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First integrals and normal forms for germs of analytic vector fields

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Abstract

For a germ of analytic vector fields, the existence of first integrals, resonance and the convergence of normalization transforming the vector field to a normal form are closely related. In this paper we first provide a link between the number of first integrals and the resonant relations for a quasi-periodic vector field, which generalizes one of the Poincaré's classical results [H. Poincaré, Sur l'intégration des équations différentielles du premier order et du premier degré I and II, Rend. Circ. Mat. Palermo 5 (1891) 161–191; 11 (1897) 193–239] on autonomous systems and Theorem 5 of [Weigu Li, J. Llibre, Xiang Zhang, Local first integrals of differential systems and diffeomorphism, Z. Angew. Math. Phys. 54 (2003) 235–255] on periodic systems. Then in the space of analytic autonomous systems in $\mathbb{C}^{2n}$ with exactly n resonances and n functionally independent first integrals, our results are related to the convergence and generic divergence of the normalizations. Lastly for a planar Hamiltonian system it is well known that the system has an isochronous center if and only if it can be linearizable in a neighborhood of the center. Using the Euler–Lagrange equation we provide a new approach to its proof.

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1. Introduction and main results

The investigation of normal forms of vector fields can be traced back to Poincaré and even earlier. This theory is extremely useful in the studies of bifurcation of periodic orbits, KAM theory, stability problem and so on (see for instance, [2,5,13] and references therein). The existence of analytic normalizations transforming an analytic vector field to a desired normal form is strongly related to the existence of first integrals and the resonance (see [9,10,13,22]).

For a given dynamical flow, what is the conditions for the flow to have the desired number of first integrals? The following Theorem 1.1 provides a partial answer. Consider the following quasi-periodic vector field:

$$\begin{aligned} &\dot{\theta} = \omega + \Omega(\theta, x), \\ &\dot{x} = Ax + f(\theta, x), \quad (\theta, x) \in \mathbb{F}^m \times \mathbb{F}^n,\ \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C} \end{aligned} \tag{1}$$

where $\Omega = O(\|x\|)$ and $f = O(\|x\|^2)$ are analytic functions in their variables, and 2π periodic in θ. In what follows we denote by $\mathcal{X}$ the vector field defined in (1).

A non-constant function $H(\theta, x)$ is an *analytic first integral* (respectively, a *formal first integral*) of $\mathcal{X}$ if it is analytic (respectively, a formal power series) in its variable and 2π periodic in θ, and the derivative of $H(\theta, x)$ along the flow of $\mathcal{X}$ vanishes, i.e. $\mathcal{X}(H) \equiv 0$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n-tuple of eigenvalues of the matrix A, and let γ denote the rank of the set $\mathcal{R} := \{(k, l);\ i\langle k, \omega\rangle + \langle l, \lambda\rangle = 0,\ k \in \mathbb{Z}^m,\ l \in \mathbb{Z}_+^n\}$, where $\mathbb{Z}$ stands for the group of integers, $\mathbb{Z}_+$ the set of non-negative integers, $i = \sqrt{-1}$ when appearing in all this paper, and $\langle \cdot, \cdot \rangle$ the usual inner product of two vectors. We have the following

Theorem 1.1. *For the vector field* (1)*, the number of functionally independent analytic first integrals in a neighborhood of the constant solution $x = 0$ is less than or equal to γ.*

We should say that this number γ is optimal, because the completely integrable non-resonant Hamiltonian vector fields are the examples (for details, see the following remarks). This last result is an extension of the following classical one due to Poincaré [18] (for a proof, see for instance [7]).

Theorem *(H. Poincaré). For an autonomous system defined by the second equation of* (1)*, if the n-tuple λ of eigenvalues of the matrix A do not satisfy any resonant conditions, i.e. $\langle l, \lambda\rangle \neq 0$ for all $l = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n$ and $|l| = l_1 + \cdots + l_n \neq 0$, then the system does not have any analytic first integrals in a neighborhood of $x = 0$.*

We note that Theorem 1 also generalizes the results given in Theorem 5 of [14] on periodic vector fields of the type $\dot{x} = A(t)x + f(t, x)$ for $x \in \mathbb{C}^n$. In addition, the condition of Theorem 1 is not necessary. For instance, a germ of planar analytic systems having a pair of pure imaginary eigenvalues at the origin, it may have no analytic first integrals in some neighborhood of the origin.

The following simple examples illustrate the relation of the first integrals and the resonant in Theorem 1.1.

Example 1. Consider the following equation

$$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2, \quad \dot{x}_1 = 2x_1 + x_2 x_3^2, \quad \dot{x}_2 = -\omega_3 i x_2, \quad \dot{x}_3 = \omega_3 i x_3, \tag{2}$$

with the frequencies ω_1 and ω_2 not satisfying any relations: $k_1\omega_1 + k_2\omega_2 = 0$, $k_1, k_2 \in \mathbb{Z}$. If $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 \neq 0$, for all $k_1, k_2, k_3 \in \mathbb{Z}$, then a basis of $\mathcal{R}$ is $(0, 0, 0, 1, 1)$. Hence the function $x_2 x_3$ is the generator of analytic first integrals of (2). If $\omega_3 = m_1\omega_1 + m_2\omega_2$ for some $m_1, m_2 \in \mathbb{Z}$, then a basis of $\mathcal{R}$ is formed by $(m_1, m_2, 0, 1, 0)$ and $(-m_1, -m_2, 0, 0, 1)$. Consequently, the analytic first integrals of (2) are the analytic functions of $e^{i(m_1\theta_1 + m_2\theta_2)} x_2$ and $e^{i(-m_1\theta_1 - m_2\theta_2)} x_3$.

Example 2. Consider the following equation

$$\dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2, \quad \dot{x}_1 = 2x_1, \quad \dot{x}_2 = (1 - \omega_3 i)x_2, \quad \dot{x}_3 = (-2 + \omega_4 i)x_3, \tag{3}$$

with the frequencies ω_1 and ω_2 not satisfying any relations: $k_1\omega_1 + k_2\omega_2 = 0$, $k_1, k_2 \in \mathbb{Z}$. If $k_1\omega_1 + k_2\omega_2 + k_3\omega_3 + k_4\omega_4 \neq 0$, for all $k_1, k_2, k_3, k_4 \in \mathbb{Z}$, $\mathcal{R}$ is an empty set. So there are no analytic first integrals. If $\omega_3 = m_1\omega_1 + m_2\omega_2$ for some $m_1, m_2 \in \mathbb{Z}$, and $\omega_4 = n_1\omega_1 + n_2\omega_2$ for some $n_1, n_2 \in \mathbb{Z}$, then a basis of $\mathcal{R}$ is formed by $(-n_1, -n_2, 1, 0, 1)$ and $(2m_1 - n_1, 2m_2 - n_2, 0, 2, 1)$. Therefore, all analytic first integrals of (3) are the analytic functions of $e^{i(-n_1\theta_1 - n_2\theta_2)} x_1 x_3$ and $e^{i((2m_1 - n_1)\theta_1 + (2m_2 - n_2)\theta_2)} x_2^2 x_3$.

