b_0^2(1-\varepsilon)^3}{L_{02}^5}\right)u_1\\ &+\frac{EAh_0}{2}\left(-\frac{1}{L_{01}^3}-\frac{1}{L_{02}^3}+\frac{3b_0^2(1+\varepsilon)^2}{L_{01}^5}+\frac{3b_0^2(1-\varepsilon)^2}{L_{02}^5}\right)u_2\end{aligned} \tag{19}$$

$$\begin{aligned}K_{2,\,11} &= EA\left(\frac{1}{2L_{01}^3}+\frac{1}{2L_{02}^3}-\frac{3b_0^2(1+\varepsilon)^2}{L_{01}^5}-\frac{3b_0^2(1-\varepsilon)^2}{L_{02}^5}+\frac{5b_0^2(1+\varepsilon)^4}{2L_{01}^7}-\frac{5b_0^2(1-\varepsilon)^4}{2L_{02}^7}\right)u_1^2\\ &+EAh_0b_0\left(\frac{3(1+\varepsilon)}{L_{01}^5}-\frac{3(1-\varepsilon)}{L_{02}^5}-\frac{5b_0^2(1+\varepsilon)^3}{L_{01}^7}+\frac{5b_0^2(1-\varepsilon)^3}{L_{02}^7}\right)u_1u_2\\ &+EA\left(\frac{1}{6L_{01}^3}+\frac{1}{6L_{02}^3}-\frac{h_0^2+b_0^2(1+\varepsilon)^2}{2L_{01}^5}-\frac{h_0^2+b_0^2(1-\varepsilon)^2}{2L_{02}^5}+\frac{5h_0^2b_0^2(1+\varepsilon)^2}{2L_{01}^7}+\frac{5h_0^2b_0^2(1-\varepsilon)^2}{2L_{02}^7}\right)u_2^2\end{aligned} \tag{20}$$

$$K_{0,\,12} = K_{0,\,21} = EAh_0b_0\left(-\frac{(1+\varepsilon)}{L_{01}^3}+\frac{(1-\varepsilon)}{L_{02}^3}\right) \tag{21}$$

$$\begin{aligned}K_{1,\,12} = K_{1,\,21} &= \frac{EAh_0}{2}\left(-\frac{1}{L_{01}^3}-\frac{1}{L_{02}^3}+\frac{3b_0^2(1+\varepsilon)^2}{L_{01}^5}+\frac{3b_0^2(1-\varepsilon)^2}{L_{02}^5}\right)u_1\\ &+\frac{EAb_0}{2}\left(\frac{(1+\varepsilon)}{L_{01}^3}-\frac{(1-\varepsilon)}{L_{02}^3}-\frac{3h_0^2(1+\varepsilon)}{L_{01}^5}+\frac{3h_0^2(1-\varepsilon)}{L_{02}^5}\right)u_2\end{aligned} \tag{22}$$

$$\begin{aligned}K_{2,\,12} = K_{2,\,21} &= \frac{EA}{2}h_0b_0\left(\frac{3(1+\varepsilon)}{L_{01}^5}-\frac{3(1-\varepsilon)}{L_{02}^5}-\frac{5b_0^2(1+\varepsilon)^3}{L_{01}^7}+\frac{5b_0^2(1-\varepsilon)^3}{L_{02}^7}\right)u_1^2\\ &+EA\left(\frac{1}{3L_{01}^3}+\frac{1}{3L_{02}^3}-\frac{h_0^2+b_0^2(1+\varepsilon)^2}{L_{01}^5}-\frac{h_0^2+b_0^2(1-\varepsilon)^2}{L_{02}^5}+\frac{5h_0^2b_0^2(1+\varepsilon)^2}{L_{01}^7}+\frac{5h_0^2b_0^2(1-\varepsilon)^2}{L_{02}^7}\right)u_1u_2\\ &+\frac{EA}{2}h_0b_0\left(\frac{3(1+\varepsilon)}{L_{01}^5}-\frac{3(1-\varepsilon)}{L_{02}^5}+\frac{5h_0^2(1+\varepsilon)}{2L_{01}^7}-\frac{5h_0^2(1-\varepsilon)}{2L_{02}^7}\right)u_2^2\end{aligned} \tag{23}$$

