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Note

On the solutions of a class of difference equations ☆

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Abstract

In this note we investigate the solutions of a class of difference equations and prove that Conjectures 4.8.2, 4.8.3, 5.4.6 and 6.10.3 proposed by M. Kulenovic and G. Ladas in [M. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures, Chapman & Hall/CRC Press, 2002] are true.

Keywords: Difference equation; Nontrivial positive solution

M. Kulenovic and G. Ladas in [1] proposed the following four conjectures:

Conjecture 4.8.2. *Show that the equation*

$$y_{n+2} = \frac{y_n}{1 + y_{n+1}}, \qquad n = 1, 2, \ldots,$$

has a solution which converges to zero.

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Conjecture 4.8.3. *Show that the equation*

$$y_{n+2} = a + \frac{y_n}{y_{n+1}}, \quad a \in (0, +\infty),\ n = 1, 2, \ldots,$$

possesses a solution $\{y_n\}_{n=1}^{\infty}$ *which remains above the equilibrium for all positive integer* n.

Conjecture 5.4.6. *Show that the equation*

$$y_{n+2} = (1 + y_n)/y_{n+1}, \quad n = 1, 2, \ldots,$$

has a nontrivial positive solution which decreases monotonically to the equilibrium of the equation.

Conjecture 6.10.3. *Assume* $a \in (0, +\infty)$. *Show that the equation*

$$y_{n+2} = (a + y_n)/(1 + y_{n+1}), \quad n = 1, 2, \ldots,$$

has a positive and monotonically decreasing solution.

In this note we shall show that the above four conjectures are true. To do this we need the following definition and some notations.

Definition 1. Let A be a closed subset of $(-\infty, +\infty) \times (-\infty, +\infty)$. A is called a simple closed arc if there exists a homeomorphism f from $[0, 1]$ to A. $E(A) = \{f(0), f(1)\}$ is called the set of end points of A.

Write $X = [0, +\infty)$ (or $(0, +\infty)$). Let $f : X \times X \to X$ be continuous such that the equation $x = f(x, x)$ has the only solution $p \in X$ and $g_i : [p, +\infty) \to X$ $(i = 1, 2)$ be continuous and satisfy:

(1) $p = g_i(p)$ $(i = 1, 2)$ and $g_1(x) \geqslant p$;
(2) $x > g_1(x) > g_2(x) > 0$ for all $x > p$.

Put

$$D_1 = \{(x, y)\colon\ g_1(x) \leqslant y \leqslant x\},$$
$$D_2 = \{(x, y)\colon\ g_2(x) \leqslant y \leqslant x\},$$
$$P = \{(x, y)\colon\ y = g_1(x)\},$$
$$Q = \{(x, y)\colon\ y = x,\ x \geqslant p\},$$
$$L = \{(x, y)\colon\ y = g_2(x)\}.$$

The following theorem is the key theorem for the proofs of the above four conjectures.

Theorem 1. *If* $F(x, y) = (y, f(x, y))$ *is a homeomorphism from* D_1 *to* D_2 *satisfying* $F(P) = Q$ *and* $F(Q) = L$, *then the equation*

$$y_{n+2} = f(y_n, y_{n+1}), \quad n = 1, 2, \ldots,$$

has a nontrivial positive solution $\{y_n\}_{n=1}^{\infty}$ *which converges to* p *such that* $(y_n, y_{n+1}) \in D_1 - P \cup Q$ *for all positive integer* n.

Proof. At first we have the following claim.

Claim 1. *Let* $A \subset D_1$ *be a simple closed arc with* $(p, p) \notin A$ *and* $E(A) = \{a, b\}$ *with* $a \in P$ *and* $b \in Q$. *Then there exists a simple closed arc* $A_1 \subset A$ *such that* $F(A_1) \subset D_1$ *is also a simple closed arc and* $F(E(A_1)) \cap P \neq \emptyset$ *and* $F(E(A_1)) \cap Q \neq \emptyset$.