We remark that for a standard Hamiltonian flow on a symplectic manifold with a Hamiltonian $H(x, y)$ starting from the second order terms, of n degree of freedom, let $\lambda = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_{2n})$ be the $2n$-tuple of eigenvalues of the linear part of the Hamiltonian flow. Without loss of generality, we can set $\lambda_k = -\lambda_{n+k}$. Assume that $\lambda_1, \ldots, \lambda_n$ are non-resonant in the sense that $n_1\lambda_1 + \cdots + n_n\lambda_n \neq 0$ for $(n_1, \ldots, n_n) \in \mathbb{Z}^n \setminus \{0\}$. This implies that the $2n$-tuple has exactly n independent resonances. Siegel [19] proved that if the symplectic transformation reducing H to the Birkhoff normal form, leaving unchanged the Hamiltonian character of the flow, is convergent, then the Hamiltonian system has exactly n functionally independent convergent first integrals. Let Ω_H be the set of Hamiltonians having the same second order terms as that of H, then there exists a dense subset of Ω_H endowed with the coefficient topology, in which every Hamiltonian vector field has only itself as the functionally independent convergent first integral, and consequently it cannot be reduced to the Birkhoff normal form by a convergent symplectic transformation. Of course, any Hamiltonian vector field in Ω_H has exactly n functionally independent formal first integrals. For the eigenvalues $\lambda_1, \ldots, \lambda_n$ not resonant or simple resonant, using the fast convergent method Ito in [9] and [10] respectively proved that if the Hamiltonian is integrable, i.e. having n functionally independent first integrals in involution, then it is analytically symplectically equivalent to the Birkhoff normal form. Recently, Zung [22] proved that any analytically integrable Hamiltonian system, without any restriction on the resonance of $\lambda_1, \ldots, \lambda_n$, is analytically symplectically equivalent to the Birkhoff normal form using a geometrical method involving homological cycles and torus actions. For Hamiltonian and non-Hamiltonian flows, Pérez-Marco [16,17] obtained some excellent results on the convergence and generic divergence of the normalizations and normal forms.

For non-Hamiltonian flows, the existence of first integrals is much more involved. In [14] we proved that for an analytic, or a formal, autonomous system with a singularity, if one of the eigenvalues vanishes and others non-resonant then the system has a formal first integral in a vicinity of the singularity if and only if the singularity is non-isolated. In the planar setting the result is in the analytic world. For a planar analytic vector field having a singularity, if the eigenvalues, denoted by λ_1, λ_2, are resonant and non-zero, then the vector field is locally analytically integrable if and only if it is analytically equivalent to

$$\dot{x} = \lambda_1 x\bigl(1 + g(z)\bigr), \qquad \dot{y} = \lambda_2 y\bigl(1 + g(z)\bigr),$$

where g is an analytic function in $z = x^r y^s$ with $r, s \in \mathbb{N}$ relatively prime and $r/s = -\lambda_2/\lambda_1$ (see for instance, [13,21]).

Associated with the above results, we have the following

Theorem 1.2. *Given an analytic vector field $\widetilde{\mathcal{X}}$ in $\mathbb{C}^{2n}$ having the origin as a singularity. Let $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ be the $2n$-tuple of eigenvalues of $\widetilde{\mathcal{X}}$ at the origin. Assume that λ_j, μ_j are non-zero and pairwise resonant for $j = 1, \ldots, n$, and $\lambda_1, \ldots, \lambda_n$ are non-resonant. If $\widetilde{\mathcal{X}}$ has n analytically functionally independent first integrals in a neighborhood of the origin, then the following hold.*

(a) *The vector field $\widetilde{\mathcal{X}}$ is formally equivalent to*

$$\begin{aligned} \dot{u}_j &= \lambda_j u_j\bigl(1 + W_j(z_1, \ldots, z_n)\bigr), \\ \dot{v}_j &= \mu_j v_j\bigl(1 + W_j(z_1, \ldots, z_n)\bigr), \quad j = 1, \ldots, n \end{aligned} \tag{4}$$

where W_j is a formal power series in $z_1, \ldots, z_n$ with $z_s = u_s^{\bar{n}_s} v_s^{\overline{m}_s}$, where $\bar{n}_s, \overline{m}_s \in \mathbb{N}$ relatively prime and $\overline{m}_s/\bar{n}_s = -\lambda_s/\mu_s$.

(b) *If either the n-tuple of eigenvalues λ belong to the Poincaré domain, or the formal power series W_j are all equal and $|\langle k, \lambda\rangle - \lambda_j| \geqslant \epsilon > 0$ for some constant ϵ and $k \in \mathbb{Q}^n$ the field of rational numbers, then the equivalence in the statement* (a) *is analytic.*

(c) *A formal power series is a first integral of* (4) *in $u_1, \ldots, u_n, v_1, \ldots, v_n$ if and only if it is a power series in the n variables $z_1, \ldots, z_n$. This kind of first integral is called universal.*

(d) *Set $\mathcal{V}$ be the set of vector fields having the same linear part as that of $\widetilde{\mathcal{X}}$. If there exists a vector field in $\mathcal{V}$ with the divergent distinguished normal form* (*respectively, normalization*)*, then generic vector fields in $\mathcal{V}$ have this property.*

We recall that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is in the *Poincaré domain* if the convex hull of the n points $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$ does not contain the origin of the complex plane. Two vector fields in $\mathbb{C}^m$ are *formally equivalent* if they can be exchanged each other by a formal series transformation f satisfying $f(0) = 0$ and $Df(0) = I$, and *analytically equivalent* if the transformation is analytic. In the statement (d), the *genericity* is in the sense of following Lemma 3.1.

We note that for planar vector fields the conditions in the statement (b) hold naturally. Consequently, in this case the vector field $\widetilde{\mathcal{X}}$ is analytically equivalent to (4).

In order to prove the statement (b) we need to use the mojarant series. In the proof of the statement (d) we will get the help of pluripotential theory in the complex domain.

On the relation between the existence of analytic first integrals and the convergence of normalizations for an analytic vector field, Zung [23] proved the following result: *Let* $\mathbf{X}$ *be a locally analytic vector field in* $(\mathbb{F}^n, 0)$ *with* $\mathbf{X}(0) = 0$*. Suppose that there are* m, $1 \leqslant m \leqslant n$, *locally analytic vector fields* $\mathbf{X}_1 = \mathbf{X}, \mathbf{X}_2, \ldots, \mathbf{X}_m$ *commuting pairwise, i.e.* $[\mathbf{X}_j, \mathbf{X}_k] = 0$*, and* *linearly independent almost everywhere, i.e.* $\mathbf{X}_1 \Lambda \dots \Lambda \mathbf{X}_m \neq 0$. *If there are* $n - m$ *locally analytic and functionally independent functions* $f_1, \dots, f_{n-m}$ *which are the common first integrals of* $\mathbf{X}_1, \dots, \mathbf{X}_m$, *i.e.* $\mathbf{X}_j(f_k) = 0$, $j = 1, \dots, m$, $k = 1, \dots, n - m$, *then the vector field* $\mathbf{X}$ *has a locally analytic normalization in* $(\mathbb{F}^n, 0)$.

For a given vector field $\mathcal{Z}$ in $\mathbb{C}^n$ with a singularity at the origin, similar to the statement (c) of the last theorem we have the following. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the n-tuple of eigenvalues of $\mathcal{Z}$ at the origin. Denote by $\mathcal{M}$ the sublattice of $k \in \mathbb{Z}_+^n$ satisfying $\langle k, \lambda \rangle = 0$ and g.c.d.$(k_1, \dots, k_n) = 1$. We note that even $\lambda \neq 0$ it is possible that $\#\mathcal{M} = n$. For instance $\lambda = (1, 1, -2)$, $\mathcal{M} = \{(1, 1, 1), (2, 0, 1), (0, 2, 1)\}$.

Proposition 1.1. *If* $\mathcal{Z}$ *is in the distinguished normal form, then its formal first integral is a formal power series in the* $\#\mathcal{M}$ *variables* $z_j = x^k$, $k \in \mathcal{M}$, *where we have used the multi-index* $x^k = x_1^{k_1} \dots x_n^{k_n}$.

This proposition can be proved easily by combining some linear algebra, the details will be omitted. The *distinguished normal form* will be defined in Section 2.

Theorem 1.3. *For a planar analytic flow with a singularity, if the eigenvalues of the flow at the singularity satisfy a unique linearly independent resonant condition and the flow has an analytic first integral in a neighborhood of the singularity, then either the singularity is non-isolated or the flow is analytically orbitally equivalent to a linear one.*

For a planar Hamiltonian system, it is always completely integrable in the convention sense. In the case that the Hamiltonian system has center, related to the periods of closed orbits in the central annulus, the following result is well known.

