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Analysis issues in Petri nets with inhibitor arcs

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Abstract

We investigate the problem of extending the analysis techniques developed for P/T systems to a proper subclass of P/T systems with inhibitor arcs. We start proposing an extension of the coverability tree construction to a subclass of P/T systems with inhibitor arcs, whose elements will be called henceforth primitive systems. We show that the coverability tree corresponding to a primitive system is finite and is a good representation of its behaviour; hence, it can be used as an analysis tool to check properties such as place boundedness, the existence of dead transitions and of a reachable marking larger than a given one.

Then we provide an encoding of primitive systems in P/T systems, which permits to retrieve the firing sequences of the primitive system from the firing sequences of the corresponding P/T system. The close correspondence between the firing sequences of the two systems is used prove the decidability of reachability, deadlock and liveness for primitive systems. We also obtain that the model checking problem for the linear time μ-calculus and labelled primitive systems is decidable. We show that primitive systems coincide with the largest class of P/T systems with inhibitor arcs whose transition sequences can be simulated by a standard P/T system; we also show that in general the step behaviour of a primitive system cannot be simulated by any P/T system. These results are then used to investigate the expressiveness of inhibitor arcs regarding the class of generated languages.

1. Introduction

Place/Transition Petri nets [22] are a widely used formalism for representing concurrent systems. However, in some cases they have revealed to be not adequate to model situations dealing e.g. with priorities. For this reason, the basic model has been extended with the so-called *inhibitor arcs* [11, 9], permitting to test for absence of a resource for a transition to fire.

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An attractive characteristic of P/T systems is the existence of a large amount of analysis techniques, permitting to decide properties of systems such as liveness, deadlock and reachability [12, 13, 17, 10, 8]. The addition of inhibitor arcs makes P/T systems Turing equivalent [1]. If on one side this greatly enhances the expressive power of the formalism, on the other side it prevents most analysis techniques for standard P/T systems from being generalized to the whole class of P/T systems with inhibitor arcs.

In this paper we face with the problem of extending the analysis techniques to a subclass of P/T systems with inhibitor arcs, for which it is still possible to decide interesting properties. To this aim, we identify a subclass of P/T systems with inhibitor arcs, that we call primitive systems, for which it is possible to construct the coverability tree [12], a useful tool to check properties such as finiteness of the state space, boundedness of places or deadness of transitions. The coverability tree is a finite approximation of the (possibly infinite) firing sequences and of the reachable markings of a P/T system; the basic idea is to represent an infinite set of markings, differing only for the contents of a place, with an extended marking, with the symbol ω associated to that place. This construction is well behaving because of the monotonicity property of P/T systems: if a sequence of transitions is firable at a given marking, then it is also firable at any bigger marking. The problem of extending such a construction to P/T systems with inhibitor arcs is due to the loss of monotonicity, caused by the introduction of inhibitor arcs: when dealing with P/T systems with inhibitor arcs, the approximation of the number of tokens in a place with ω causes a loss of information about the firability of a transition inhibited by that place. To overcome this problem, we restrict ourselves to a class of P/T systems with inhibitor arcs, called primitive systems, satisfying the following property: it is possible to associate an emptiness limit to each inhibiting place such that, whenever the number of tokens in the place exceeds the limit, there is no subsequent transition testing that place for absence. We modify the construction of the coverability tree by approximating the contents of an inhibiting place with ω only after the corresponding limit has been exceeded. We show that the coverability tree of a primitive system is finite; moreover, we show also that the coverability tree is a good representation of the behaviour of the primitive system, in the sense that each reachable marking is represented in the coverability tree, directly or by an ω approximation, and that ω symbols appearing in the coverability tree represent places that will hold an unbounded number of tokens. Finally, we show that the coverability tree can still be used to check the properties of place boundedness, the existence of dead transitions and of a reachable marking containing a given marking.

Note that, in general, it is impossible to decide if a given P/T system with inhibitor arcs is primitive (see the discussion in Section 4.1); however, in practical cases we may know the emptiness limits because of the particular structure of the net. For example, in [2] we show that the subclass of P/T systems with inhibitor arcs, corresponding to π-calculus [18] terms according to the net semantics for the π-calculus proposed in [3], contains only primitive systems; hence, in the case the obtained system is also finite, we can adopt the analysis techniques developed in this paper to study the behaviour of π-calculus terms. In Section 4.1 we list also some sufficient conditions, based on invariants or on the coverability tree of the P/T system obtained by dropping the inhibitor arcs, for a system to be primitive.

We have found a way for constructing the coverability tree for primitive systems, but, due to the ω-approximations, the coverability tree cannot be used to decide many interesting properties, such as reachability and liveness. In Section 5 we show a way to decide such properties by providing an encoding of primitive systems in P/T systems, which permits to retrieve the firing sequences of the primitive system from the firing sequences of the corresponding P/T system. The construction consists in representing each inhibiting place s by a set of places $s^0, s^1, \ldots, s^{EL(s)}, s^\omega$, where $EL(s)$ is the emptiness limit of s; the fact that place s contains k tokens is represented either by 1 token in place s^k (if $k \leqslant EL(s)$) or by $k+1$ tokens in place s^ω; each transition connected with s is split in a set of transitions, each one managing a specific representation of the contents of place s in the corresponding P/T system; the inhibitor arcs on s are replaced by a self-loop on s^0. We show the existence of a close correspondence between the firing sequences of the primitive system and the firing sequences of the corresponding P/T systems; this permits to prove the decidability of reachability, deadlock and liveness for primitive systems, by reduction to a similar problem for P/T systems; moreover, we also obtain that the model checking problem for the linear time μ-calculus and labelled primitive systems is decidable.

We introduce a notion of simulation of P/T systems with inhibitor arcs by P/T systems (consisting of a labelling of transitions of the P/T system by transitions of the P/T system with inhibitor arcs) which reflects and preserves transition sequences and we show that primitive systems are the largest class of P/T systems with inhibitor arcs that can be simulated by a P/T system. The construction we have introduced permits to simulate the firing sequences, but in general step firing sequences are not preserved; we show that we cannot to better, by exhibiting a primitive system for which there exists no P/T system with the same step behaviour. Finally, we make use of these results to compare the expressiveness of P/T systems and P/T systems with inhibitor arcs regarding the set of generated languages: we have that primitive systems are equivalent to P/T systems, whereas the class of languages generated by P/T systems with inhibitor arcs is strictly greater than the class of languages generated by P/T systems. To obtain a gap in expressiveness between P/T and primitive systems, we have to move to step-languages, that are sets of sequences of multisets of labels: the class of step-languages generated by primitive systems is strictly larger than the class of step-languages generated by P/T systems.

The paper is organized as follows. After recalling some basic definitions in Section 2 and some analysis techniques developed for P/T systems in Section 3, in Section 4 we extend the coverability tree construction to deal with primitive systems. In Section 5 we present an encoding of primitive systems in P/T systems, and we show the decidability of reachability, deadlock and liveness for primitive systems; we also show that primitive systems are the largest class of systems whose behaviour can be simulated by a P/T system. Section 6 contains expressiveness results on the class of languages generated by primitive systems and in Section 7 we report some conclusive remark. The appendix contains a sketch of the proof of Turing equivalence of P/T systems with inhibitor arcs.

2. Basic definitions

In this section we give some preliminary definition, followed by the definitions of P/T systems and of P/T systems with inhibitor arcs.

Let ω be the set of natural numbers and $\omega^+ = \omega \backslash \{0\}$.

2.1. Relations

A relation over a set X is a subset of $X \times X$. The composition of two relations is defined as $R \circ R' = \{(x,z) \mid \exists y(xRy \wedge yR'z)\}$. We denote by R^+ (R^*) the transitive (reflexive and transitive) closure of R.

2.2. Sequences and languages

A finite *sequence* (word, string), of length n, over a set X is a function from $\{1,\dots,n\}$ to X; it is usually represented as $x_1x_2\dots x_n$, with $x_i \in X$ for $i = 1,\dots,n$. The empty sequence, i.e. the sequence of length 0, is denoted by ε. Given a finite sequence σ, we denote by σ^k the sequence obtained by concatenating k occurrences of σ, i.e. $\underbrace{\sigma\dots\sigma}_{k}$.

An infinite sequence over X is a function from ω^+ to X. We usually represent it as $x_1x_2\dots x_i\dots$, with $x_i \in X$ for $i \in \omega^+$.

We say that $x_{j_1}x_{j_2}\dots x_{j_i}\dots$ is an infinite *subsequence* of $x_1\dots x_n\dots$ if $1 \leqslant j_1 < j_2 < \cdots < j_i < \cdots$.

We denote by X^* and X^ω the set of respectively finite and infinite sequences over X, and $X^\infty = X^* \cup X^\omega$.

A *language* over X is a subset of X^*. A ∞-*language* over X is a subset of X^∞. A ω-*language* over X is a subset of X^ω.

2.3. Multisets

Definition 2.1. Given a finite set S, a *multiset* over S is a function $m: S \to \omega$. The *domain* of a multiset is the set $dom(m) = \{s \in S \mid m(s) \neq 0\}$; a multiset m such that $dom(m) = \emptyset$ is said to be *empty*. The *multiplicity* of an element s in m is given by the natural number $m(s)$. The set of all finite multisets over S, denoted by $\mathscr{M}_{\text{fin}}(S)$, is ranged over by m. A multiset m such that $dom(m) = \emptyset$ is called *empty*. The set of all finite sets over S is denoted by $\wp_{\text{fin}}(S)$.

The cardinality of a multiset is defined as $|m| = \sum_{s \in S} m(s)$. We write $m \subseteq m'$ if $m(s) \leqslant m'(s)$ for all $s \in S$. The operator $\oplus$ denotes *multiset union*: $m \oplus m'(s) = m(s) + m'(s)$. The operator $\backslash$ denotes *multiset difference*: $m \backslash m'(s) =$ if $m(s) > m'(s)$ then

$m(s) - m'(s)$ else 0. The *scalar product* of a number j with a multiset m is $(j \cdot m)(s) = j \cdot (m(s))$.

2.4. Graphs

Definition 2.2. A *rooted graph* labelled over the set Act is a tuple (N, A, x_0), where
- N is the set of nodes;
- $A \subseteq N \times Act \times N$ is the set of labelled arcs;
- $x_0 \in N$ is the root.

We denote arc (x, a, y) by $x \xrightarrow{a} y$. Given $n \geqslant 0$, a *path* from x_1 to x_{n+1} is a (possibly empty) sequence of arcs $x_1 \xrightarrow{a_1} \cdots x_n \xrightarrow{a_n} x_{n+1}$.

Definition 2.3. A *tree* (N, A, x_0) is a rooted graph such that, for each $x \in N$, there exists a unique path from x_0 to x.

Definition 2.4. Let $(N_i, A_i, x_{0,i})$, $i = 1, 2$, be two graphs. A *bisimulation* is a relation $\mathscr{R} \subseteq N_1 \times N_2$ such that
- $(x_{0,1}, x_{0,2}) \in \mathscr{R}$;
- if $(x, y) \in \mathscr{R}$ and $x \xrightarrow{a} x'$ then there exists y' such that $y \xrightarrow{a} y'$ and $(x', y') \in \mathscr{R}$;
- if $(x, y) \in \mathscr{R}$ and $y \xrightarrow{a} y'$ then there exists x' such that $x \xrightarrow{a} x'$ and $(x', y') \in \mathscr{R}$.

2.5. P/T nets

Definition 2.5. A P/T net is a tuple $N = (S, T, F)$, where
- S and T are finite sets of *places* and *transitions*, such that $S \cap T = \emptyset$;
- $F : (S \times T) \cup (T \times S) \to \omega$ is the *flow function*.

A multiset over the set S of places is called a *marking*. Given a marking m and a place s, we say that the place s contains $m(s)$ *tokens*. If $F(x, y) > 0$, we say that there is an arc from x to y with weight $F(x, y)$.

The *preset* of a transition t is the multiset ${}^\bullet t(s) = F(s, t)$, and represents the tokens to be “consumed”; the *postset* of t is the multiset $t^\bullet(s) = F(t, s)$, and represents the tokens to be “produced”. If $F(s, t) = F(t, s) > 0$, we say that there is a *self-loop* on s and t.

A transition t is *enabled* at m if ${}^\bullet t \subseteq m$. The execution of a transition t enabled at m produces the marking $m' = (m \setminus {}^\bullet t) \oplus t^\bullet$. This is usually written as $m[t\rangle m'$. A non empty multiset over the set T is called a *step*.