Proof of Claim 1. Let $g : [0, 1] \to A$ be a homeomorphism with $a = g(0)$ and $b = g(1)$. Write $h = F \circ g$. Then $F(A) = h([0, 1]) \subset D_2$ is also a simple closed arc with $h(0) = F(a) \in Q$ and $h(1) = F(b) \in L$. Since $F(A)$ is a connected closed set, we see that $F(A) \cap P \neq \emptyset$ and $F(A) \cap P$ is also a closed set. Let

$$t = \min\big\{x\colon\ h(x) \in P\big\}.$$

Then $h(t) \in P$ and $h([0, t]) \subset D_1$. Let $A_1 = g([0, t])$ and $u : [0, 1] \to [0, t]$ be a homeomorphism with $0 = u(0)$ and $t = u(1)$, then $A_1 = g \circ u([0, 1]) \subset A$ is a simple closed arc, $F(A_1) = h \circ u([0, 1]) \subset D_1$ and $h \circ u(0) \in Q$ and $h \circ u(1) \in P$. Claim 1 is proven. □

Let

$$B_1 = \big\{(x, y)\colon\ x = p + 1,\ g_1(p + 1) \leqslant y \leqslant p + 1\big\}.$$

It follows from Claim 1 that for any positive integer n, there exist simple closed arcs $B_i \subset D_1$ $(i = 1, 2, \ldots, n)$ and $C_i \subset B_i$ $(i = 1, 2, \ldots, n - 1)$ such that $F(C_{i-1}) = B_i$ $(i = 2, \ldots, n)$. Write $S_n = F^{-(n-1)}(B_n)$, then $S_n \neq \emptyset$ is a closed subset of B_1 and $S_1 = B_1 \supset S_2 \supset S_3 \supset \cdots \supset S_n \supset \cdots$. Let $D = \bigcap_{n=1}^{\infty} S_n$, then $D \neq \emptyset$. Choose $(y_1, y_2) \in D$, we can show that for all positive integer n,

$$F^n(y_1, y_2) = (y_{n+1}, y_{n+2}) \in B_{n+1} \subset D_1 \quad \text{and} \quad p < y_{n+1} < y_n.$$

Let $\lim_{n\to\infty} y_n = e$, then $e = f(e, e)$. Hence $e = p$ since $x = f(x, x)$ has the only solution $p \in X$. □

Proof of Conjecture 4.8.2. Let $p = 0$ and $f(x, y) = x/(1 + y)$ $(x \geqslant 0$ and $y \geqslant 0)$. Write

$$D_1 = \left\{(x, y)\colon\ \sqrt{x + \frac{1}{4}} - \frac{1}{2} \leqslant y \leqslant x\right\},$$

$$D_2 = \left\{(x, y)\colon\ \frac{x}{1 + x} \leqslant y \leqslant x\right\},$$

$$P = \left\{(x, y)\colon\ y = \sqrt{x + \frac{1}{4}} - \frac{1}{2},\ x \geqslant p\right\},$$

$$Q = \big\{(x, y)\colon\ y = x,\ x \geqslant p\big\},$$

$$L = \left\{(x, y)\colon\ y = \frac{x}{1 + x},\ x \geqslant p\right\}.$$

It is easy to see that $F(x, y) = (y, x/(1 + y))$ is a homeomorphism from D_1 to D_2 satisfying $F(P) = Q$ and $F(Q) = L$. Then it follows from Theorem 1 that the equation $y_{n+2} = y_n/(1 + y_{n+1})$ has a nontrivial positive solution $\{y_n\}_{n=1}^{\infty}$ which converges to $p = 0$. $\square$