Theorem 1.4. *A planar analytic Hamiltonian system has an isochronous center if and only if it can be analytically linearizable.*

We will provide a new approach to its proof by using the Euler–Lagrange equation. On the characterization of isochronous centers, we refer to [4,6,11] and the references therein.

This paper is organized as follows. In Section 2 we prove Theorem 1. The proof of Theorems 1.2 and 1.3 are given in Sections 3 and 4, respectively. In the last section we provide the new approach to the proof of Theorem 1.4.

2. Proof of Theorem 1.1

For the vector field $\mathcal{X}$ given in (1) we say that the n-tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ of eigenvalues of the matrix A is *non-resonant* if for all $k \in \mathbb{Z}^m$, $l \in \mathbb{Z}_+^n$ and $|l| = l_1 + \dots + l_n > 1$, the following hold

$$\langle l, \lambda \rangle \neq -i \langle k, \omega \rangle, \qquad \langle l, \lambda \rangle - \lambda_j \neq -i \langle k, \omega \rangle, \quad j = 1, \dots, n. \tag{5}$$

The n-tuple λ is *weekly non-resonant* if the conditions (5) hold except for the case $k = 0$.

Set $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_h$, where $\mathcal{X}_1 = \langle \omega, \partial_\theta \rangle + \langle Ax, \partial_x \rangle$ and $\mathcal{X}_h$ are the higher order terms. Since the algebra of linear vector fields in $\mathbb{F}^n$, under the standard Lie bracket, is nothing but the reductive algebra $gl(n, \mathbb{F}) = sl(n, \mathbb{F}) \oplus \mathbb{F}$, we write $A = A_1 + A_2$ with A_1 semisimple and A_2 nilpotent.

Correspondingly we separate $\mathcal{X}_1 = \mathcal{X}_1^s + \mathcal{X}_1^n$ with $\mathcal{X}_1^s = \langle \omega, \partial_\theta \rangle + \langle A_1 x, \partial_x \rangle$ called the *semisimple part* of $\mathcal{X}_1$ and $\mathcal{X}_1^n = \langle A_2 x, \partial_x \rangle$ called the *nilpotent part* of $\mathcal{X}_1$. Without loss of generality, we can assume that

$$\mathcal{X}_1^s = \langle \omega, \partial_\theta \rangle + \langle \lambda x, \partial_x \rangle,$$

where $\lambda x = (\lambda_1 x_1, \ldots, \lambda_n x_n)$.

The vector field $\mathcal{X}$ is in *normal form* if the Lie bracket of $\mathcal{X}_1^s$ and $\mathcal{X}_h$ vanishes, i.e. $[\mathcal{X}_1^s, \mathcal{X}_h] = 0$. We note that for a vector field of type (1) in normal form, all *pseudomonomials*, $e^{i\langle k,\theta\rangle} x^l$, are resonant, in the sense that if $e^{i\langle k,\theta\rangle} x^l$ is in the component ∂_{θ_j} then $i\langle k, \omega \rangle + \langle l, \lambda \rangle = 0$ called *in the first resonant*; and in the component ∂_{x_j} we should have $i\langle k, \omega \rangle + \langle l, \lambda \rangle = \lambda_j$ called *in the second resonant*. A pseudomonomial $e^{i\langle k,\theta\rangle} x^l$ of an analytic or a formal quasi-periodic function is called *resonant* if $i\langle k, \omega \rangle + \langle l, \lambda \rangle = 0$.

Usually, a transformation reducing a vector field to its normal form is not unique. In what follows, we call such a transformation *distinguished normalization* if the transformation contains non-resonant terms only. The distinguished normalization is unique. Correspondingly, the normal form is called a *distinguished normal form*.

The following result due to Bibikov [3] is the key point to prove the following Lemma 2.2.

Lemma 2.1. *Denote by $\mathcal{G}^r(\mathbb{F})$ the linear space of n-dimensional vector-valued homogeneous polynomials of degree r in n variables with coefficients in $\mathbb{F}$. Let A and B be two nth square matrices with entries in $\mathbb{F}$, and their n-tuple of eigenvalues be λ and κ, respectively. Define a linear operator L on $\mathcal{G}^r(\mathbb{F})$ as follows,*

$$Lh = \langle \partial_x h, Ax \rangle - Bh, \quad h \in \mathcal{G}^r(\mathbb{F}).$$

Then the spectrum of the operator L is

$$\big\{ \langle l, \lambda \rangle - \kappa_j;\ l \in \mathbb{Z}_+^n,\ |l| = r,\ j = 1, \ldots, n \big\}.$$

Our next result will be used in the proof of Theorem 1.1. We note that it is an extension of the classical Poincaré–Dulac normal form on autonomous systems, and of Lemma 6 in [15] on periodic systems to quasi-periodic systems.

Lemma 2.2. *The vector field $\mathcal{X}$ defined in* (1) *can be formally normalized by a distinguished normalization.*

Proof. Assume that the vector field $\mathcal{X}$ is transformed to

$$\dot{\beta} = \omega + \Lambda(\beta, y), \qquad \dot{y} = Ay + g(\beta, y), \tag{6}$$

under the transformation

$$\theta = \beta + \phi(\beta, y), \qquad x = y + \psi(\beta, y), \tag{7}$$

where $\Lambda, \phi = O(\|y\|)$ and $g, \psi = O(\|y\|^2)$ are 2π-periodic in β. Then ϕ, ψ satisfy the following

$$\langle \partial_\beta \phi, \omega\rangle + \langle \partial_y \phi, Ay\rangle = \Omega(\beta+\phi, y+\psi) - \Lambda(\beta, y) - \langle \partial_\beta \phi, \Lambda\rangle - \langle \partial_y \phi, g\rangle,$$
$$\langle \partial_\beta \psi, \omega\rangle + \langle \partial_y \psi, Ay\rangle - A\psi = f(\beta+\phi, y+\psi) - g(\beta, y) - \langle \partial_\beta \psi, \Lambda\rangle - \langle \partial_y \psi, g\rangle. \tag{8}$$

Expanding the considered functions in Taylor series in y

$$V(\beta, y) = \sum_r V_r(\beta, y) \quad \text{for } V \in \{\Lambda, g, \phi, \psi, \Omega, f\} \tag{9}$$

where V_r is a homogeneous polynomial of degree r in y with 2π periodic coefficients in β. The system of equations (8) is equivalent to

$$\langle \partial_\beta \phi_r, \omega\rangle + \langle \partial_y \phi_r, Ay\rangle = \Omega_r - \Lambda_r - p_r,$$
$$\langle \partial_\beta \psi_{r+1}, \omega\rangle + \langle \partial_y \psi_{r+1}, Ay\rangle - A\psi_{r+1} = f_{r+1} - g_{r+1} - q_{r+1}, \quad r = 1, 2, \ldots \tag{10}$$

where p_r, q_{r+1} are known inductively. In precisely, p_r is a polynomial in $\phi_s, \Lambda_s, g_{s+1}$ with $s = 1, \ldots, r-1$; q_{r+1} is a polynomial in $\psi_s, \Lambda_{s-1}, g_s$ with $s = 2, \ldots, r$.