A step G is enabled at m if $m_1 \subseteq m$, where $m_1 = \bigoplus_t G(t) \cdot {}^\bullet t$. The execution of a step G enabled at m produces the marking $m' = (m \setminus m_1) \oplus m_2$, where $m_2 = \bigoplus_t G(t) \cdot t^\bullet$. This is written as $m[G\rangle m'$. A P/T system is a tuple $N(m_0) = (S, T, F, m_0)$, where (S, T, F) is a P/T net and m_0 is a multiset over S, called the *initial marking*. A *labelled* P/T net (system) over a set Act of labels is a tuple (S, T, F, l) $((S, T, F, m_0, l))$, where (S, T, F) is a P/T net and $l : T \to Act$ is the labelling function.

Any P/T system can be regarded as a labelled P/T system over the set T of transitions, where the labelling function is the identity function over T.

Definition 2.6. A *firing sequence starting at marking m* is defined inductively as follows:

- m is a firing sequence;
- if $m[t_1\rangle m_1 \ldots [t_{n-1}\rangle m_{n-1}$ is a firing sequence and $m_{n-1}[t_n\rangle m_n$ then $m[t_1\rangle m_1 \ldots [t_{n-1}\rangle m_{n-1}[t_n\rangle m_n$ is a firing sequence.

We simply call *firing sequence* a firing sequence starting at the initial marking m_0.

Given a firing sequence $m[t_1\rangle \ldots [t_n\rangle m_n$, we call $t_1 \ldots t_n$ a *transition sequence starting at m*. We often write $m[t_1 \ldots t_n\rangle m'$ to mean that there exist $m_1, \ldots, m_{n-1}$ such that $m[t_1\rangle m_1 \ldots m_{n-1}[t_n\rangle m'$.

The set of markings *reachable* from m, denoted by $[m\rangle$, is defined as the least set of markings such that

- $m \in [m\rangle$,
- if $m_1 \in [m\rangle$ and, for some transition $t \in T$, $m_1[t\rangle m_2$ then $m_2 \in [m\rangle$.

We say that a marking m is reachable if m is reachable from the initial marking m_0.

The *interleaving marking graph* of N is $IMG(N) = (\mathscr{M}_{\mathrm{fin}}(S), \rightarrow, m_0)$, where $\rightarrow \subseteq \mathscr{M}_{\mathrm{fin}}(S) \times Act \times \mathscr{M}_{\mathrm{fin}}(S)$ is defined by $m \xrightarrow{l(t)} m'$ iff there exists a transition $t \in T$ such that $m[t\rangle m'$.

The systems N_1 and N_2 are *bisimilar* ($N_1 \sim N_2$) iff there exists a bisimulation R relating $IMG(N_1)$ and $IMG(N_2)$.

A *step firing sequence starting at marking m* is defined inductively as follows:

- m is a step firing sequence;
- if $m[G_1\rangle m_1 \ldots [G_{n-1}\rangle m_{n-1}$ is a step firing sequence and $m_{n-1}[G_n\rangle m_n$ then $m[G_1\rangle m_1 \ldots [G_{n-1}\rangle m_{n-1}\ [G_n\rangle m_n$ is a step firing sequence.

Given a step firing sequence $m[G_1\rangle \ldots [G_n\rangle m_n$, we call $G_1 \ldots G_n$ a *step transition sequence*.

The *step marking graph* of N is $SMG(N) = (\mathscr{M}_{\mathrm{fin}}(S), \rightarrow, m_0)$, where $\rightarrow \subseteq \mathscr{M}_{\mathrm{fin}}(S) \times \mathscr{M}_{\mathrm{fin}}(Act) \times \mathscr{M}_{\mathrm{fin}}(S)$ is defined by $m \xrightarrow{A} m'$ iff there exists a step G such that $m[G\rangle m'$ and $A = l(G)$.

The systems N_1 and N_2 are *step bisimilar* ($N_1 \sim_{\mathrm{step}} N_2$) iff there exists a bisimulation R relating $SMG(N_1)$ and $SMG(N_2)$.

2.6. P/T nets with inhibitor arcs

Definition 2.7. A *P/T net with inhibitor arcs* (PTI net for short) is a tuple $N = (S, T, F, I)$ where

- (S, T, F) is a P/T net;
- $I \subseteq S \times T$ is the *inhibiting relation*.

The *inhibitor set* of a transition t is the set ${}^{\circ}t = \{s \in S \mid (s, t) \in I\}$, and represents the places to be "tested for absence" of tokens.

This changes the definition of enabling: a transition t is enabled at m if ${}^\bullet t \subseteq m$ and $dom(m) \cap {}^\circ t = \emptyset$. Any transition t for which ${}^\circ t \cap dom({}^\bullet t) \neq \emptyset$ can never fire, thus it is called *blocked*.

The execution of a transition t enabled at m producing the marking m', written $m[t\rangle m'$, is defined as for P/T nets.

We say that s is an *inhibiting place* if there exists a transition t such that $s \in {}^\circ t$. We denote with $Inib(N)$ the set of inhibiting places of the net N.

A PTI system is a tuple $N(m_0) = (S, T, F, I, m_0)$, where (S, T, F, I) is a contextual P/T net and m_0 is a multiset over S, called the *initial marking*.

We adopt the usual notation to draw P/T nets: places are represented as circles, transitions as segments, flow arcs as directed segments (i.e. with an arrow at one end) and tokens as black dots inside the place. We represent an inhibitor arc as a line terminating with a small circle on the transition side.

2.6.1. Step semantics

The definition of step semantics we present here is inspired by the definition given for contextual C/E nets [1] in [20]. According to our definition, two transitions can happen in the same step iff they can happen in either order. We have to check that not all tokens in a place tested for presence by (an occurrence of) a transition are consumed by the others and that (an occurrence of) a transition does not produce tokens in a place tested for absence by another.

A step G is enabled at m iff

- $m_1 \subseteq m$, where $m_1 = \bigoplus_t G(t) \cdot {}^\bullet t$,
- for all $t \in dom(G)$ ${}^\circ t \cap dom(m) = \emptyset$,
- for all $t_1, t_2 \in dom(G)$, such that $t_1 = t_2 \Rightarrow G(t_1) \geqslant 2$, we have that $dom(t_1^\bullet) \cap {}^\circ t_2 = \emptyset$.

The third condition ensures that, for each pair of occurrences of transitions in the step, it never happens that one occurrence puts a token in a place inhibiting the other one. The execution of a step G enabled at m producing the marking m', written $m[G\rangle m'$, is defined as for P/T nets.

The following proposition enunciates an important property of our definition of step enabling: if a step G can fire then, for any way of dividing it in two substeps, these substeps can fire in sequence.

Proposition 2.8. *Let G_1, G_2 be steps and $G = G_1 \oplus G_2$. Then $m[G\rangle m'$ iff there exists m_1 such that $m[G_1\rangle m_1[G_2\rangle m'$.*

Hence, given a firable step, any firing sequence obtained by sequentializing that step is firable:

[1] C/E nets are essentially nets with arc weights not greater than one and containing at most one token per place in any reachable marking; the latter condition is realized by imposing a constraint on the firability of transitions, i.e. by forbidding the firing of a transition if some place in its postset (but not in its preset) already contain a token. Contextual C/E nets are C/E nets enriched with inhibitor arcs and read as arcs (i.e., arcs testing for presence of tokens).

Corollary 2.9. *Let G be a step of cardinality n. If $m[G\rangle m'$ then for any sequence of transitions $t_1,\ldots,t_n$, such that $G(t)=|\{i \mid 1\leqslant i\leqslant n \wedge t_i=t\}|$, we have $m[t_1\rangle\ldots[t_n\rangle m'$.*

Finally, any marking reached by a step firing sequence can by reached from a firing sequence:

Corollary 2.10. *If $m[G_1\rangle m_1\ldots[G_n\rangle m_n$ then there m_n is reachable from m.*

A comparison of this notion of step with other notions appeared in the literature can be found in [5].

3. Analysis of P/T systems

We recall some classic results on the analysis of P/T systems (see e.g. [21]).

3.1. Coverability tree

The coverability tree is a finite representation of the firing sequences, hence also of the reachable markings, of a P/T system. It has been introduced, under the name of reachability tree, in [12]. As the set of reachable markings is in general infinite, we need to find a finite approximation, that is sufficiently good to permit to decide interesting properties about the system. In the following we first illustrate the intuition and then we give the algorithm for the construction of the coverability tree; hence we list the satisfied properties and the decision procedures that can be deduced.

The coverability tree is a finite representation of the reachable markings of a net: a node represents a set of (approximation of) markings, whereas arcs represent the firing of transitions: an arc, labelled with a transition, represents the fact that the transition is enabled at the marking(s) corresponding to the source node, whereas the target node represents the marking produced after the firing of that transition. The naive construction proceeds as follows: initially, the coverability tree contains only one node, corresponding to the initial marking of the net; for each transition enabled in this marking, we add an arc and a new node corresponding to the marking that we reach after firing the transition, and repeat this step for all the new nodes. Unfortunately this procedure may easily lead to an infinite tree. The reduction to a finite structure is accomplished in the following way. First of all, we need to limit the new markings (called *frontier nodes*) added at each step. Two classes of nodes are useful for this task:

- nodes corresponding to dead markings, i.e. marking at which no transition is enabled, do not produce further nodes; thus they are called *terminal nodes*;
- nodes corresponding to markings previously appeared in the tree; they are called *duplicate nodes* and need not to be considered: all its successors have already been produced from the first occurrence of the marking in the tree.

This is not sufficient for unbounded nets, i.e. nets whose places can be filled with an unlimited number of tokens. The problem is that the set of reachable markings of these nets is infinite: we need to abstract from markings without loosing the firability information.

Consider a transitions sequence σ which starts at a marking m and ends at a marking m', with $m' \supseteq m$; m' is the same as m except it has some "extra" token in some places, that is, $m' = m \oplus (m' \backslash m)$, with $m' \backslash m \neq \emptyset$. Since transition firings are not affected by extra tokens, the sequence σ can be fired at marking m', leading to a marking m''. Since the effect of the sequence of transitions σ is to add $m' \setminus m$ tokens to the marking m', so $m'' = m' \oplus (m' \backslash m) = m \oplus 2 \cdot (m' \setminus m)$. In general, we can fire the sequence σ n times to produce a marking $m \oplus n \cdot (m' \setminus m)$. Thus, we can produce an arbitrarily large number of tokens in each place s such that $(m' \backslash m)(s) > 0$ by iterating the firing of the sequence σ. We represent the infinite set of markings which result from these types of loops by using the symbol ω to represent a number of tokens that can be made arbitrarily large. We extend markings to allow the number of tokens in a place to be either a natural number or the ω symbol:

Definition 3.1. For any natural number n, we define $\omega + n = \omega - n = \omega$, $n < \omega$ and $\omega \leqslant \omega$.

An *extended marking* over the set S is a function $m : S \to \omega \cup \{\omega\}$.

We say that an extended marking m' *covers* a marking m if $m \subseteq m'$ and, for all places $s \in S$, $m'(s) \neq m(s)$ implies $m'(s) = \omega$.

In this way, each reachable marking either appears explicitly in the tree or there exists an extended marking that covers it.

The algorithm for the construction of the coverability tree, reported in Table 1, proceeds as follows: each node x in the tree is associated with an extended marking $M[x]$ and is classified as a frontier, terminal, duplicated or internal node. Frontier nodes are those which have not been processed yet; they are converted by the algorithm in terminal, duplicated or internal nodes. The algorithm begins by defining the initial marking to be the root of the tree and, initially, a frontier node. As long as frontier nodes remain, they are processed by the algorithm. The algorithm terminates when no frontier nodes are left in the coverability tree.

The finiteness of the coverability tree generated by the algorithm can be shown with the help of the following lemmata:

Lemma 3.2 (König). *In any infinite, finitely branching tree there exists an infinite path starting from the root.*

Lemma 3.3. *Every infinite sequence of elements in $\omega \cup \{\omega\}$ contains an infinite subsequence which satisfies one of the following conditions*:

- *either the elements in the subsequence are all equal*,
- *or they appear in a strictly increasing order.*

Table 1
Algorithm for the construction of the coverability tree

Let $N=(S,T,F,m_0)$ be a finite P/T system.
Create a tree formed by a single frontier node x_0 (the root node) with extended marking $M[x_0]=m_0$.
While the set of frontier nodes is not empty do the following:
Let x be a frontier node to be processed.