$$K_{0,22} = EAh_0^2\left(\frac{1}{L_{01}^3}+\frac{1}{L_{02}^3}\right) \tag{24}$$

$$\begin{aligned}K_{1,\,12} = K_{1,\,21} &= \frac{EAb_0}{2}\left(\frac{(1+\varepsilon)}{L_{01}^3}-\frac{(1-\varepsilon)}{L_{02}^3}-\frac{3h_0^2(1+\varepsilon)}{L_{01}^5}+\frac{3h_0^2(1-\varepsilon)}{L_{02}^5}\right)u_1\\ &+\frac{3EAh_0}{2}\left(-\frac{1}{L_{01}^3}-\frac{1}{L_{02}^3}+\frac{h_0^2}{L_{01}^5}+\frac{h_0^2}{L_{02}^5}\right)u_2\end{aligned} \tag{25}$$

$$\begin{aligned} K_{2,\,22} = EA&\left(\frac{1}{6L_{01}^3}+\frac{1}{6L_{02}^3}-\frac{h_0^2+b_0^2\left(1+\varepsilon\right)^2}{2L_{01}^5}-\frac{h_0^2+b_0^2\left(1-\varepsilon\right)^2}{2L_{02}^5}+\frac{5h_0^2b_0^2\left(1+\varepsilon\right)^2}{2L_{01}^7}+\frac{5h_0^2b_0^2\left(1-\varepsilon\right)^2}{2L_{02}^7}\right)u_1^2 \\ &+EA\left(\frac{3b_0\left(1+\varepsilon\right)}{L_{01}^5}-\frac{3b_0\left(1-\varepsilon\right)}{L_{02}^5}-\frac{5h_0^3b_0\left(1+\varepsilon\right)}{L_{01}^7}+\frac{5h_0^3b_0\left(1-\varepsilon\right)}{L_{02}^7}\right)u_1u_2 \\ &+EA\left(\frac{1}{2L_{01}^3}+\frac{1}{2L_{02}^3}-\frac{3h_0^2}{L_{01}^5}-\frac{3h_0^2}{L_{02}^5}+\frac{5h_0^4}{2L_{01}^7}-\frac{5h_0^4}{2L_{02}^7}\right)u_2^2 \end{aligned} \tag{26}$$

$$K_{t,\,11} = K_{0,\,11} + 2K_{1,\,11} + 3K_{2,\,11} \tag{27}$$

$$K_{t,\,12} = K_{t,\,21} = K_{0,\,12} + 2K_{1,\,12} + 3K_{2,\,12} \tag{28}$$

$$K_{t,\,22} = K_{0,\,22} + 2K_{1,\,22} + 3K_{2,\,22} \tag{29}$$

Obviously, the mathematical relationship meets the deduced conclusion of this study.

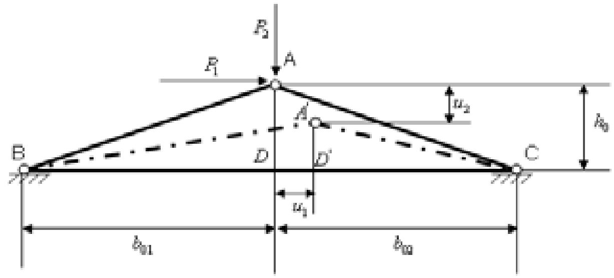


Figure1: The diagram of the unsymmetrical simply supported arch

## 4. Conclusions

The conclusion is deduced in the level structure, but it is suitable in the overall system, therefore may apply conveniently similarly in the complex engineering structure such as plate and shell and so on, so it has certain application value. Considering arbitrary initial defect by the parameter variable's form, it meets real engineering structure in this study. This general conclusion may apply in both discrete system and discretization continuous system.

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