Make the Fourier expansions on V_r,

$$V_r(\beta, y) = \sum_{k\in\mathbb{Z}^m} V_r^k(y) e^{i\langle k,\beta\rangle}, \quad \text{for } V \in \{\Lambda, g, \phi, \psi, \Omega, f\}. \tag{11}$$

From (10) we get

$$\mathcal{A}_0 \phi_r^k = \Omega_r^k - \Lambda_r^k - p_r^k,$$
$$\mathcal{A}_1 \psi_{r+1}^k = f_{r+1}^k - g_{r+1}^k - q_{r+1}^k, \quad r = 1, 2, \ldots \tag{12}$$

where $\mathcal{A}_s = i\langle k, \beta\rangle + L_s$, $s = 0, 1$, and L_0 and L_1 are the linear operators on $\mathcal{G}^r(y)$ and $\mathcal{G}^{r+1}(y)$ respectively, defined by

$$L_0 h(y) = \langle \partial_y h, Ah\rangle, \quad h \in \mathcal{G}^r(y),$$
$$L_1 h(y) = \langle \partial_y h, Ah\rangle - Ah, \quad h \in \mathcal{G}^{r+1}(y).$$

Applying Lemma 2.1 to the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ in (12), we obtain

$$\text{the spectrum of } \mathcal{A}_0 = \big\{ i\langle k,\beta\rangle + \langle l, \lambda\rangle;\ l \in \mathbb{Z}_+^n,\ |l| = r \big\},$$
$$\text{the spectrum of } \mathcal{A}_1 = \big\{ i\langle k,\beta\rangle + \langle l, \lambda\rangle - \lambda_j;\ l \in \mathbb{Z}_+^n,\ |l| = r+1,\ j = 1, \ldots, n \big\}.$$

According to the operator $\mathcal{A}_0$ (respectively, $\mathcal{A}_1$), we separate the space $\mathcal{G}^r(y)$ into the direct sum $\mathcal{G}^r(y) = \mathcal{G}_{0,1}^r(y) \oplus \mathcal{G}_{0,2}^r(y)$ (respectively, $\mathcal{G}^{r+1}(y) = \mathcal{G}_{1,1}^{r+1}(y) \oplus \mathcal{G}_{1,2}^{r+1}(y)$) in such a way that $\mathcal{A}_0$ restricted to $\mathcal{G}_{0,1}^r(y)$, denoted by $\mathcal{A}_0^1$, is invertible, and restricted to $\mathcal{G}_{0,2}^r(y)$, denoted by $\mathcal{A}_0^2$, is degenerated, i.e. having only zero spectrum (respectively, $\mathcal{A}_1$ restricted to $\mathcal{G}_{1,1}^{r+1}(y)$, denoted by $\mathcal{A}_1^1$, is invertible, and restricted to $\mathcal{G}_{1,2}^{r+1}(y)$, denoted by $\mathcal{A}_1^2$, is degenerated). Decompose the right-hand side of (12) as $\Omega_r^k - \Lambda_r^k - p_r^k = (\Omega_r^{k,1} - \Lambda_r^{k,1} - p_r^{k,1}) \oplus (\Omega_r^{k,2} - \Lambda_r^{k,2} - p_r^{k,2}) \in$

$\mathcal{G}_{0,1}^r(y) \oplus \mathcal{G}_{0,2}^r(y)$, and $f_{r+1}^k - g_{r+1}^k - q_{r+1}^k = (f_{r+1}^{k,1} - g_{r+1}^{k,1} - q_{r+1}^{k,1}) \oplus (f_{r+1}^{k,2} - g_{r+1}^{k,2} - q_{r+1}^{k,2}) \in \mathcal{G}_{1,1}^{r+1}(y) \oplus \mathcal{G}_{1,2}^{r+1}(y)$. Then Eqs. (12) are the same as the following

$$\begin{aligned}\mathcal{A}_0^s \phi_r^{k,s} &= \Omega_r^{k,s} - \Lambda_r^{k,s} - p_r^{k,s},\\ \mathcal{A}_1^s \psi_{r+1}^{k,s} &= f_{r+1}^{k,s} - g_{r+1}^{k,s} - q_{r+1}^{k,s}, \quad r = 1, 2, \ldots, s = 1, 2.\end{aligned} \tag{13}$$

For $s = 1$, since the operators in (13) are invertible, for any choice of $\Lambda_r^{k,s}$ and $g_{r+1}^{k,s}$ the equations have a unique solution. In order for obtaining the distinguished normal form, we choose $\Lambda_r^{k,1} = g_{r+1}^{k,1} = 0$, then we get a unique solution $\phi_r^{k,1}$ and $\psi_{r+1}^{k,1}$ corresponding to the two equations in (13), respectively. For $s = 2$, since the operators in (13) are degenerated, choosing $\Lambda_r^{k,2} = \Omega_r^{k,2} - p_r^{k,2}$ and $g_{r+1}^{k,2} = f_{r+1}^{k,2} - q_{r+1}^{k,2}$, we get $\phi_r^{k,2} = \psi_{r+1}^{k,2} = 0$.

Summarizing the above process, we get a formal transformation

$$\theta = \beta + \sum_{k \in \mathbb{Z}_+^m,\ r \geqslant 1} \phi_r^{k,1}(y) e^{i\langle k, \beta\rangle}, \qquad x = y + \sum_{k \in \mathbb{Z}_+^m,\ r \geqslant 2} \psi_r^{k,1}(y) e^{i\langle k, \beta\rangle},$$

where all the components in the summations are non-resonant, under which the vector field $\mathcal{X}$ is transformed into

$$\dot\beta = \omega + \sum_{k \in \mathbb{Z}^m,\, r \geqslant 1} \Lambda_r^{k,2}(y) e^{i\langle k, \beta\rangle}, \qquad \dot y = Ay + \sum_{k \in \mathbb{Z}^m,\, r \geqslant 2} g_r^{k,2}(y) e^{i\langle k, \beta\rangle},$$

where each component in the summations is resonant.

Denote by $\mathcal{Y}$ this last vector field, and write it in the form $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_h$ with $\mathcal{Y}_1 = \langle \omega, \partial_\beta\rangle + \langle Ay, \partial_y\rangle$ and

$$\mathcal{Y}_h = \sum_{p=1}^m \Biggl(\sum_{k \in \mathbb{Z}^m,\ l \in \mathbb{Z}_+^n} \xi_p^{k,l} y^l e^{i\langle k, \beta\rangle} \Biggr) \partial_{\beta_p} + \sum_{q=1}^n \Biggl(\sum_{k' \in \mathbb{Z}^m,\ l' \in \mathbb{Z}_+^n} \eta_q^{k',l'} y^{l'} e^{i\langle k', \beta\rangle} \Biggr) \partial_{y_q},$$

where $\xi_p^{k,l}, \eta_q^{k',l'} \in \mathbb{F}$, and k, l, k', l' satisfy $|l| \neq 0$, $|l'| > 1$, $i\langle k, \omega\rangle + \langle l, \lambda\rangle = 0$ and $i\langle k', \omega\rangle + \langle l', \lambda\rangle = \lambda_q$. Then the Lie bracket

$$\begin{aligned}\bigl[\mathcal{Y}_1^s, \mathcal{Y}_h\bigr] = {} & \sum_{p=1}^m \Biggl(\sum_{k \in \mathbb{Z}^m,\ l \in \mathbb{Z}_+^n} \xi_p^{k,l} \bigl(i\langle k, \omega\rangle + \langle l, \lambda\rangle \bigr) y^l e^{i\langle k, \beta\rangle} \Biggr) \partial_{\beta_p} \\ & + \sum_{q=1}^n \Biggl(\sum_{k' \in \mathbb{Z}^m,\ l' \in \mathbb{Z}_+^n} \eta_q^{k',l'} \bigl(i\langle k', \omega\rangle + \langle l', \lambda\rangle - \lambda_q \bigr) y^{l'} e^{i\langle k', \beta\rangle} \Biggr) \partial_{y_q} = 0,\end{aligned}$$

where $\mathcal{Y}_1^s$ is the semisimple part of $\mathcal{Y}_1$. This proves the lemma. □

Corollary 2.1. *If the n-tuple of eigenvalues of the matrix A is non-resonant, then the vector field $\mathcal{X}$ is formally equivalent to its linear part $\mathcal{X}_1$. If the n-tuple is weekly non-resonant, then the vector field $\mathcal{X}$ is formally equivalent to an autonomous system.*

Proof. If the n-tuple of eigenvalues of the matrix A is non-resonant, then the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ in (12) are both invertible. So for any choice of Λ_r^k and g_{r+1}^k Eqs. (12) have a unique solution. By choosing all $\Lambda_r^k = g_{r+1}^k = 0$, we get the desired normal form.