(1) If there exists another node y in the tree which is not a frontier node, such that $M[y]=M[x]$, then classify x as a duplicate node.
(2) If no transitions are enabled for the marking $M[x]$, then classify x as a terminal node.
(3) For all transitions $t\in T$ which are enabled at marking $M[x]$, let $m'=(M[x]\backslash{}^\bullet t)\oplus t^\bullet$ be the marking reached after the firing of transition t at marking $M[x]$. Create a new node z, classified as a frontier node, in the coverability tree. The extended marking $M[z]$ associated with the new node is defined as follows: for each place $s\in S$,
   (a) if $m'(s)\neq\omega$ and there exists a node y on the path from the root node to x with $M[y]\subset m'$ and $M[y](s)<m'(s)$, then $M[z](s)=\omega$;
   (b) otherwise, $M[z](s)=m'(s)$.
   Add an arc, labelled with t and directed from node x to node z. Node x is redefined as an internal node.

Lemma 3.4 (Dickson). *Every infinite sequence of extended markings contains an infinite subsequence ordered w.r.t.* $\subseteq$.

Theorem 3.5. *The coverability tree of a P/T system is finite.*

The coverability tree satisfies the properties listed below.

All reachable markings are covered by some marking in the coverability tree, and each firing sequence is represented in the coverability tree in the following way:

Lemma 3.6. *For each firing sequence* $m_0[t_1\rangle\dots[t_n\rangle m_n$ *there exists a sequence of nodes and arcs* $x_0\xrightarrow{t_1}y_1x_1\xrightarrow{t_2}y_2\dots x_{n-1}\xrightarrow{t_n}y_n$ *in the coverability tree such that*:

- x_0 *is the root of the tree*;
- $M[y_i]=M[x_i]$ *for* $i=1,\dots,n-1$;
- m_i *is covered by* $M[x_i]$ *for* $i=0,\dots,n-1$ *and* m_n *is covered by* $M[y_n]$.

The ω-components in the extended markings appearing in the coverability tree effectively correspond to places that will hold an unlimited number of tokens:

Lemma 3.7. *Let* z *be a node of the coverability tree and* $k>0$. *Then there exists a marking* $m\in[m_0\rangle$ *such that, for all* $s\in S$,

- *if* $M[z](s)<\omega$ *then* $m(s)=M[z](s)$;
- *if* $M[z](s)=\omega$ *then* $m(s)>k$.

The coverability tree is a useful tool for the analysis of nets. Indeed, some problems about coverability, boundedness and liveness can be reduced to properties of the coverability tree. In the following we illustrate some decision procedures for properties of a P/T system, based on the coverability tree.

Definition 3.8. A place s is *bounded* if there exists $k>0$ such that, for all $m \in [m_0\rangle$, $m(s) \leqslant k$. The P/T system N is bounded if there exists $k>0$ such that, for all $s \in S$ and $m \in [m_0\rangle, m(s) \leqslant k$.

Boundedness is decidable.

Theorem 3.9. *A place s is bounded iff, for all nodes x in the coverability tree*, $M[x](s) \neq \omega$.

Corollary 3.10. *The P/T system N is bounded if the symbol ω does not appear in any node of the coverability tree.*

We have that boundedness of a P/T system corresponds to have a finite set of reachable markings; it is easy to see that, if the system is bounded, the coverability tree contains a node corresponding to each reachable marking; in that case, a larger number of properties are decidable.

From the coverability tree we can find dead transitions, i.e. transitions that can never fire:

Definition 3.11. A transition t is dead if, for all $m \in [m_0\rangle$, t is not enabled at m.

Theorem 3.12. *The transition t is dead if no t-labelled arc occurs in the coverability tree.*

The coverability problem consists in finding a reachable marking m' that is larger than a given marking m, i.e. $m \subseteq m'$. This problem can be solved with the inspection of the nodes of the coverability tree:

Theorem 3.13. *Let m be a marking. There exists a marking* $m' \in [m_0\rangle$, *such that* $m \subseteq m'$, *iff there exists a node x in the coverability tree such that* $m \subseteq M[x]$.

A net is conservative w.r.t. a (nonnegative) weighting vector if the weighted sum of tokens is constant over all reachable markings. We can check if a net is conservative by computing the weighted sum for all markings associated to the nodes of the coverability tree: if the sums are the same for all nodes, then the net is conservative w.r.t. the given weighting vector. Note that, by Lemma 3.7, if there exists a node x such that $M[x](s)=\omega$, then the weight of place s must be 0.

3.2. Reachability

We recall that a marking is reachable if there exists a firing sequence (starting from the initial marking) leading to it.

The reachability problem for a net consists in deciding if a given marking is reachable from the initial marking. The reachability problem is decidable [13, 17] and known to require exponential space [16], but none of the algorithms known so far is primitive recursive.

Theorem 3.14. *The reachability problem is decidable for P/T systems.*

The submarking reachability problem is a generalization of the above problem; it amounts to decide if, given a subset $S' \subseteq S$ of places and a marking m, there exists a marking $m' \in [m_0\rangle$ such that $m'(s) = m(s)$ for all $s \in S'$. The submarking reachability problem is reducible to the reachability problem, hence it is decidable.

3.3. Liveness

A transition t is *live* if for each marking m reachable from m_0 there exists a marking m' reachable from m such that t is enabled at m. A net is *live* if each of its transitions is live. The liveness problem for a net consists of deciding if it is live; it has been shown to be decidable by reduction to the reachability problem (see e.g. [10]).

Theorem 3.15. *The liveness problem is reducible to the reachability problem*, *hence it is decidable.*

3.4. Deadlock

A marking m is *dead* if it does not enable any transition, i.e. $\neg m[t\rangle$ for all $t \in T$. A net has a *deadlock* if there exists a dead marking reachable from m_0. The deadlock problem for a net consists in deciding if it has a deadlock.

The deadlock problem for a finite P/T system is decidable; this can be proved by reducing it to the liveness problem, which is known to be decidable.

The theorem below is a slight generalization of the result proved in [6] for nets where the preset and postset of transitions are sets.

Theorem 3.16. *Deadlock is reducible to liveness.*

Proof. Let $N = (S, T, F, m_0)$ be a P/T system. We construct a net $N' = (S', T', F', m'_0)$, where $S' = S \cup \{ok\}$, $T' = \{t', t'' \mid t \in T\} \cup \{live\}$,

$$F'(x,y) = \begin{cases} F(x,y) & \text{if } x \in S \wedge \exists t \in T(y = t' \vee y = t''), \\ F(x,y) & \text{if } \exists t \in T(x = t') \wedge y \in S, \\ 1 & \text{if } \exists t \in T(x = t'') \wedge y = ok, \\ 1 & \text{if } x = ok \wedge y = live, \\ 1 & \text{if } x = live \wedge y = ok, \\ \sum_{t \in T} F(y,t) & \text{if } x = live \wedge y \in S, \\ 0 & \text{otherwise} \end{cases}$$

and $m'_0 = m_0$.

We show that N has no reachable dead marking iff N' is live.

Suppose that N has a reachable dead marking m; also N' can reach the marking m by firing the corresponding t' transitions. We have $m(ok) = 0$, then the transition *live* is not enabled at m; as transitions t' and t'' have the same preset of the corresponding transition t, also these transitions are not enabled at m in N'; then m is a dead marking for N', hence N' is not live.

Suppose N has no reachable dead markings. Let m be a reachable marking in N'; two cases can happen:

- $m(ok) > 0$; the transition *live* is enabled at m, and after the firing of *live* each transition in T' is enabled;
- $m(ok) = 0$; note that if we produce a token in the place ok it will remain always marked; then only transitions t' have been fired to reach m, hence m is reachable also in N; as N has no reachable dead markings, there exists $t \in T$ such that $m[t\rangle$ in N; then $m[t''\rangle m'$ in N', and $m'(ok) = 1$, leading to the previous case. □

3.5. Model checking

We recall the decidability of the model checking problem for P/T systems and closed formulas of the linear time μ-calculus [8].

The linear time μ-calculus is a powerful linear time logics, largely used for verification.

In the following we assume that *Act* is a denumerable set of symbols.

Definition 3.17. Let N be a labelled P/T system. We can extend the notion of firing sequences to deal with infinite firing sequences. We say that $m_0[t_1\rangle m_1 \ldots [t_i\rangle m_i \ldots$ is an infinite firing sequence if $m_0[t_1\rangle m_1 \ldots [t_i\rangle m_i$ is a firing sequence, for each $i \in \omega$.

Definition 3.18. Let N be a labelled P/T system. The language, ω-language and ∞-language of N are defined respectively as

- $L(N) = \{a_1 \ldots a_n \mid m_0[t_1\rangle \ldots [t_n\rangle m_n$ is a firing sequence of N and $l(t_i) = a_i$ for $i = 1, \ldots, n\}$.
- $L^\omega(N) = \{a_1 \ldots a_i \ldots \mid m_0[t_1\rangle \ldots [t_i\rangle m_i \ldots$ is an infinite firing sequence of N and $l(t_i) = a_i$ for $i \in \omega^+\}$.
- $L^\infty(N) = L(N) \cup L^\omega(N)$.

The syntax of the modal μ-calculus is the following:

$$\phi = Z | \neg\phi | \phi \wedge \phi | (a)\phi | \nu Z.\phi$$

where a ranges over a set *Act* of actions and Z over a set of propositional variables. Free and bound occurrences of variables are defined as usual. A formula is closed if no variable occurs free in it. Formulas are generated by the grammar above, and subject to the monotonicity condition that all free occurrences of a variable Z lie inside the scope of an even number of negations.

A *valuation* V of the logics maps each variable Z on a subset of Act^{∞}. The valuation $V[A/Z]$ is defined as

$$V[A/Z](Z') = \begin{cases} A & \text{if } Z' = Z, \\ V(Z') & \text{otherwise.} \end{cases}$$

Given a word $\sigma = a_1 \ldots a_i \ldots$ over Act^{∞}, with $\sigma(1)$ we denote the first action of σ, i.e. a_1, and with σ^1 we denote the word obtained from σ by dropping the first action, i.e. $a_2 \ldots a_i \ldots$.

The denotation of a formula consists of the set of word satisfying it. The denotation $[\![\phi]\!]_V$ of a formula ϕ according to valuation V is defined inductively as follows:

$$[\![Z]\!]_V = V(Z),$$

$$[\![\neg\phi]\!]_V = Act^{\infty} \backslash [\![\phi]\!]_V,$$

$$[\![\phi \wedge \psi]\!]_V = [\![\phi]\!]_V \cap [\![\psi]\!]_V,$$

$$[\![(a)\phi]\!]_V = \{\sigma \in Act^{\infty} \mid \sigma(1) = a \wedge \sigma^1 \in [\![\phi]\!]_V\},$$

$$[\![\nu Z.\phi]\!]_V = \bigcup \{A \subseteq Act^{\infty} \mid A \subseteq [\![\phi]\!]_{V[A/Z]}\}.$$

The denotation of a closed formula ϕ does not depend on the valuation, hence we drop it and use the symbol $[\![\phi]\!]$.

A labelled P/T system N satisfies ϕ if $L^{\infty}(N) \subseteq [\![\phi]\!]$.

The model checking problem for the linear time μ-calculus and P/T systems is defined as as follows: given a P/T system N and a closed formula ϕ, determine if N satisfies ϕ. This problem is decidable [8]:

Theorem 3.19. *Let N be a P/T system and ϕ a closed formula of the linear time μ-calculus. It is decidable if N satisfies ϕ, i.e. $L^{\infty}(N) \subseteq [\![\phi]\!]$.*

3.6. Incidence matrix and invariants

We illustrate an approach for the analysis of P/T nets based on linear algebraic techniques and we show that it can be used for PTI nets. The idea is to collect the changes of tokens in the places, due to transition firings, in a matrix, called the *incidence matrix* of the net. The rows and the columns of the matrix are respectively the places and the transitions; an entry at position (s,t) denotes the change of the number of tokens in place s caused by the firing of transition t, i.e. the difference between the number of occurrences of s in the postset and in the preset of t. The incidence matrix can be used to solve the conservation problem and gives sufficient or necessary conditions for liveness, reachability and boundedness. We claim that the same methods can also be used for PTI nets. One can argue that the incidence matrix does not reflect the structure of the net, because no information is kept regarding inhibitor arcs, thus losing any information about the firability of a transition. The same problem arises in P/T nets: in fact, the incidence matrix keeps information only about the difference between the postset and the preset of a transition; take a transition t_1 that produces n tokens in a place s while not consuming tokens from that place, and a transition t_2 which consumes 1 token from s and produces $n+1$ tokens in s: these two transitions are treated in the same way in the incidence matrix w.r.t. place s, that is, the two entries (s,t_1) and (s,t_2) contain the same value. So, also in absence of inhibitor arcs, the information about firability is lost, because there exists a marking m, with $m(s)=0$, such that $m[t_1\rangle$ but not $m[t_2\rangle$.

The following definitions and results are borrowed from [7]; the notion of net invariant was introduced for the first time in [15].