Proof of Conjecture 4.8.3. Let $p = a + 1$ and $f(x, y) = a + x/y$ $(x > 0$ and $y > 0)$. Write

$$D_1 = \left\{(x, y)\colon \sqrt{x + \frac{a^2}{4}} + \frac{a}{2} \leqslant y \leqslant x\right\},$$
$$D_2 = \left\{(x, y)\colon p \leqslant y \leqslant x\right\},$$
$$P = \left\{(x, y)\colon y = \sqrt{x + \frac{a^2}{4}} + \frac{a}{2},\ x \geqslant p\right\},$$
$$Q = \left\{(x, y)\colon y = x,\ x \geqslant p\right\},$$
$$L = \left\{(x, y)\colon y = p,\ x \geqslant p\right\}.$$

It is easy to see that $F(x, y) = (y, a + x/y)$ is a homeomorphism from D_1 to D_2 satisfying $F(P) = Q$ and $F(Q) = L$. Then it follows from Theorem 1 that the equation $y_{n+2} = a + y_n/y_{n+1}$ has a nontrivial positive solution $\{y_n\}_{n=1}^{\infty}$ which decreases monotonically to p. $\square$

Proof of Conjecture 5.4.6. Let $p = (\sqrt{5} + 1)/2$ and $f(x, y) = (1 + x)/y$ $(x > 0$ and $y > 0)$. Write

$$D_1 = \left\{(x, y)\colon \sqrt{x + 1} \leqslant y \leqslant x\right\},$$
$$D_2 = \left\{(x, y)\colon \frac{x + 1}{x} \leqslant y \leqslant x\right\},$$
$$P = \left\{(x, y)\colon y = \sqrt{x + 1},\ x \geqslant p\right\},$$
$$Q = \left\{(x, y)\colon y = x,\ x \geqslant p\right\},$$
$$L = \left\{(x, y)\colon y = \frac{1 + x}{x},\ x \geqslant p\right\}.$$

It is easy to see that $F(x, y) = (y, (1 + x)/y))$ is a homeomorphism from D_1 to D_2 satisfying $F(P) = Q$ and $F(Q) = L$. Then it follows from Theorem 1 that the equation $y_{n+2} = (1 + y_n)/y_{n+1}$ has a nontrivial positive solution which decreases monotonically to p. $\square$

Proof of Conjecture 6.10.3. Let $p = \sqrt{a}$ and $f(x, y) = (a + x)/(1 + y)$ $(x \geqslant 0$ and $y \geqslant 0)$. Write

$$D_1 = \left\{(x, y)\colon \sqrt{x + a + \frac{1}{4}} - \frac{1}{2} \leqslant y \leqslant x\right\},$$
$$D_2 = \left\{(x, y)\colon \frac{x + a}{x + 1} \leqslant y \leqslant x\right\},$$

$$P = \left\{(x, y)\colon\ y = \sqrt{x + a + \frac{1}{4}} - \frac{1}{2},\ x \geqslant p\right\},$$
$$Q = \left\{(x, y)\colon\ y = x,\ x \geqslant p\right\},$$
$$L = \left\{(x, y)\colon\ y = \frac{x + a}{x + 1},\ x \geqslant p\right\}.$$

It is easy to see that $F(x, y) = (y, (a + x)/(1 + y))$ is a homeomorphism from D_1 to D_2 satisfying $F(P) = Q$ and $F(Q) = L$. Then it follows from Theorem 1 that the equation $y_{n+2} = (a + y_n)/(1 + y_{n+1})$ has a nontrivial positive solution which decreases monotonically to p. □

Remark 1. After we submitted this paper, we became aware that some results of the paper have already been published by J.T. Hoag and C.M. Kent in the last two years (see [2–4], pointed out to us by the referee(s)).

Remark 2. After we submitted this paper, using arguments different to ones developed in the proof of Theorem 1, we showed that the equation

$$y_{n+k+1} = a + \frac{y_n}{y_{n+k}}, \quad a \in (0, +\infty),\ n = 1, 2, \ldots,$$

which is the more general case of the equation $y_{n+2} = a + \frac{y_n}{y_{n+1}}$, has also nontrivial positive solutions which decrease monotonically to the equilibrium $a + 1$ of the equation, where $k \in \{1, 2, \ldots\}$. This new result will be published in J. Differ. Equations Appl. soon.

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