Assume that the n-tuple of eigenvalues of the matrix A is weekly non-resonant. For $k \neq 0$, Eqs. (12) have a unique solution for any given Λ_r^k and g_{r+1}^k. In these cases, set $\Lambda_r^k = g_{r+1}^k = 0$. For the terms related to $k = 0$, they are independent of β. Hence, we get a normal form which is autonomous. □

Lemma 2.3. *Assume that $\mathcal{H}(\theta, x)$ is an analytic (or a formal) first integral, with 2π period in θ, of the vector field $\mathcal{X}$. Let $\mathcal{Y}$ be the distinguished normal form associated to $\mathcal{X}$, and let $\overline{\mathcal{H}}(\beta, y)$ be of $\mathcal{H}(\theta, x)$ written in the normalized coordinates β, y. Then $\overline{\mathcal{H}}(\beta, y)$ is a first integral of $\mathcal{Y}$, and it contains resonant terms only, i.e. if we expand $\overline{\mathcal{H}}$ in Fourier series*

$$\overline{\mathcal{H}}(\beta, y) = \sum_{\mu \in \mathbb{Z}^m,\ \nu \in \mathbb{Z}_+^n} \bar{h}^{\mu,\nu} y^\nu e^{i\langle \mu, \beta\rangle},$$

then we should have $i\langle \mu, \omega\rangle + \langle \nu, \lambda\rangle = 0$.

Proof. Here we still use the notations given in the proof of Lemma 2.2. The first statement is obvious. Without loss of generality, in what follows we can assume that $\mathcal{X}$ is in the distinguished normal form. To prove the second statement, we expand $\mathcal{H}$ into Taylor series in x,

$$\mathcal{H}(\theta, x) = \sum_{p=r}^{\infty} H_p(\theta, x),$$

where H_r is the first non-zero terms with $r \geqslant 0$, and H_p is homogeneous in x. Then we have

$$\mathcal{L} H_p = -\sum_{q=1}^{p-r} \big(\langle \partial_\theta H_{p-q}, \Omega_q\rangle + \langle \partial_x H_{p-q}, f_{q+1}\rangle\big), \quad p = r, r+1, \ldots \tag{14}$$

where $\mathcal{L}$ is the linear operator defined by $\mathcal{L} H_p = \langle \partial_\theta H_p, \omega\rangle + \langle \partial_x H_p, Ax\rangle$.

Eq. (14) with $p = r$ is a linear homogeneous equation, it follows from the spectrum of the linear operator that its non-trivial solution $H_r(\theta, x)$ should be composed of the resonant terms.

Consider Eq. (14) with $p = r + 1$. From the construction of the distinguished normal form, we know that each pseudomonomial in Ω_q, e.g. $\Omega_q^{k,l} x^l e^{i\langle k,\theta\rangle}$, is in the first resonant, i.e. $i\langle k, \omega\rangle + \langle l, \lambda\rangle = 0$ and that each pseudomonomial in the jth component of f_{q+1} for $j = 1, \ldots, n$, e.g. $f_{q+1,j}^{k,l} x^l e^{i\langle k,\theta\rangle}$, is in the second resonant, i.e. $i\langle k, \omega\rangle + \langle l, \lambda\rangle = \lambda_j$. Hence, all the terms in the right-hand side of (14) with $p = r + 1$ is in the first resonant. Thus, the terms in the left-hand side, consequently the solution H_{r+1} of (14), should be in the first resonant.

By induction we can prove that for each p the solution H_p of (14) is composed of the resonant terms. We complete the proof of the lemma. □

Proof of Theorem 1.1. Working in a similar way to the proof of Lemma 2.3, we can assume that the vector field $\mathcal{X}$ is in the distinguished normal form, and its functionally independent analytic (or formal) first integrals are $\mathcal{H}_1, \ldots, \mathcal{H}_\tau$. Since all pseudomonomials in each of $\mathcal{H}_j$

for $j = 1, \dots, \tau$ are resonant, it implies that $\mathcal{X}_1^s(\mathcal{H}_j) = 0$, i.e. each $\mathcal{H}_j$ is also a first integral of $\mathcal{X}_1^s$. Obviously, the set of analytic and formal first integrals of $\mathcal{X}_1^s$ is generated by $\{x^l e^{i\langle k,\theta\rangle};\ i\langle k,\omega\rangle + \langle l,\lambda\rangle = 0,\ l \in \mathbb{Z}_+^m,\ k \in \mathbb{Z}^n\}$, denote S. Then, the number of functionally independent elements of S is exactly γ. This proves that the maximum number of functionally independent first integrals of $\mathcal{X}$ is less than or equal to γ. □

3. Proof of Theorem 1.2

(a) From the assumption on the resonance of eigenvalues of the vector field $\widetilde{\mathcal{X}}$ at the origin and the proof of Theorem 1.1, it follows that the vector field $\widetilde{\mathcal{X}}$ is formally equivalent to the following

$$\begin{aligned} \dot{u}_j &= \lambda_j u_j \big(1 + F_j(z_1, \dots, z_n)\big), \\ \dot{v}_j &= \mu_j v_j \big(1 + G_j(z_1, \dots, z_n)\big), \quad j = 1, \dots, n \end{aligned} \tag{15}$$

where z_s is defined in Theorem 1.2, and F_j, G_j are formal power series in $z_1, \dots, z_n$. Indeed, for each monomial $u_1^{\alpha_1} \dots u_n^{\alpha_n} v_1^{\beta_1} \dots v_n^{\beta_n}$ in the component ∂_{u_j} we have $\langle \alpha, \lambda\rangle + \langle \beta, \mu\rangle = \lambda_j$. By the resonant relations it follows that $\sum_{s=1}^n (\alpha_s - \frac{\bar{m}_s}{\bar{n}_s}\beta_s - \sigma_{sj})\lambda_s = 0$ with $\sigma_{sj} = 1$ if $s = j$, or $\sigma_{sj} = 0$ if $s \neq j$. So, for $s \neq j$ there exists a $k_s \in \mathbb{Z}_+$ for which $\alpha_s = k_s \bar{m}_s$ and $\beta_s = k_s \bar{n}_s$; for $s = j$ there exists $k_s \in \mathbb{Z}_+$ for which $\alpha_s = k_s \bar{m}_s + 1$ and $\beta_s = k_s \bar{n}_s$. This proves the claim.

Since $\widetilde{\mathcal{X}}$ has n functionally independent analytic first integrals, the vector field (15) has n functionally independent formal first integrals. Lemma 2.3 tells us the first integrals of (15) contain resonant terms only. So, if $\mathcal{H}$ is a first integral of (15), we can assume without loss $\mathcal{H} = \mathcal{H}(z_1, \dots, z_n)$. Then direct calculations show that the first integral $\mathcal{H}$ satisfies

$$\bar{n}_1\lambda_1 z_1(F_1 - G_1)\frac{\partial H}{\partial z_1} + \dots + \bar{n}_n\lambda_n z_n(F_n - G_n)\frac{\partial H}{\partial z_n} \equiv 0.$$

This implies that every first integral of (15) is a first integral of the vector field

$$\mathcal{X}^* = \bar{n}_1\lambda_1 z_1(F_1 - G_1)\frac{\partial}{\partial z_1} + \dots + \bar{n}_n\lambda_n z_n(F_n - G_n)\frac{\partial}{\partial z_n},$$

in the n-dimensional space. It is well known that if a vector field in an n-dimensional space is not trivial, it has at most $n - 1$ functionally independent first integrals. But $\mathcal{X}^*$ has n functionally independent first integrals, it should be trivial. Hence, we have $F_j = G_j$, $j = 1, \dots, n$. This proves the statement (a).