Definition 3.20. Given a finite set $A=\{a_1,\ldots,a_n\}$ and a set X, every mapping $f:A\rightarrow X$ can be represented by the vector $(f(a_1),\ldots,f(a_n))$. $f\cdot g$ represents the scalar product of two vectors. If C is a matrix, then $f\cdot C$ and $C\cdot f$ denote the left and right products of f and C. With abuse of notation, we denote with 0 the empty vector.

Definition 3.21. Let $N=(S,T,F,I)$ be a PTI net. The incidence matrix of N, $D_N:S\times T\rightarrow\mathbf{Z}$, is defined as $D_N(s,t)=t^\bullet(s)-{}^\bullet t(s)$.

The column vector of D_N associated to a transition t is denoted by $D_N(t)$, whereas the row vector associated to a place s is denoted by $D_N(s)$. A marking m can be seen as a vector indexed on S. Let $\boldsymbol{t}$ be the $|T|$-dimensional vector which is zero everywhere except in the t component. If the transition t is enabled at marking m, and $m[t\rangle m'$, then we have that $m'=m+D_N\cdot\boldsymbol{t}$. In fact, $m'(s)=m(s)+t^\bullet(s)-{}^\bullet t(s)=m(s)+D_N(s,t)=m(s)+D_N(s)\cdot\boldsymbol{t}$. This fact can be generalized to transition sequences:

Definition 3.22. Let $N=(S,T,F,I)$ be a PTI net and σ a transitions sequence. The *Parikh vector* $\boldsymbol{\sigma}:T\rightarrow\omega$ maps each transition t to the number of occurrences of t in σ.

Lemma 3.23. *Given an occurrence sequence $m[\sigma\rangle m'$, the following marking equation holds*: $m'=m+D_N\cdot\boldsymbol{\sigma}$.

Proof. By induction on the length of σ. $\square$

This lemma tells us that the marking reached after the firing of a transition sequence only depends on the number of occurrences of each transition fired, and not on the actual ordering of the transitions in the sequence.

The incidence matrix is useful to solve the conservation problem, i.e. to find a (nonzero) weighting vector, indexed on places, such that the weighted sum of tokens is constant over all reachable markings. In the following we show how to derive a sufficient condition for a weighting vector to satisfy the above property. Let m_0 be the initial marking of N and $m\in[m_0\rangle$. We are looking for a weighting vector x such that $x\cdot m_0=x\cdot m$. As m is a reachable marking, there exists a transition sequence σ such that $m_0[\sigma\rangle m$, so, by Lemma 3.23, $m=m_0+D_N\cdot\boldsymbol{\sigma}$. We have that $x\cdot m=x\cdot m_0+x\cdot D_N\cdot\boldsymbol{\sigma}$.

To obtain $x \cdot m = x \cdot m_0$, we need that $x \cdot D_N \cdot \boldsymbol{\sigma} = 0$. If we require that $x \cdot D_N = 0$, then the condition holds for all $\boldsymbol{\sigma}$, then also for all reachable markings.

The (nonempty) solutions of the equation above are very useful to study properties of nets; they are called *S-invariants*:

Definition 3.24. An S-invariant of a CPT net N is a rational-valued solution of the equation $x \cdot D_N = 0$.

Each S-invariant I satisfied the property that the scalar product $I \cdot m$ remains constant for every reachable marking m:

Proposition 3.25. *Let $N = (S, T, F, I, m_0)$ be a PTI system and I be an S-invariant of N. If $m \in [m_0\rangle$ then $I \cdot m = I \cdot m_0$.*

Proof. If $m \in [m_0\rangle$ then there exists a transition sequence σ such that $m_0[\sigma\rangle m$. By Lemma 3.23, $m = m_0 + D_N \cdot \boldsymbol{\sigma}$. Then $I \cdot m = I \cdot m_0 + I \cdot D_N \cdot \boldsymbol{\sigma}$. By definition of S-invariant, $I \cdot D_N = 0$, thus $I \cdot m = I \cdot m_0$. $\square$

Now we define semipositive and positive invariants.

Definition 3.26. An S-invariant I is *semipositive* if $\forall s \in S(I(s) \geqslant 0)$ and $I \neq 0$. The *support* of a semipositive S-invariant I, denoted by $\langle I \rangle$, is the set of places satisfying $I(s) > 0$.

An S-invariant I is *positive* if $\forall s \in S(I(s) > 0)$.

The following result gives a sufficient condition for boundedness of a place of the system.

Theorem 3.27. *Let $N = (S, T, F, I, m_0)$ be a PTI system and $s \in S$. If N has a semi-positive S-invariant I, such that $I(s) > 0$, then place s is bounded.*

Proof. Let $m \in [m_0\rangle$. Since I is an S-invariant, $I \cdot m_0 = I \cdot m$. As I is semipositive, we have that $I(s) \cdot m(s) \leqslant I \cdot m = I \cdot m_0$. As $I(s) > 0$ by hypothesis, we have $m(s) \leqslant (I \cdot m_0)/I(s)$. $\square$

The incidence matrix can be used to obtain a necessary condition for reachability. If a marking m is reachable from m_0, then there exists a transitions sequence σ such that $m_0[\sigma\rangle m$. By Lemma 3.23, the following equation holds: $m = m_0 + D_N \cdot \boldsymbol{\sigma}$. This means that $\boldsymbol{\sigma}$ is a solution, in nonnegative integers, of the matrix equation $m = m_0 + D_N \cdot x$. Thus, if m is reachable from m_0 then the above equation has a solution in nonnegative integers.

Another necessary condition for reachability arises from Proposition 3.25: if I is an S-invariant and $m \in [m_0\rangle$, from that proposition we obtain $I \cdot m = I \cdot m_0$. Thus we have the following: if $m \in [m_0\rangle$ then, for all S-invariants I, $I \cdot m = I \cdot m_0$.

4. Coverability tree and primitive systems

In this section we show how to construct a coverability tree, joining the properties mentioned in Section 3.1, for a subclass of finite P/T nets with inhibitor arcs.

Note that for a generic net with inhibitor arcs it may not be possible to construct a (finite) coverability tree, because these nets are Turing powerful and the construction of the coverability tree would permit to test termination. More precisely, it is possible to simulate any RAM with a PTI system which is deterministic, i.e. at most one transition is firable at each reachable marking (see Appendix A for a sketch of the proof); a coverability tree permits to test if there exist no infinite computation, then it could test for termination of the PTI system corresponding to such a RAM; thus, in general it is not possible to construct a coverability tree useful to verify properties of the net.

The problem for the construction of a finite coverability tree for PTI systems is the loss of monotonicity: it is no longer true that if $m \subseteq m'$ and $m[\sigma\rangle$ then also $m'[\sigma\rangle$. Thus, the approximation with ω causes a loss of information regarding the firability of transitions inhibited by some place.

For this reason, we impose the following constraint on systems: it is possible to know a limit for each inhibiting place, in such a way that, if the number of tokens in the place exceeds the limit at some stage of the computation, then that place cannot be tested for absence of tokens any more. A system satisfying the above constraint will be called primitive.

Definition 4.1. The PTI system $N = (S, T, F, I, m_0)$ is *primitive* if we can compute $EL : Inib(N) \to \omega$ such that

$$\forall s \in Inib(N) \forall m \in [m_0\rangle (m(s) > EL(s) \Rightarrow \forall m' \in [m\rangle \forall t \in T(m'[t\rangle \Rightarrow s \notin {}^{\circ}t)).$$

Given an inhibiting place s, we call $EL(s)$ the *emptiness limit* of s.

For example, consider the net in Fig. 1: at the initial marking, only transition t can fire; after two firings of t, transition v becomes enabled; if v fires, then place c becomes empty and transition u can fire; otherwise, if a third occurrence of t fires, place c contains 3 tokens and cannot be emptied any more, thus preventing u to fire. Hence, the emptiness limit of place c is 2, because if we reach a marking with more than 2 tokens in c then the place cannot be emptied.

Two other examples of primitive system, with $EL(c) = 2$, are reported in Fig. 2.

In Fig. 3 a non-primitive PTI system is depicted: place s can be filled with any number of tokens by repeated firings of transition a, hence it can be emptied by the repeated firings of b, and finally tested for absence of tokens by c.

Coming back to the construction of the coverability tree, as previously said it is no longer true that a transition that can be fired in a marking is firable in every bigger marking: for example, take a transition t, enabled at marking m, that tests for absence a place s; we have that t is not enabled at marking $m' = m \oplus \{s\}$, even if $m' \supseteq m$. We need to modify the ordering on marking to take into account the introduction of zero

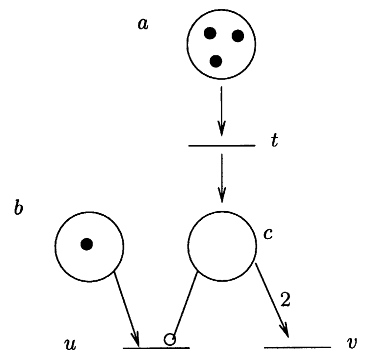

Fig. 1. A primitive system ($EL(c)=2$).

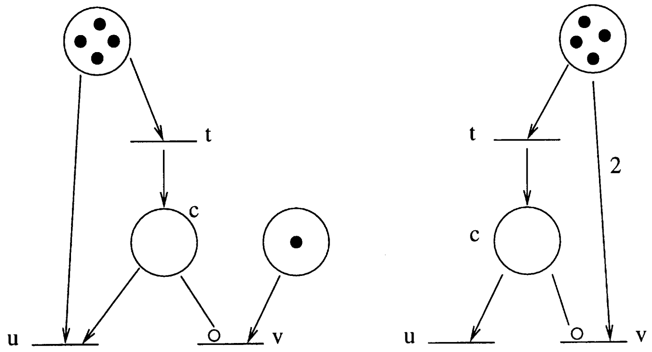

Fig. 2. Two primitive systems ($EL(c)=2$).

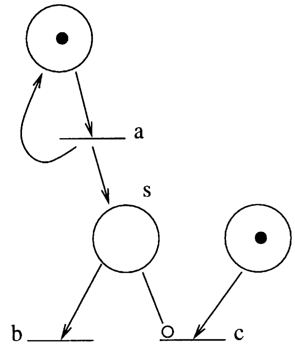

Fig. 3. A PTI system that is not primitive.

testing: for two markings to be comparable, they must have the same inhibiting places empty. At this point, given an extended marking m with $m(s) = \omega$ for some place s and a transition t which consumes some token from s without reproducing them (i.e. $({}^\bullet t \setminus t^\bullet)(s) > 0$), we do not know if after the firing of t at m the place s will become empty or not; thus we cannot know if at this point a transition t' with s in its inhibitor set can fire or not. For this reason, we need to explicitly represent in the tree the actual marking of inhibiting places that may be tested for emptiness. In other words, we do not abstract with the symbol ω the number of tokens in an inhibiting place until this number exceeds the emptiness limit of the place; once the limit has been exceeded, we are sure that the place cannot be tested for emptiness any more and we can make use of ω to represent an unbounded, positive number of tokens in the place. Besides coinciding on empty places, for two markings to be in the "precedence" relation we require that their value coincides also for those inhibiting places whose token number has not exceeded the emptiness limit.

Definition 4.2. Let m be a marking. With $Z(m)$ we denote the set of inhibiting places that can be tested for emptiness in a marking reachable from m:
$Z(m) = \{s \in Inib(N) \mid \exists m' \in [m\rangle \exists t(m'[t\rangle \wedge s \in {}^{\circ}t)\}$.

The "precedence" relation on markings is defined in the following way:

Definition 4.3. Let m_1 be a marking and m_2 an extended marking. $m_1 \preccurlyeq m_2$ iff, for all places $s \in S$:

- $m_1(s) \leqslant m_2(s)$,
- $s \in Z(m_1)$ implies $m_1(s) = m_2(s)$.

Note that relation $\preccurlyeq$ is not transitive.

We need to modify the covering notion: an extended marking with a symbol ω corresponding to a place s represents a set of markings such that the place s cannot be emptied any more.

Definition 4.4. We say that an extended marking m' *covers* a marking m if $m \preccurlyeq m'$ and, for all $s \in S$, $m'(s) \neq m(s)$ implies $m'(s) = \omega$.