(b) In order to prove the statement, we need to refine the normalization process. Under the assumption of the theorem, without loss of generality we can set the vector field $\widetilde{\mathcal{X}}$ be of the form

$$\dot{x}_j = \lambda_j x_j + p_j(x, y), \qquad \dot{y}_j = \mu_j y_j + q_j(x, y), \quad j = 1, \dots, n$$

where p_j, q_j are analytic functions in x, y. Assume that it is reduced, by the formal transformation

$$x_j = u_j + \phi_j(u, v), \qquad y_j = v_j + \psi_j(u, v), \quad j = 1, \dots, n$$

to the following formal vector field

$$\dot{u}_j = \lambda_j u_j + \alpha_j(u, v), \qquad \dot{v}_j = \mu_j v_j + \beta_j(u, v), \quad j = 1, \dots, n.$$

Using the multi-index notation, for $w \in \{p, q, \phi, \psi, \alpha, \beta\}$ we denote by

$$w_j(u, v) = \sum_{k,l} w_j^{(k,l)} u^k v^l,$$

where $u^k = u_1^{k_1} \dots u_n^{k_n}$ and $v^l = v_1^{l_1} \dots v_n^{l_n}$. Then from the proof of Theorem 1.1 we get that

$$\begin{aligned}
\big(\langle k, \lambda\rangle + \langle l, \mu\rangle - \lambda_j\big)\phi_j^{(k,l)} &= \big[p_j(u + \phi, v + \psi)\big]^{(k,l)} - \alpha_j^{(k,l)} \\
&\quad - \sum_{s=1}^{n} \sum_{(a,b)} \phi_j^{(a,b)} \big(a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)}\big), \\
\big(\langle k, \lambda\rangle + \langle l, \mu\rangle - \mu_j\big)\psi_j^{(k,l)} &= \big[q_j(u + \phi, v + \psi)\big]^{(k,l)} - \beta_j^{(k,l)} \\
&\quad - \sum_{s=1}^{n} \sum_{(a,b)} \psi_j^{(a,b)} \big(a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)}\big)
\end{aligned} \tag{16}$$

where $[p_j(u+\phi, v+\psi)]^{(k,l)}$ and $[q_j(u+\phi, v+\psi)]^{(k,l)}$ are obtained after we re-expand $p_j(u+\phi, v+\psi), q_j(u+\phi, v+\psi)$ in power series in u and v, e_s the n-dimensional vector equal to 1 at the sth entry and equal to 0 for otherwise, and in the summation (a, b) are taken over all the vectors in $\mathbb{Z}_+^{2n}$ for which $(k-a, l-b) \in \mathbb{Z}_+^{2n}$. For simplicity to notations, set $[p_j]^{(k,l)} = [p_j(u+\phi, v+\psi)]^{(k,l)}$ and $[q_j]^{(k,l)} = [q_j(u+\phi, v+\psi)]^{(k,l)}$.

For (k, l) in the resonant cases, i.e. $\langle k, \lambda\rangle + \langle l, \mu\rangle - \lambda_j = 0$ or $\langle k, \lambda\rangle + \langle l, \mu\rangle - \mu_j = 0$, in order for the normalization transformation to be distinguished, we choose $\phi_j^{(k,l)} = 0$ or $\psi_j^{(k,l)} = 0$. Correspondingly we have

$$\alpha_j^{(k,l)} = [p_j]^{(k,l)} - \sum_{s=1}^{n} \sum_{(a,b)} \phi_j^{(a,b)} \big(a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)}\big), \tag{17}$$

or

$$\beta_j^{(k,l)} = [q_j]^{(k,l)} - \sum_{s=1}^{n} \sum_{(a,b)} \psi_j^{(a,b)} \big(a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)}\big). \tag{18}$$

If (k, l) is not in the resonant case, we choose $\alpha_j^{(k,l)} = \beta_j^{(k,l)} = 0$. Then Eq. (16) has a unique solution

$$\phi_j^{(k,l)} = \frac{[p_j]^{(k,l)} - \sum_{s=1}^{n} \sum_{(a,b)} \phi_j^{(a,b)} (a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)})}{\langle k, \lambda\rangle + \langle l, \mu\rangle - \lambda_j}, \tag{19}$$

$$\psi_j^{(k,l)} = \frac{[q_j]^{(k,l)} - \sum_{s=1}^{n} \sum_{(a,b)} \psi_j^{(a,b)} (a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)})}{\langle k, \lambda\rangle + \langle l, \mu\rangle - \mu_j}. \tag{20}$$

We claim that $\alpha_j^{(k,l)} = [p_j]^{(k,l)}$ in (17) and $\beta_j^{(k,l)} = [q_j]^{(k,l)}$ in (18). Indeed, since $\alpha_s^{(k+e_s-a,l-b)}$ and $\beta_s^{(k-a,l+e_s-b)}$ are the coefficients of the resonant terms (otherwise, they are zero by the construction), we have $\langle k+e_s-a,\lambda\rangle + \langle l-b,\mu\rangle = \lambda_s$. This is equivalent to $\langle k-a,\lambda\rangle + \langle l-b,\mu\rangle = 0$. Using this equality and (k,l) in the resonant case, we get that $0 = \langle k,\lambda\rangle + \langle l,\mu\rangle - \lambda_j = \langle a,\lambda\rangle + \langle b,\mu\rangle - \lambda_j$. This proves that (a,b) is also in the resonant case. Therefore, we should have $\phi_j^{(a,b)} = 0$. Working in a similar way we get that $\psi_j^{(a,b)} = 0$. This proves the claim.

Summarizing the above construction, we obtain a distinguished formal transformation

$$x_j = u_j + \sum_{(k,l)} \phi_j^{(k,l)} u^k v^l, \qquad y_j = v_j + \sum_{(k,l)} \psi_j^{(k,l)} u^k v^l,$$

with all (k,l) in non-resonant cases and $\phi_j^{(k,l)}$ and $\psi_j^{(k,l)}$ given in (19) and (20), respectively. Under the action of this transformation, the distinguished normal form satisfies

$$\alpha_j = \sum_{(k,l)} [p_j]^{(k,l)} u^k v^l, \qquad \beta_j = \sum_{(k,l)} [q_j]^{(k,l)} u^k v^l,$$

where (k,l) are in resonant cases.

Now we prove the convergence of the distinguished transformation. Comparing with the formal normal form (4), we have

$$\alpha_s^{(k+e_s-a,l-b)} = \lambda_s w_s^{(k-a,l-b)}, \qquad \beta_s^{(k-a,l+e_s-b)} = \mu_s w_s^{(k-a,l-b)},$$

where $w_s^{(k-a,l-b)}$ are the coefficients of monomials in W_s, and $\langle k-a,\lambda\rangle + \langle l-b,\mu\rangle = 0$. Then

$$\sum_{s=1}^{n} \bigl(a_s \alpha_s^{(k+e_s-a,l-b)} + b_s \beta_s^{(k-a,l+e_s-b)}\bigr) = \sum_{s=1}^{n} (k_s\lambda_s + l_s\mu_s) w_s^{(k-a,l-b)}. \tag{21}$$

If the eigenvalues $\lambda = (\lambda_1,\dots,\lambda_n)$ belong to the Poincaré domain, then there exists a δ_2 such that

$$\bigl(\langle k,\lambda\rangle + \langle l,\mu\rangle - \lambda_j\bigr)^{-1},\ \bigl(\langle k,\lambda\rangle + \langle l,\mu\rangle - \mu_j\bigr)^{-1} \geqslant \delta_2.$$

Moreover, since λ_j and μ_j for $j = 1,\dots,n$ are pairwise resonant, there exists a constant C_1 for which

$$\frac{k_s + l_s}{\langle k,\lambda\rangle + \langle l,\mu\rangle - \mu_j},\ \frac{k_s + l_s}{\langle k,\lambda\rangle + \langle l,\mu\rangle - \lambda_j} \leqslant C_1.$$

If all W_j are equal to W, then there exists a C_2 such that

$$\left| \frac{\sum_{s=1}^{n} (k_s\lambda_s + l_s\mu_s) w_s^{(k-a,l-b)}}{\langle k,\lambda\rangle + \langle l,\mu\rangle - \nu_j} \right| \leqslant \left(1 + \frac{|\lambda_j| + |\mu_j|}{\epsilon}\right) \sum_{s=1}^{n} w^{(k-a,l-b)},$$

where $\nu \in \{\lambda, \nu\}$.

Thus, in any cases there exist $\delta > 0$ and $C > 0$ such that

$$\left|\phi_j^{(k,l)}\right| \leqslant \delta\left|\left[p_j(u+\phi, v+\psi)\right]^{(k,l)}\right| + C\sum_{s=1}^{n}\sum_{(a,b)}\left|\phi_j^{(a,b)} w_s^{(k-a,l-b)}\right|, \tag{22}$$

$$\left|\psi_j^{(k,l)}\right| \leqslant \delta\left|\left[q_j(u+\phi, v+\psi)\right]^{(k,l)}\right| + C\sum_{s=1}^{n}\sum_{(a,b)}\left|\psi_j^{(a,b)} w_s^{(k-a,l-b)}\right|. \tag{23}$$

For p_j and q_j to be analytic in a neighborhood of the origin, there exists a polydisc $\mathcal{D}$: $|x_s|, |y_s| \leqslant r$ on which p_j and q_j are analytic. From the Cauchy inequality we have

$$\left|p_j^{(k,l)}\right|,\ \left|q_j^{(k,l)}\right| \leqslant M r^{-|k|-|l|},$$

where $M = \max_j \sup_{\partial\mathcal{D}}\{|p_j|, |q_j|\}$. Set

$$\hat{p} = M\sum_{|k|+|l|=2}^{\infty} r^{-|k|-|l|} x^k y^l.$$

Then $\hat{p}$ is an analytic function in the interior of $\mathcal{D}$, and it is a majorant series of p_j, q_j for $j = 1, \ldots, n$. Consider the following majorant relations

$$\begin{aligned}\sum_{j=1}^{n}(\phi_j+\psi_j+\alpha_j+\beta_j) &\preccurlyeq \sum_{j=1}^{n}(\hat{\phi}_j+\hat{\psi}_j+\hat{\alpha}_j+\hat{\beta}_j)\\ &\preccurlyeq 2n(1+\delta)\hat{p}(u+\hat{\phi}, v+\hat{\psi}) + C\sum_{j=1}^{n}\sum_{s=1}^{n}(\hat{\phi}_j+\hat{\psi}_j)\hat{W}_s,\end{aligned} \tag{24}$$

where $\hat{\omega}$ denotes the corresponding majorant series of ω with $\omega \in \{\alpha, \beta, \phi, \psi, W\}$, and $\preccurlyeq$ shows the majorant relations between two power series (see for instance, [8]).

Set

$$\Pi(u,v) = \sum_{j=1}^{n}(\hat{\phi}_j+\hat{\psi}_j+\hat{\alpha}_j+\hat{\beta}_j).$$

Since all coefficients in Π are non-negative, it is sufficient to consider the case $u_1 = \cdots = u_n = v_1 = \cdots = v_n = \theta$. Let $\Pi(u,v) = R(\theta)\theta$ with R a function in the single variable θ. Then by the construction we have $R(0) = 0$. From the relation (24), we get the following

$$R(\theta)\theta \preccurlyeq 2n(1+\delta)\theta^2\hat{p}^*\big(1+R(\theta), 1+R(\theta)\big) + CR^2(\theta)\theta, \tag{25}$$

where we have used $W_s = \alpha_s/u_s$ or β_s/v_s, $\hat{p}(u+\hat{\phi}, v+\hat{\psi}) \preccurlyeq \hat{p}(u_1+\Pi(u,v), \ldots, u_n+\Pi(u,v), v_1+\Pi(u,v), \ldots, v_n+\Pi(u,v))$, and $\hat{p}^* = \hat{p}(\theta+\Pi, \ldots, \theta+\Pi, \theta+\Pi, \ldots, \theta+\Pi)/\theta^2$ a power series.

Consider the following function

$$\Phi(\theta,h) = h - 2n(1+\delta)\theta\hat{p}^*(1+h, 1+h) - Ch^2. \tag{26}$$

Clearly, the function Φ is analytic in θ, h. Since $\Phi(0,0)=0$ and $(\partial\Phi/\partial h)|_{(0,0)}=1$, the Implicit Function Theorem tells us the equation $\Phi(\theta,h)=0$ has an analytic solution $h(\theta)$ in a neighborhood of the origin.

Comparing (25) and (26), we get that $h(\theta)$ majorizes $R(\theta)$. This proves that $R(\theta)$ is analytic in some neighborhood of the origin. Therefore, $\Pi(u,v)$ is analytic, and consequently the power series $\phi_j, \psi_j, \alpha_j, \beta_j$ are analytic. Thus, we have proved that the vector field $\widetilde{\mathcal{X}}$ is analytically equivalent to the distinguished analytic normal form by the analytic distinguished normalization.

(c) Obviously, $z_1, \dots, z_n$ are the first integrals of (4). So, any formal power series in $z_1, \dots, z_n$ is a formal first integral. Conversely, if H is a formal first integral of (4), then working in a similar way to the proof of (15) we get the desired form of H.

(d) In order to prove this statement, we need some elementary facts on pluripolar set. A set $\mathbf{E} \subset \mathbb{C}^m$ is called *pluripolar* if for each $z \in \mathbf{E}$, there exists a neighborhood U of z and a plurisubharmonic function u on U for which $\mathbf{E} \cap U \subset u^{-1}(-\infty)$ (see for instance, [12,20]). Given an open subset Ω in $\mathbb{C}^m$. A function $u : \Omega \to [-\infty, \infty)$ is *plurisubharmonic* if it is upper semicontinuous, i.e. $\{z \in \Omega\colon\ u(x) < c\}$ is open for each $c \in \mathbb{R}$ and not identically $-\infty$ on any connected component of Ω, and for any $x \in \Omega$ we have

$$u(x) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u\big(x + e^{it} y\big)\, dt,$$

where y is any number in $\mathbb{C}^m$ satisfying $x + \delta y \in \Omega$ and $|\delta| \leqslant 1$. A pluripolar set in $\mathbb{C}^m$ has $2n$-dimensional Lebesgue measure 0. The countable union of pluripolar subsets is again pluripolar.

Let $\mathcal{L}$ be the set of plurisubharmonic functions in $\mathbb{C}^m$ with minimal growth in the sense that $u(z) - \log\|z\|$ is bounded above for $\|z\| \to \infty$. For any given subset $\mathbf{E} \subset \mathbb{C}^m$, define

$$V_{\mathbf{E}}(z) = \sup\big\{u(z);\ u \in \mathcal{L},\ u \leqslant 0 \text{ on } \mathbf{E}\big\}.$$

Then we have

Lemma *(Bernstein–Walsh). If $\mathbf{E} \subset \mathbb{C}^m$ is not pluripolar and $P(z)$ is a polynomial of degree d, then for $z \in \mathbb{C}^m$*

$$\big|P(z)\big| \leqslant \|P\|_{\mathbf{E}} \exp\big(d V_{\mathbf{E}}(z)\big).$$

This lemma is the key point to prove the following result, its proof follows from the idea of Pérez-Marco [17].