If $m_1 \preccurlyeq m_2$, then each transition firable at a marking m_1 is firable also at marking m_2:

Proposition 4.5. *Let m_1, m_2 be markings and t a transition. If $m_1[t\rangle m_1'$ and $m_1 \preccurlyeq m_2$, then $m_2[t\rangle m_2'$ and $m_1' \preccurlyeq m_2'$.*

Corollary 4.6. *Let m_1, m_2 be markings and t a transition. If $m_1[t\rangle m_1'$ and m_2 covers m_1, then $m_2[t\rangle m_2'$ and m_2' covers m_1'.*

Given two markings m_1 and m_2, in general we have no way to know if $m_1 \preccurlyeq m_2$, because this information depends on all the firing sequences starting at m_1. Hence, in the algorithm we will use a finer relation on markings, which depends only on the current contents of places and does not involve properties of markings reachable from them.

Definition 4.7. Let m_1 and m_2 be markings. We define $m_1 \sqsubseteq m_2$ iff, for all places $s \in S$: if $s \in Inib(N)$ and $m_1(s) \leqslant EL(s)$ then $m_1(s) = m_2(s)$ else $m_1(s) \leqslant m_2(s)$.

Moreover, $m_1 \sqsubset m_2$ if $m_1 \sqsubseteq m_2$ and $m_1 \neq m_2$.

Note that $\sqsubset$ ($\sqsubseteq$) is a strict (weak) ordering relation.

Proposition 4.8. *Let m_1 and m_2 be markings. If $m_1 \sqsubseteq m_2$ then $m_1 \leqslant m_2$.*

Given a primitive net, we can apply a variant of the algorithm in Table 1 obtained by replacing the inclusion ordering ($\subset$) on markings with the new ordering $\sqsubset$ in step 3a.

We now show that the constructed coverability tree is finite, using (generalizations of) the auxiliary lemmata recalled in Section 3.1:

Lemma 4.9. *Every infinite sequence of extended markings contains an infinite subsequence ordered w.r.t. $\sqsubseteq$.*

Proof. By induction on the cardinality of the set of places S. Let $\{m_i \mid i \in \omega\}$ be a sequence of extended markings on S. If $S = \emptyset$, then all markings in the sequence are empty, and form a monotonous and nondecreasing sequence.

If $S \neq \emptyset$, take $s' \in S$ and consider the sequence $\{m_i(s') \mid i \in \omega\}$. By Lemma 3.3 there exists a constant infinite subsequence or a strictly increasing infinite subsequence. In the second case, if $s' \in Inib(N)$ then extract the subsequence of elements that are greater than $EL(s')$. In either cases, we have found a sequence of markings such that, for every pair of consecutive markings, place s' satisfies the conditions in the definition of $\sqsubseteq$. If we apply the induction hypothesis on the sequence of markings resulting from ignoring place s', we obtain an infinite subsequence of markings ordered with $\sqsubseteq$. $\square$

Lemma 4.10. *The coverability tree of a primitive system N is finite.*

Proof. Suppose that the coverability tree is infinite. The coverability tree is finitely branching, because, by construction, the number of arcs exiting from each node is limited by the number of transitions of the net, which is finite. Then, by König Lemma, there is an infinite path $x_0 \dots x_n \dots$ starting from the root. $M[x_0] \dots M[x_n] \dots$ is a sequence of extended markings, and by Lemma 4.9 there exists an infinite monotonous and nondecreasing subsequence $M[x_{i_0}] \sqsubseteq M[x_{i_1}] \sqsubseteq M[x_{i_2}] \sqsubseteq \cdots$. By construction, we cannot have $M[x_i] = M[x_j]$, otherwise one of them is a duplicate node and has no successors. Thus, we have $M[x_{i_0}] \sqsubset M[x_{i_1}] \sqsubset M[x_{i_2}] \sqsubset \cdots$. As $M[x_{i_j}] \sqsubset M[x_{i_{j+1}}]$, by construction there exists at least one place s for which $M[x_{i_j}](s) < \omega$ and $M[x_{i_{j+1}}] = \omega$. Thus, each $M[x_{i_j}]$ has at least j places with associated symbol ω. Let k be the cardinality of the set of places S. We have that all places in $M[x_{i_k}]$ have associated the symbol ω. We have $M[x_{i_k}] \sqsubset M[x_{i_{k+1}}]$, reaching a contradiction. Thus, the coverability tree is finite. $\square$

We now show that the properties enunciated in Section 3.1 continue to hold.

Each reachable marking either appears or it is covered by some marking in the coverability tree:

Lemma 4.11. *For each firing sequence* $m_0[t_1\rangle\ m_1 \ldots [t_n\rangle m_n$ *there exists a sequence of nodes and arcs* $x_0 \xrightarrow{t_1} y_1\ \ x_1 \xrightarrow{t_2} y_2 \ldots x_{n-1} \xrightarrow{t_n} y_n$ *in the coverability tree, such that*:
- x_0 *is the root of the tree*,
- $M[y_i] = M[x_i]$ *for* $i = 1, \ldots, n-1$,
- m_i *is covered by* $M[x_i]$ *for* $i = 0, \ldots, n-1$ *and* m_n *is covered by* $M[y_n]$.

Proof. By induction on the length of the firing sequence. If the firing sequence is m_0, as x_0 is the root of the tree we have $M[x_0] = m_0$.

Given a firing sequence $m_0[t_1\rangle m_1 \ldots [t_n\rangle m_n[t_{n+1}\rangle m_{n+1}$, by induction hypothesis there exists a sequence $x_0 \xrightarrow{t_1} y_1\ \ x_1 \xrightarrow{t_2} y_2 \cdots x_{n-1} \xrightarrow{t_n} y_n$ in the coverability tree, such that:
- x_0 is the root of the tree,
- $M[y_i] = M[x_i]$ for $i = 1, \ldots, n-1$,
- m_i is covered by $M[x_i]$ for $i = 0, \ldots, n-1$ and m_n is covered by $M[y_n]$.

If node y_n is a duplicate node, then there exists a node x_n in the tree, that is not a duplicate node, such that $M[y_n] = M[x_n]$. Otherwise, take $x_n = y_n$.

As m_n is covered by $M[x_n]$ and $m_n[t_{n+1}\rangle m_{n+1}$, by Corollary 4.6 we obtain $M[x_n]$ $[t_{n+1}\rangle m'$, and m' covers m_{n+1}.

During the processing of node x_n, a new node y_{n+1}, and a t_{n+1}-labelled arc from x_n to y_{n+1}, are created.

It remains to show that m_{n+1} is covered by $M[y_{n+1}]$. Let $s \in S$.
- We prove that $m_{n+1}(s) \leqslant M[y_{n+1}](s)$. If $M[y_{n+1}](s) = \omega$, we surely have $m_{n+1}(s) \leqslant \omega$. If $M[y_{n+1}](s) \neq \omega$, then $M[y_{n+1}](s) = m'(s)$; as m' covers m_{n+1}, we have $m_{n+1}(s) \leqslant m'(s) = M[y_{n+1}](s)$.
- We prove that if $s \in Z(m_{n+1})$ then $m_{n+1}(s) = M[y_{n+1}](s)$. $M[y_{n+1}](s)$ can be calculated in the following ways:
  - — It is calculated in case 3a of the algorithm; then, $m'(s) \neq \omega$, a node v exists such that $M[v] \sqsubseteq m'$ and $M[v](s) < m'(s)$. We show that in this case we have $s \notin Z(m_{n+1})$. As $m'(s) \neq \omega$, from the fact that m' covers m_{n+1} it follows that $m'(s) = m_{n+1}(s)$. From $M[v] \sqsubseteq m'$, by definition of $\sqsubseteq$ we have that $M[v](s) \leqslant EL(s) \Rightarrow M[v](s) = m'(s)$. Thus, from $M[v](s) < m'(s)$ we obtain $M[v](s) > EL(s)$. We have $m_{n+1}(s) = m'(s) > M[v](s) > EL(s)$, and by primitivity we obtain $s \notin Z(m_{n+1})$.
  - — It is calculated in case 3b of the algorithm; then, $M[y_{n+1}](s) = m'(s)$. As m' covers m_{n+1}, from $s \in Z(m_{n+1})$ we obtain $m'(s) = m_{n+1}(s)$, thus $M[y_{n+1}](s) = m_{n+1}(s)$.
- We prove that if $m_{n+1}(s) \neq M[y_{n+1}](s)$ then $M[y_{n+1}](s) = \omega$. $M[y_{n+1}](s)$ can be calculated in the following ways:
  - — If it is calculated in case 3a of the algorithm, then $M[y_{n+1}](s) = \omega$.
  - — If it is calculated by case 3b of the algorithm, then $M[y_{n+1}](s) = m'(s)$. If $M[y_{n+1}](s) \neq m_{n+1}(s)$, then also $m'(s) \neq m_{n+1}(s)$. As m' covers m_{n+1}, it follows that $m'(s) = \omega$, thus also $M[y_{n+1}](s) = \omega$. □

Now we show that ω symbols in the extended markings appearing in the covering tree do indeed represent places that will hold an unlimited number of tokens: given such an extended marking m', we show that, for any positive number k, there exists a reachable marking m such that the places holding the symbol ω in m' are filled with at least k tokens in m.

Lemma 4.12. *Let z be a node of the coverability tree and $k>0$. Then there exists a marking $m \in [m_0\rangle$ such that, for all $s \in S$,*
- *if $M[z](s) < \omega$ then $m(s) = M[z](s)$,*
- *if $M[z](s) = \omega$ then $m(s) \geqslant k$.*

Proof. Let $EL = \max\{EL(s) \mid s \in Inib(s)\}$. W.l.o.g. we assume $k > EL$.

The proof proceeds by induction on the generation ordering of nodes in the tree.

If z is the root node, then $M[z] = m_0$ and the condition is satisfied. If z is generated during the processing of node x, then there exist an extended marking m' and a transition t such that $M[x][t\rangle m'$; moreover, there is an arc $x \xrightarrow{t} z$ in the coverability tree.

By inductive hypothesis, given $k_1 > EL$, there exists a marking $m_1 \in [m_0\rangle$ such that
- if $M[x](s) < \omega$ then $m_1(s) = M[x](s)$,
- if $M[x](s) = \omega$ then $m_1(s) \geqslant k_1$.

Let $k_2 > EL$; if we take $k_1 = k_2 + \max\{{}^{\bullet}t(s) \mid s \in S\}$ and the corresponding marking m_1 which satisfies the conditions above, we have that $m_1[t\rangle m_2$ and m_2 satisfies the following conditions:
- if $M[x](s) < \omega$ then $m_2(s) = m'(s)$,
- if $M[x](s) = \omega$ then $m_2(s) \geqslant k_2$.

From the construction of $M[z]$, we have that $M[z](s) < \omega$ implies $M[z](s) = m'(s)$; thus m_2 satisfies the conditions required by the lemma for all s such that $M[z](s) < \omega$ or $M[x](s) = \omega$.

Now we show how to obtain a marking which satisfies the lemma.

The problem is due to those places s such that $M[x](s) < \omega$ and $M[z](s) = \omega$; it may happen that some of those places do not contain at least k tokens in marking m_2. If $M[x](s) < \omega$ and $M[z](s) = \omega$, then there exists a node y in the path between the root and x such that $M[y] \sqsubset m'$; the idea is to take a k_2 sufficiently large, to permit to iterate the transitions, occurring in the path from y to z, a sufficient number of times.

We define an upper limit to the number of tokens that can be consumed from a place by the firing of a transition sequence obtained by the labels of a path in the part of the tree we have constructed until now. Let $maxcons = \max_{n \in \omega} \{\sum_{i=1}^{n} {}^{\bullet}t_i(s) \mid s \in S \wedge x_0 \xrightarrow{t_1} x_1 \ldots \xrightarrow{t_n} x_n$ is a path of the tree$\}$.

For each marking m we define $A_{m,k} = \{s \in S \mid M[z](s) = \omega \wedge m(s) < k\}$ and $h_{m,k} = k \times maxcons \times |A_m|$.

We prove the following:

Claim. *Let $k > EL$ and $m \in [m_0\rangle$, such that*:
- $M[z](s) < \omega$ *implies* $m(s) = M[z](s)$,
- $M[z](s) = \omega \wedge M[x](s) < \omega$ *implies* $m(s) \geqslant m'(s)$,
- $M[x](s) = \omega$ *implies* $m(s) \geqslant k + h_{m,k}$.

Then there exists $\bar{m} \in [m\rangle$ *such that*
- $M[z](s) < \omega$ *implies* $\bar{m}(s) = M[z](s)$,
- $M[z](s) = \omega$ *implies* $\bar{m}(s) \geqslant k$.

Proof of the Claim. By induction on $|A_{m,k}|$.

If $|A_{m,k}| = 0$, then take $\bar{m} = m$.