Lemma 3.1. *Each vector field in any affine finite-dimensional subspace $\mathcal{F}$ of $\mathcal{V}$ has a convergent distinguished normal form* (*respectively, normalization*), *or only the vector fields in an exceptional pluripolar subset of $\mathcal{F}$ have this property.*

Proof. If the second statement of Lemma 3.1 holds, we are done. So, we assume that there is a subset of $\mathcal{F}$ not pluripolar in which every vector field has a convergent distinguished normal form. Let $\mathcal{X}_1, \dots, \mathcal{X}_m$ be the $2n$-dimensional vector fields with the starting terms of order at least two, and let $\widetilde{\mathcal{X}}_0$ be the linear part of $\widetilde{\mathcal{X}}$. Consider $\mathcal{F}$ to be the m-dimensional vector space

$\{\widetilde{\mathcal{X}}_0 + t_1\mathcal{X}_1 + \cdots + t_m\mathcal{X}_m;\ t = (t_1, \ldots, t_m) \in \mathbb{C}^m\}$. Then $\mathcal{F} \subset \mathcal{V}$ is isomorphic to $\mathbb{C}^m$. Denote by $\mathcal{X}_t$ the vector field in $\mathcal{F}$.

Let $\mathcal{C}$ be the set of $t \in \mathbb{C}^m$ for which the corresponding vector field $\mathcal{X}_t \subset \mathcal{F}$ has a convergent distinguished normalization, and assume that it is not pluripolar. Write $\mathcal{C} = \bigcup_{r \geqslant 1} \mathcal{C}_r$, where $\mathcal{C}_r$ is set of t for which the vector field $\mathcal{X}_t$ has a convergent distinguished normalization Φ_t at least in the polydisc D_r of the radius $1/r$, and the normalization is bounded by 1 in D_r. By the assumption there exists a $\mathcal{C}_r$ which is non-pluripolar (otherwise $\mathcal{C}$ should be pluripolar).

According to the proof of statement (b), we write the normalization to be the form

$$\Phi_t(u, v) = \sum_{j \in \mathbb{Z}_+^{2n}} \Phi_j(t)(u, v)^j,$$

where $(u, v)^j$ is the multi-index, and $\Phi_j(t)$ are $2n$-dimensional vector-valued functions. From the construction of the normalization, especially the formulae (17)–(20), it follows that $\Phi_j(t)$ is a vector-valued polynomial of degree at most $|j|$. Since Φ_t is analytic in $\mathcal{C}_r$ by the construction, it follows from the Cauchy inequality that there exists a $\rho_0 > 0$ for which

$$\Psi(t) = \sup_j \big\|\Phi_j(t)\big\|_\infty \rho_0^{-|j|} < \infty, \quad t \in \mathcal{C}_r$$

where the norm $\|\cdot\|_\infty$ denotes the summation of the absolute values of components of a vector. Now the non-pluripolar set $\mathcal{C}_r$ can be represented as the union of the subsets $\{t \in \mathcal{C}_r;\ \Psi(t) \leqslant s,\ s \in \mathbb{N}\}$, in which there is a non-pluripolar set. Denote by $\mathcal{D}_s$ one of the non-pluripolar subsets. Choose $\Omega \subset \mathcal{D}_s$ to be a non-pluripolar compact set for which there exists $\rho_1 > 0$ such that for all $t \in \Omega$ and all j we have

$$\big\|\Phi_j(t)\big\|_\infty \leqslant \rho_1^{|j|}.$$

So, it follows from the Bernstein–Walsh lemma that for any compact subset $C \subset \mathbb{C}^m$ and $|j| \geqslant 2$ there exists a $\rho_2 > 0$ depending on C only for which the following holds

$$\|\Phi_j\|_C \leqslant \|\Phi_j\|_\Omega \exp\Big(|j| \max_{t \in C} V_\Omega(t)\Big) \leqslant \rho_1^{|j|}\rho_2^{|j|},$$

where $\|\Phi_j\|_\Omega = \max_{t\in\Omega} \|\Phi_j(t)\|_\infty$. This implies that for arbitrary t on any compact subset of $\mathbb{C}^m$, $\Phi_t(u, v)$ is convergent on the polydisc $\{(u, v):\ |u_i|, |v_i| \leqslant \min\{\frac{1}{\rho_1}, \frac{1}{\rho_2}\},\ i = 1, \ldots, n\}$. Consequently, it is an analytic diffeomorphism in a neighborhood of the origin in the (u, v) space for all $t \in \mathbb{C}^m$. We complete the proof of the lemma. □

Now the proof of statement (d) follows from Lemma 3.1 and the assumption that $\mathcal{V}$ contains a vector field having the divergent distinguished normalization or normal form.

4. Proof of Theorem 1.3

Denote by $\mathcal{Y}$ the planar analytic flow. Under the assumption of the theorem, the flow $\mathcal{Y}$ has an analytic first integral. We have the following three cases.

Case 1. One of the eigenvalues is zero. Then the other does not vanish, it follows from [14] that the singularity is non-isolated.

Case 2. The two eigenvalues are a pair of pure imaginary numbers. The classical Poincaré's result, see for instance [3], tells us the vector field $\mathcal{Y}$ is analytically equivalent, with a possible time rescaling by a non-zero constant, to

$$\dot{x} = x\big(i + g(xy)\big), \qquad \dot{y} = -y\big(i + g(xy)\big).$$

Obviously, it is Hamiltonian, and is orbitally equivalent to the linear vector field.

Case 3. The two eigenvalues are real, and their ratio is a negative rational number. Then we get from Theorem 1 of [21] that the vector field is analytically equivalent, with a possible time rescaling by a non-zero constant, to

$$\dot{x} = nx\big(1 + g\big(x^m y^n\big)\big), \qquad \dot{y} = -my\big(1 + g\big(x^m y^n\big)\big).$$

This proves the theorem.

5. Proof of Theorem 1.4

Recall that for a given differentiable function $L(x, y, t)$ of three variables, a curve $\gamma : x = x(t)$ for $t \in [t_0, t_1]$ is an extremal of the functional $\Phi(\gamma) = \int_{t_0}^{t_1} L(x, \dot{x}, t)\, dt$ on the space of curves passing through the points $x(t_0) = x_0$ and $x(t_1) = x_1$, if and only if $x(t)$ satisfies the Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{27}$$

Given a planar Hamiltonian vector field $\mathcal{X}$ with the Hamiltonian function $H(x, y)$. Assume that the origin is a linear center of $\mathcal{X}$. For otherwise, it is not isochronous. We now construct the action-angle coordinates (I, ϕ) according to the method of Arnold [1]. In the neighborhood of the origin, every closed orbit is a level curve $H = h$, denoted by C_h, for $h \in (0, h_0)$ with h_0 finite or infinite. Set

$$I = \frac{1}{2\pi}\Pi(h) = \frac{1}{2\pi}\int\limits_{C_h} y\, dx.$$

We remark that $\Pi(h)$ is the area of the domain enclosed by C_h. Choosing ϕ as the usual angle variable. Clearly, the transformation from (x, y) to (I, ϕ) is analytic. The Hamiltonian vector field $\mathcal{X}$ under this action-angle coordinates is of the form

$$\dot{I} = 0, \qquad \dot{\phi} = \partial_I H(I).$$

The period of the closed orbit C_h is

$$T(h) = \int\limits_0^{2\pi} \frac{d\phi}{\partial_I H(I)}.$$

Set $L(I, \dot{I}, \phi) = (\partial_I H(I))^{-1}$. The center is isochronous if and only if $T(h)$ is constant, and if and only if all the closed orbits C_h are the extremal of the functional $T(h)$. So on all the closed orbits the Euler–Lagrange equation holds, i.e.

$$\frac{\partial L}{\partial I} = \frac{d}{d\phi}\left(\frac{\partial L}{\partial \dot{I}}\right) = 0,$$

because in this case L is independent of $\dot{I}$. This last equation means that L is independent of I, too. Therefore, $H(I)$ is a linear function in I. This proves the theorem.

Remark 5.1. Let $I(h)$ be the inverse function of $H(I) = h$. Since

$$T(h) = \frac{2\pi}{\dot{\phi}} = \frac{2\pi}{\partial_I H(I)} = 2\pi\,\partial_h I(h) = \partial_h \Pi(h),$$

the origin is an isochronous center if and only if the area of the domain enclosed by the closed orbit C_h is a linear function of h.

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