Otherwise, let $\bar{s} \in A_{m,k}$. We have $m(\bar{s}) < k$, hence $M[x](\bar{s}) < \omega$; thus, we have $m'(\bar{s}) = M[x](\bar{s}) - {}^{\bullet}t(\bar{s}) + t^{\bullet}(\bar{s}) < \omega$. We have $M[z](\bar{s}) = \omega$ and $m'(\bar{s}) = \omega$, hence case 3a of the algorithm has been used, i.e. there exists a node y in the path from the root to x such that $M[y] \sqsubset m'$ and $M[y](\bar{s}) < m'(\bar{s})$.

We have a path $y = x_1 \xrightarrow{t}{}_1 \cdots x_n \xrightarrow{t}{}_n z$, with $x_n = x$, in the tree.

We show that the transition sequence $t_1 \ldots t_n$ is firable at m.

Let $s \in {}^{\bullet}t_i$; we show that the marking reached after firing $t_1 \ldots t_{i-1}$ contains at least ${}^{\bullet}t_i(s)$ tokens; the following cases can occur:
- $M[z](s) < \omega$: By hypothesis of the claim we have $m(s) = M[z](s)$; as $M[z](s) < \omega$, by construction of the tree we have $M[z](s) = m'(s)$; from $M[y] \sqsubset m'$ we get $M[y](s) \leqslant m'(s)$, hence we have $M[y](s) \leqslant m(s)$. As $M[z](s) < \omega$, we have also $M[x_i](s) < \omega$; as t_i is enabled at $M[x_i]$ we have ${}^{\bullet}t_i(s) \leqslant M[x_i](s)$; we have $M[x_i](s) = M[y](s) - \sum_{j=1}^{i-1} {}^{\bullet}t_j(s) + \sum_{j=1}^{i-1} t_j^{\bullet}(s)$; as $m(s) \geqslant M[y](s)$, we have that the number of tokens in s at the marking reached from m after firing $t_1 \ldots t_{i-1}$ is greater or equal to ${}^{\bullet}t_i(s)$.
- $M[z](s) = \omega$ and $M[x](s) < \omega$: By hypothesis of the claim we have $m(s) \geqslant m'(s)$; from $M[y] \sqsubset m'$ we get $M[y](s) \leqslant m'(s)$, hence we have $M[y](s) \leqslant m(s)$. As $M[x](s) < \omega$, we have also $M[x_i](s) < \omega$; the proof proceeds as for the item above.
- $M[x](s) = \omega$: By hypothesis of the claim we have that $m(s) \geqslant k + h_{m,k} \geqslant maxcons$, hence there are enough tokens for t_i to fire at the marking reached after firing $t_1 \ldots t_{i-1}$.

Let $s \in {}^{\circ}t_i$; we show that s is empty in the marking reached after the firing of $t_1 \ldots t_{i-1}$ from marking m. As t_i is enabled at $M[x_i]$, we have $M[x_i](s) = 0 < \omega$, hence also $M[y](s) < \omega$. We show that $M[y](s) \leqslant EL(s)$: suppose $M[y](s) > EL(s)$: by inductive hypothesis of the lemma, there exists a marking $m'' \in [m_0\rangle$ such that, for all $s \in S$:
- if $M[y](s) < \omega$ then $m''(s) = M[y](s)$,
- if $M[y](s) = \omega$ then $m''(s) \geqslant maxcons$.

It is easy to see that $t_1 \ldots t_i$ is enabled at m'', thus obtaining an inhibiting place s and a marking $m'' \in [m_0\rangle$ such that $m''(s) > EL(s)$ and s is tested for absence of tokens in a subsequent transition (t_i), contradicting the primitivity of the net. Hence, $M[y](s) \leqslant EL(s)$; as $M[y](s) \sqsubset m'$, we have $M[y](s) = m'(s)$; the following cases can happen:
- $M[z](s) < \omega$: in this case, $m(s) = M[z](s)$ by hypothesis of the claim, and $m'(s) = M[z](s)$ by construction; hence, we have $m(s) = m'(s) = M[y](s)$; as

$M[x_i](s) = 0$, we have that place s becomes empty after the firing of $t_1 \dots t_{i-1}$ from marking m.

- $M[z](s) = \omega$ and $M[x](s) < \omega$: we show that this case cannot happen; we have $m'(s) < \omega$, hence by construction there exists a node w such that $M[w] \sqsubset m'$ and $M[w](s) < m'(s)$: We have seen above that $M[y](s) \leqslant EL(s)$, from which follows that $M[y](s) = m'(s) \leqslant EL(s)$. As $M[w] \sqsubset m'$, we have $M[w](s) \leqslant m'(s) \leqslant EL(s)$, hence, by definition of $\sqsubset$, $M[w](s) = m'(s)$, contradiction.
- $M[x](s) = \omega$: we show that this case cannot happen; we have $m'(s) = \omega$; we have seen above that $M[y](s) \leqslant EL(s)$, from which follows that $M[y](s) = m'(s) \leqslant EL(s)$, contradiction.

We have shown that the transition sequence $t_1 \dots t_n$ is firable at m.

Now we show that $m[t_1 \dots t_n\rangle \bar{m}'$, with $\bar{m}'$ satisfying the following:

- $M[z](s) < \omega$ implies $\bar{m}'(s) = M[z](s)$,
- $M[z](s) = \omega \wedge M[x](s) < \omega$ implies $\bar{m}'(s) \geqslant m'(s)$,
- $M[x](s) = \omega$ implies $\bar{m}'(s) \geqslant k + h_{m,k} - maxcons$,
- $M[y](s) < m'(s) < \omega$ implies $\bar{m}'(s) \geqslant m(s) + 1$,
- $M[y](s) = m'(s) < \omega$ implies $\bar{m}'(s) = m(s)$.

Let $s \in S$:

- If $M[z](s) < \omega$, then by hypothesis of the claim we have $m(s) = M[z](s)$; by construction, we have $M[z](s) = m'(s)$. We know that $M[y] \sqsubset m'$; if $M[y](s) < m'(s)$, then by step 3a of the algorithm we get $M[z](s) = \omega$, contradiction; hence we have $M[y](s) = m'(s)$. We have $M[z](s) = M[y](s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) = m'(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) = \bar{m}'(s)$.
- If $M[z](s) = \omega$ and $M[x](s) < \omega$, by hypothesis of the claim we have that $m(s) \geqslant m'(s)$. As $M[y] \sqsubset m'$, we have $m'(s) \geqslant M[y](s)$, hence $m(s) \geqslant M[y](s)$. We have $\bar{m}'(s) = m(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) \geqslant M[y](s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) = m'(s)$, i.e. $\bar{m}'(s) \geqslant m'(s)$.
- If $M[x](s) = \omega$, by hypothesis of the claim we have $m(s) \geqslant k + h_{m,k}$; we have $\bar{m}'(s) = m(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) \geqslant m(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) \geqslant m(s) - maxcons \geqslant k + h_{m,k} - maxcons$, i.e. $\bar{m}'(s) \geqslant k + h_{m,k} - maxcons$.
- If $M[y](s) < m'(s) < \omega$, then we have $m'(s) = M[y](s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s)$; as $M[y](s) < m'(s)$, we have $\sum_{i=1}^{n} t_i^\bullet(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) \geqslant 1$; we have that $\bar{m}'(s) = m(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) \geqslant m(s) + 1$, i.e. $\bar{m}'(s) \geqslant m(s) + 1$.
- If $M[y](s) = m'(s) < \omega$, then we have $m'(s) = M[y](s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s)$; as $M[y](s) = m'(s)$, we have $\sum_{i=1}^{n} t_i^\bullet(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) = 0$; we have that $\bar{m}'(s) = m(s) - \sum_{i=1}^{n} {}^\bullet t_i(s) + \sum_{i=1}^{n} t_i^\bullet(s) = m(s)$, i.e. $\bar{m}'(s) = m(s)$.

By repeating the firing of the transition sequence $t_1 \dots t_n$ for k times, we reach a marking $\bar{m}''$ such that

- $M[z](s) < \omega$ implies $\bar{m}''(s) = M[z](s)$,
- $M[z](s) = \omega \wedge M[x](s) < \omega$ implies $\bar{m}''(s) > m'(s)$,
- $M[x](s) = \omega$ implies $\bar{m}''(s) \geqslant k + h_{m,k} - k \times maxcons$,
- $M[y](s) < m'(s) < \omega$ implies $\bar{m}''(s) \geqslant m(s) + k$,
- $M[y](s) = m'(s) < \omega$ implies $\bar{m}''(s) = m(s)$.

We have seen at the beginning of the inductive step of the proof that there exists a place $\bar{s} \in A_{m,k}$ such that $M[y](\bar{s}) < m'(\bar{s}) < \omega$; hence, we have $\bar{m}''(\bar{s}) \geqslant m(\bar{s}) + k$; as $M[y] \sqsubset m'$, from the last two conditions satisfied by $\bar{m}''$ it is easy to see that $\bar{m}''(s) \geqslant m(s)$ for all $s \in S$; thus $\bar{m}''(s) \leqslant k$ implies $m(s) \leqslant k$, hence we have $A_{\bar{m}'',k} \subseteq A_{m,k}$; moreover, we have that $\bar{s} \in A_{m,k} \backslash A_{\bar{m}'',k}$; hence $|A_{\bar{m}'',k}| < |A_{m,k}|$.

We show that $\bar{m}''$ satisfies the hypothesis of the claim:

- we have that $M[z](s) < \omega$ implies $\bar{m}''(s) = M[z](s)$;
- we have that $M[z](s) = \omega \wedge M[x](s) < \omega$ implies $\bar{m}''(s) > m'(s)$;
- if $M[x](s) = \omega$, then $\bar{m}''(s) \geqslant k + h_{m,k} - k \times \mathit{maxcons}$; as $h_{m,k} = k \times \mathit{maxcons} \times |A_{m,k}|$, we have $k + h_{m,k} - k \times \mathit{maxcons} = k + k \times \mathit{maxcons} \times (|A_{m,k}| - 1)$; we have shown above that $|A_{\bar{m}'',k}| < |A_{m,k}|$, hence $k + k \times \mathit{maxcons} \times (|A_{m,k}| - 1) \geqslant k + k \times \mathit{maxcons} \times |A_{\bar{m}'',k}| = k + h_{\bar{m}'',k}$; thus, we have obtained $\bar{m}''(s) \geqslant k + h_{\bar{m}'',k} \geqslant h_{\bar{m}'',k}$.

As $\bar{m}''$ satisfies the hypothesis of the claim, and $|A_{\bar{m}'',k}| < |A_{m,k}|$, we can apply the inductive hypothesis: we obtain a marking $\bar{m} \in [\bar{m}''\rangle$ such that

- $M[z](s) < \omega$ implies $\bar{m}(s) = M[z](s)$,
- $M[z](s) = \omega$ implies $\bar{m}(s) \geqslant k$.

We know that $\bar{m}''$ is obtained by firing k times the transition sequence $t_1 \ldots t_n$, starting from marking m; hence, $\bar{m}'' \in [m\rangle$; from $\bar{m}'' \in [m\rangle$ and $\bar{m} \in [\bar{m}''\rangle$ we get $\bar{m} \in [m\rangle$.

This ends the proof of the claim.

Now, let $k_2 = k + k \times \mathit{maxcons} \times |S|$; we show that m_2 satisfies the hypothesis of the claim:

- If $M[z](s) < \omega$, then also $M[x](s) < \omega$, hence, by the conditions satisfied by m_2, we have $m_2(s) = m'(s)$; as $M[z](s) < \omega$, by construction we have $m'(s) = M[z](s)$, hence we obtain $m_2(s) = M[z](s)$.
- If $M[z](s) = \omega$ and $M[x](s) < \omega$, by the condition on m_2 we have $m_2(s) = m'(s)$.
- If $M[x](s) = \omega$, we have by the condition on m_2 we have $m_2(s) \geqslant k_2$; we have taken $k_2 = k + k \times \mathit{maxcons} \times |S|$; as $A_{m_2,k_2} \subseteq S$, we have $k + k \times \mathit{maxcons} \times |S| \geqslant k + k \times \mathit{maxcons} \times |A_{m_2,k_2}| = k + h_{m_2,k_2}$, hence $m_2(s) \geqslant k + h_{m_2,k_2}$.

Hence, by the claim, we obtain a marking $m \in [m_2\rangle$ such that

- if $M[z](s) < \omega$ then $m(s) = M[z](s)$,
- if $M[z](s) = \omega$ then $m(s) \geqslant k$.

From $m_1 \in [m_0\rangle$, $m_1[t\rangle m_2$ and $m \in [m_2\rangle$ we get $m \in [m_0\rangle$. $\square$

Now we show that the decision procedures presented in Section 3.1 are valid also for primitive systems.

It is possible to decide if a place is bounded.

Theorem 4.13. *A place s is bounded iff, for all nodes x in the coverability tree, $M[x](s) \neq \omega$.*

Proof. Suppose s is bounded. If there exists a node x such that $M[x](s) = \omega$ then, by Lemma 4.12, there exists a marking $m \in [m_0\rangle$ such that, for each k, $m(s) > k$. Thus, s is not bounded.

Suppose that $M[x](s) \neq \omega$ for all nodes x in the coverability tree. If s is unbounded, then, for each k, there exists a marking $m \in [m_0\rangle$ such that $m(s) > k$. Take $k > \max\{M[x](s) \mid x \text{ is a node}\}$. By Lemma 4.11, there exists a node z in the tree such that $M[z]$ covers m; thus, $M[z](s) \geqslant m(s) > k$, which is a contradiction. □

Boundedness for single places can be useful to reduce the size of the net: for example, if an inhibiting place is bounded by 0, then we can remove the inhibiting arcs exiting from that place.

Corollary 4.14. *The net N is bounded if the symbol ω does not appear in any node of the coverability tree.*

From the coverability tree we can find dead transitions:

Theorem 4.15. *t is dead iff no t-labelled arc occurs in the coverability tree.*

Proof. We show that if there exists a t-labelled arc in the coverability tree then t is not dead. Suppose an arc $x \xrightarrow{t} y$ occurs in the tree. By construction, t is enabled at $M[x]$. Let $k = \max\{{}^\bullet t(s) \mid s \in S\}$; by Lemma 4.12, there exists $m \in [m_0\rangle$ such that, for all $s \in S$,
- $M[x](s) < \omega$ implies $m(s) = M[x](s)$,
- $M[x](s) = \omega$ implies $m(s) \geqslant k$.

then, t is enabled at m, and it is not dead.

Now we show that if t is not dead then there exists a t-labelled transition in the coverability tree. Suppose t is not dead. Then there exists $m \in [m_0\rangle$ such that t is enabled at m. By Lemma 4.11 there exists a node x such that m is covered by $M[x]$. By Corollary 4.6, from t enabled at m we obtain t enabled at $M[x]$. If x is a duplicate node, then take the internal node z such that $M[z] = M[x]$; otherwise, take $z = x$. By construction, we have that a t-labelled arc exiting from node z occurs in the coverability tree. □

The coverability problem can be solved with the inspection of the nodes of the coverability tree.

Theorem 4.16. *Let m be a marking. A marking $m' \in [m_0\rangle$, such that $m \subseteq m'$, exists iff there exists a node x in the coverability tree such that $m \subseteq M[x]$.*

Proof. Suppose that there exists $m' \in [m_0\rangle$ such that $m \subseteq m'$. By Lemma 4.11 there exists a node x such that m' is covered by $M[x]$. Then we have that $m' \subseteq M[x]$, and also $m \subseteq M[x]$.

Suppose that there exists a node x such that $m \subseteq M[x]$. Take $k = max\{m(s) \mid s \in S\}$. By Lemma 4.12 there exists a marking $m' \in [m_0\rangle$ such that, for all $s \in S$,
- $M[x](s) < \omega$ implies $m'(s) = M[x](s)$,
- $M[x](s) = \omega$ implies $m'(s) \geqslant k$.

It is easy to see that $m \subseteq m'$. □

The theorem above gives information on the contents of places, thus can be useful to check properties dealing with the quantity of resources involved in a system, often represented as tokens in some place.

4.1. Some sufficient conditions for a PTI system to be primitive

Note that, in general, it is not possible to decide if a given PTI system is primitive, otherwise we can test a RAM program for termination: given a RAM program, we construct the corresponding (deterministic) PTI system, as shown in Appendix A; if the obtained PTI system is primitive, then we can construct the coverability tree, and the RAM program terminates if and only if no ω symbol appears in the extended markings labelling the nodes of the coverability tree, as we already said at the beginning of this section; if the obtained PTI system is not primitive, then there exists a place in which the number of tokens can become unlimitedly large and the place can still be tested for emptiness; as the system is deterministic, it has no dead markings, hence the RAM program does not terminate.

We will show two sufficient conditions for a PTI system to be primitive.

The first technique is based on S-invariants; if, for each place $s \in Inib(N)$, there exists a semipositive S-invariant I, such that $I(s) > 0$, then, by Theorem 3.27, place s is bounded; we can take the emptiness limit equal to the upper bound to the number of tokens in place s that can be found in the proof of the Theorem; clearly, boundedness of the inhibiting places implies primitivity.

Another technique is based on the analysis of the coverability tree of the P/T system obtained by dropping the inhibitor arcs; it is easy to see that each firing sequence of a PTI system N is also a firing sequence of the P/T system N' obtained by dropping the inhibitor arcs from N, hence each firing sequence of N is represented in the coverability tree of N', in the sense of Lemma 3.6; we inspect the coverability tree of N', looking for a set of emptiness limits $EL(s)$, for all $s \in Inib(N)$, satisfying the following: for each $s \in Inib(N)$ and for each node x of the tree, if $M[x](s) > EL(s)$ and there exists a sequence of arcs $x \xrightarrow{t_1} y_1 = x_1 \xrightarrow{t_2} y_2 \ldots x_{n-1} \xrightarrow{t_n} y_n$ such that $s \in {}^{\circ}t_n$, then $0 < M[x_{n-1}](s) < \omega$. The above condition amount to check that, if there exists a firing sequence in N' violating the primitivity requirement, then that sequence is not firable in N.

5. Simulating primitive systems by P/T systems

We have seen in the previous section how to use the coverability tree to solve some analysis problems, such as coverability and boundedness. However, in general, it does not contain a sufficient amount of information to solve problems such as reachability and liveness, or to check if a given firing sequence is possible. This is due to the presence of the ω symbol, representing a set of markings with a sufficiently large number of tokens, which causes the loss of the information regarding the effective

Fig. 4. An inhibiting arc on place s of a primitive system is represented as a self-loop on place s^0 in the corresponding P/T system.

contents of places. In this section we reduce the reachability problem for primitive systems to the reachability for P/T systems, which is known to be decidable. This is made by constructing a corresponding P/T system for each primitive system in such a way that there exists a correspondence between the respective markings and firing sequences. The construction is based on the same idea underlying the covering relation in the definition of the coverability tree, that is we need to know the exact contents of each inhibiting place, until it has not exceeded its emptiness limit. An inhibiting place s is mapped on a set of places $\{s^i \mid 0 \leqslant i \leqslant EL(s)\} \cup \{s^\omega\}$; a token in a place s^i represents the fact that place s contains i tokens; $k+1$ tokens in place s^ω represent the fact that place s contains k tokens and that it cannot be emptied any more, because its contents has exceeded the emptiness limit during the execution. The net is constructed in such a way that, for each reachable marking m and for each inhibiting place s in the original net, either a single token is contained in exactly one place s^i and the other places s^j, with $j \neq i$, and s^ω are empty, or place s^ω contains any number of tokens and all places s^i are empty. An inhibiting arc connecting a place s with a transition t, such that s is not in the postset of t, is substituted by a self loop on place s^0, as illustrated in Fig. 4.

Each transition is split in a set of transitions, each one managing a specific representation of the contents of each inhibiting place s in the original net by means of the contents of places s^i and s^ω.

Consider a transition t which consumes only one token from an inhibiting place s: it is mapped on a set of transitions $t^1, \dots, t^{EL(s)}, t^\omega$: each transition t^i manages the fact that place s contains i tokens, and that this is represented, in the net we are constructing, by one token in place s^i; transition t^ω manages the fact that the number of tokens in place s is one token less than the ones in place s^ω. When t^i fires, it removes the token from place s^i and produces a token in place s^{i-1}, representing the fact that, after the firing of transition t in a marking with i tokens in place s, the place s contains $i-1$ tokens. If transition t^ω fires, then two tokens are removed from place s^ω, and one is produced in that place.

Note that, for the property on markings above enunciated, exactly one of the transitions $t^1, \dots, t^{ELs}, t^\omega$ is enabled in a marking, if transition t was enabled in the corresponding marking of the original net.

Consider now a transition t which only produces one token in an inhibiting place s: it is mapped on a set of transitions $t^0, \dots, t^{ELs}, t^\omega$: each transition t^i manages the fact that place s contains i tokens, whereas t^{ω} manages the fact that the contents of place s is one token less than number of tokens in place s^{ω}. When t^i fires, it removes the token from place s^i; if $i < ELs$ then it puts a token in place s^{i+1}, representing the fact that, after the firing of transition t in a marking with i tokens in place s, that place contains $i+1$ tokens; if $i = ELs$, then it puts $ELs + 2$ tokens in s^{ω}. When t^{ω} fires, a token is consumed, and two are produced, in place s^{ω}. The necessity to represent the presence of k tokens in place s with $k+1$ tokens in place s^{ω}, and to add a self loop on place s^{ω} to each transition connected to s^{ω} becomes clear now: consider e.g. the situation where place s^i contains one token; in absence of the self-loop, both transitions t^i and t^{ω} could fire. As place s^{ω} contains one token more than those actually contained in s, a transition t consuming k tokens from place s needs to be mapped on a transition t^{ω} which consumes $k+1$ tokens, and reproduced the "enabling" token, in place s^{ω}. Generalizing the above discussion, a transition t is split into a set of transitions, each one managing one of the different representations of the contents of the inhibiting places from which t removes or produces tokens in the original net. This is obtained by associating to each transition t a set of transitions t^{μ}, where $\mu : Inib(N) \to \omega \cup \{\omega\}$ represents some information about the contents of the set of places corresponding to the inhibiting places involved in the firing of t in the original net: for each inhibiting place $s \in dom({}^{\bullet}t \oplus t^{\bullet})$, $\mu(s) = i \leqslant EL(s)$ represents a marking with one token in place s^i, whereas $\mu(s) = \omega$ represents a marking where the contents of place s is represented by the same number of tokens in place s^{ω}. As we are interested only in the contents of places involved in the production or consumption of tokens during firing of t, we take only those μ such that $dom(\mu) \subseteq dom({}^{\bullet}t \oplus t^{\bullet})$. Moreover we take only those μ representing markings in which transition t is firable, that is, that satisfy $\mu(s) \geqslant {}^{\bullet}t(s)$. Note that, if t does not produce/consume tokens from inhibiting places, then a single transition $t^{\emptyset}$ is associated to it. If a transition t removes $i \geqslant 0$ tokens from place s and produces $j \geqslant 0$ tokens in the same place, then

- Each transition t^{μ} with $i \leqslant \mu(s) \leqslant EL(s)$ removes the token from place $s^{\mu(s)}$; let $k = \mu(s) - i + j$; k is the number of tokens in place s after the firing of transition t, if the contents of that place was $\mu(s)$ before the firing; if $k \leqslant EL(s)$, then a token is produced in place s^k, otherwise $k+1$ tokens are produced in place s^{ω}.
- Each transition t^{μ} with $\mu(s) = \omega$ removes $i+1$ tokens from place s^{ω} and produces $j+1$ tokens in the same place.

See Fig. 5 for a graphical representation.

In the following, we will use α to range over $\omega \cup \{\omega\}$.

Given a primitive net N, we construct a corresponding net $Norm(N)$ in the following way:

Definition 5.1. Let $N = (S, T, F, I, m_0)$ be a primitive PTI system.

The P/T system $Norm(N) = (S', T', F', m_0')$ is defined as

$$S' = (S \setminus Inib(N)) \cup S_{inib},$$

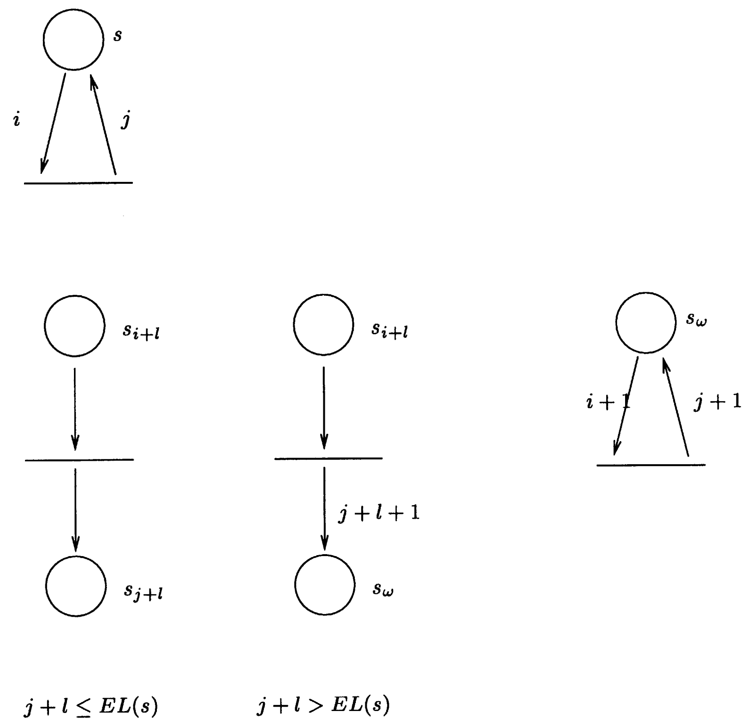

Fig. 5. Representation of the flow arcs connecting a place s and a transition t of a primitive system in the corresponding P/T system.

where

$$S_{inib} = \bigcup_{s \in Inib(N)} \{s^\alpha \mid 0 \leqslant \alpha \leqslant EL(s) \vee \alpha = \omega\},$$

$$T' = \bigcup_{t \in T} \{t^\mu \mid \mu : Inib(N) \to \omega \cup \{\omega\} \wedge dom(\mu) \subseteq dom({}^\bullet t \oplus t^\bullet) \wedge \\ \forall s \in dom(\mu)({}^\bullet t(s) \leqslant \mu(s) \leqslant EL(s) \vee \mu(s) = \omega)\}.$$

Given $s' \in S'$, two cases can happen:

- if $s' \in S \backslash Inib(N)$, then $F'(s', t^\mu) = F(s', t)$ and $F'(t^\mu, s') = F(t, s')$ for all $t^\mu \in T'$
- if $s' \in S_{inib}$, then $s' = s^\alpha$ for some $s \in Inib(N)$:
  if $s \notin {}^\circ t \cup dom({}^\bullet t) \cup dom(t^\bullet)$ then $F'(s^\alpha, t^\mu) = 0$ and $F'(t^\mu, s^\alpha) = 0$, else

$$F'(s^\alpha, t^\mu) = \begin{cases} 1 & \text{if } \alpha = 0 \wedge s \in^\circ t \\ 1 & \text{if } \alpha = \mu(s) \wedge \alpha < \omega \\ F(s,t)+1 & \text{if } \alpha = \mu(s) \wedge \alpha = \omega \\ 0 & \text{otherwise} \end{cases}$$

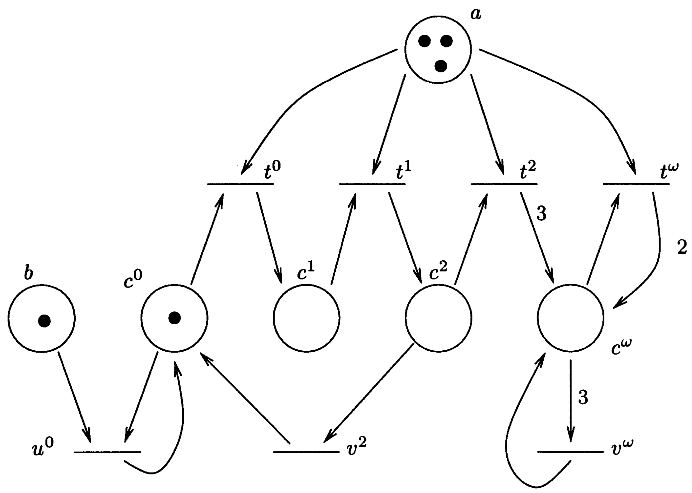

Fig. 6. The P/T system corresponding to the primitive system of Fig. 1.

$$F'(t^\mu, s^\alpha) = \begin{cases} 1 & \text{if } \alpha < \omega \wedge \\ & \alpha = \mu(s) - F(s,t) + F(t,s) \\ \mu(s) - F(s,t) + F(t,s) + 1 & \text{if } \alpha = \omega \wedge \mu(s) < \omega \wedge \\ & \mu(s) - F(s,t) + F(t,s) > EL(s) \\ F(t,s) + 1 & \text{if } \alpha = \mu(s) \wedge \alpha = \omega \\ 0 & \text{otherwise.} \end{cases}$$

The initial marking of $Norm(N)$ is defined as follows:

Given $s' \in S'$, two cases can happen:

- if $s' \in S \setminus Inib(N)$, then $m'_0(s') = m_0(s')$,
- if $s' \in S_{inib}$, then $s' = s^\alpha$ for some $s \in Inib(N)$ and

$$m'_0(s^\alpha) = \begin{cases} 1 & \text{if } \alpha = m_0(s), \\ m_0(s) + 1 & \text{if } \alpha = \omega \wedge m_0(s) > EL(s), \\ 0 & \text{otherwise.} \end{cases}$$

We illustrate the construction above with an example; consider the primitive system in Fig. 1: the corresponding P/T system $Norm(N)$ is shown in Fig. 6.

For each firing sequence of a primitive net N there exists a firing sequence in $Norm(N)$, such that the reached markings are related, and vice versa.

Lemma 5.2. *Let $m_0[t_1\rangle m_1 \dots [t_n\rangle m_n$ be a firing sequence of N. Then there exists a firing sequence $m'_0[t_1^{\mu_1}\rangle m'_1 \dots [t_n^{\mu_n}\rangle m'_n$ of $Norm(N)$ such that, for $i = 0, \dots, n$, and for all*

$s' \in S'$,

- *if* $s' \in S \backslash Inib(N)$ *then* $m'_i(s') = m_i(s')$
- *if* $s' \in S_{inib}$ *and* $s' = s^\alpha$ *then*

$$m'_i(s^\alpha) = \begin{cases} 1 & \textit{if } \alpha = m_i(s) \wedge \forall j \ (0 \leqslant j \leqslant i \Rightarrow m_j(s) \leqslant EL(s)), \\ m_i(s)+1 & \textit{if } \alpha = \omega \wedge \exists j (0 \leqslant j \leqslant i \wedge m_j(s) > EL(s)), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. By induction on the length of the firing sequence.

If the firing sequence is m_0, it is easy to see that m'_0 satisfies the required condition.

Let $m_0[t_1\rangle m_1 \ldots m_n[t\rangle m_{n+1}$ be a firing sequence; by inductive hypothesis, there exists a firing sequence $m'_0[t_1^{\mu_1}\rangle m'_1 \ldots m'_n$ which satisfies the condition. Moreover, t is enabled at m_n (i.e. ${}^\bullet t \subseteq m_n$ and ${}^\circ t \cap dom(m_n) = \emptyset$) and $m_{n+1} = (m_n \backslash {}^\bullet t) \oplus t^\bullet$.

We want to find a μ such that $m'_n[t^\mu\rangle m'_{n+1}$.

Take μ defined as follows:

if $s \notin dom({}^\bullet t \oplus t^\bullet)$ then $\mu(s) = 0$, else

$$\mu(s) = \begin{cases} m_n(s) & \text{if } \forall j (0 \leqslant j \leqslant n \Rightarrow m_j(s) \leqslant EL(s), \\ \omega & \text{otherwise.} \end{cases}$$

We show that $t^\mu \in T'$: we have $\mu(s) = 0$ if $s \notin dom({}^\bullet t \oplus t^\bullet)$, hence $dom(\mu) \subseteq dom({}^\bullet t \oplus t^\bullet)$. Let $s \in dom(\mu)$; if $\mu(s) \leqslant \omega$, then by definition $\mu(s) = m_n(s)$; as t is enabled at $m_n(s)$, we have ${}^\bullet t(s) \leqslant m_n(s)$; moreover, by definition we have $m_n(s) \leqslant EL(s)$, hence ${}^\bullet t(s) \leqslant \mu(s) \leqslant EL(s)$. So $t^\mu \in T'$.

Now we show that t^μ is enabled at m'_n, i.e. ${}^\bullet t^\mu \subseteq m'_n$, i.e. ${}^\bullet t^\mu(s) = F'(s, t^\mu) \leqslant m'_n(s)$ for each $s' \in S'$.

Let $s' \in S'$; two cases can occur:

- If $s' \in S \backslash Inib(N)$, then $F'(s', t^\mu) = F(s', t) \leqslant m_n(s') = m'_n(s')$.
- If $s' \in S_{inib}$ and $s' = s^\alpha$, then we proceed by case analysis on the definition of $F'(s^\alpha, t^\mu)$: if $s \notin {}^\circ t \cup dom({}^\bullet t) \cup dom(t^\bullet)$ then $F'(s^\alpha, t^\mu) = 0$ and the condition is satisfied; otherwise the following cases can happen:
  - — $\alpha = 0$ and $s \in {}^\circ t$: in this case we have $F'(s^\alpha, t^\mu) = 1$. From $s \in {}^\circ t$ and t enabled at m_n it follows that $m_n(s) = 0$. The net N is primitive, hence the emptiness of place s in the current marking implies that the emptiness limit has not been exceeded during the computation, i.e. $\forall j \ (0 \leqslant j \leqslant n \Rightarrow m_j(s) \leqslant EL(s))$; then, by inductive hypothesis, we have that $m'_n(s^\alpha) = 1 \geqslant F'(s^\alpha, t^\mu)$.
  - — $\alpha = \mu(s)$ and $\alpha < \omega$: we have $F'(s^\alpha, t^\mu) = 1$. As $\mu(s) < \omega$, by definition of μ we have that $\mu(s) = m_n(s)$ and $\forall j \ (0 \leqslant j \leqslant n \Rightarrow m_j(s) \leqslant EL(s)$. From this and $\alpha = \mu(s) = m_n(s)$, by inductive hypothesis we have that $m'_n(s^\alpha) = 1$, hence $m'_n(s^\alpha) \geqslant F'(s^\alpha, t^\mu)$.
  - — $\mu(s) = \alpha = \omega$: then $F'(s^\alpha, t^\mu) = F(s, t) + 1$; From $\mu(s) = \omega$, by definition of μ we obtain that $\exists j \ (0 \leqslant j \leqslant n \wedge m_j(s) > EL(s)$. From this and $\alpha = \omega$, by inductive hypothesis we obtain $m'_n(s^\alpha) = m_n(s) + 1$. As t is enabled at m_n we

have that $F(s,t) \leqslant m_n(s)$, from which we obtain $F'(s^\alpha,t^\mu)=F(s,t)+1 \leqslant m_n(s)+1=m'_n(s)$.
— otherwise we have $F'(s^\alpha,t^\mu)=0$ and the condition is obviously satisfied.

Let $m'=m'_n\setminus{}^\bullet(t^\mu)\oplus(t^\mu)^\bullet$ and m'_{n+1} as defined by the condition of the lemma. Finally, we show that $m'=m'_{n+1}$.

If $s'\in S\setminus Inib(N)$ then $m'(s')=m'_n(s')-F'(s',t^\mu)+F'(t^\mu,s')=m_n(s')-F(s',t)+F(t,s')=m_{n+1}(s')$ and $m'_{n+1}(s')=m_{n+1}(s')$.

Let $s'\in S_{inib}$ and $s'=s^\alpha$; the following cases can occur:

- $\forall j\ (0\leqslant j\leqslant n+1 \Rightarrow m_j(s)\leqslant EL(s)).$ (1)

By definition of μ we have that $\mu(s)=m_n(s)$, hence $m_{n+1}(s)=m_n(s)-F(s,t)+F(t,s)=\mu(s)-F(s,t)+F(t,s)$, i.e.

$$m_{n+1}(s)=\mu(s)-F(s,t)+F(t,s). \tag{2}$$

Three cases can occur:

∘ $\alpha=m_{n+1}(s)$: by (2) we have that $\alpha=\mu(s)-F(s,t)+F(t,s)$ hence, by definition of F', $F'(t^\mu,s^\alpha)=1$. By inductive hypothesis and condition (1) we have that $m'_n(s^\alpha)\in\{0,1\}$.
 — If $m'_n(s^\alpha)=0$ then $m'(s^\alpha)=m'_n(s^\alpha)-F'(s^\alpha,t^\mu)+F'(t^\mu,s^\alpha)=1$.
 — If $m'_n(s^\alpha)=1$ then by inductive hypothesis we have $\alpha=m_n(s)=\mu(s)$, hence $F'(s^\alpha,t^\mu)=1$ and $m'(s^\alpha)=m'_n(s^\alpha)-F'(s^\alpha,t^\mu)+F'(t^\mu,s^\alpha)=1$.
 By condition (1) and $\alpha=m_{n+1}(s)$ we have that $m'_{n+1}(s^\alpha)=1$.

∘ $\alpha<\omega$ and $\alpha\neq m_{n+1}(s)$: by (2) we have that $\alpha\neq\mu(s)-F(s,t)+F(t,s)$, hence, by definition of F' and $\alpha<\omega$, $F'(t^\mu,s^\alpha)=0$. By inductive hypothesis and condition (1) we have that $m'_n(s^\alpha)\in\{0,1\}$.
 — If $m'_n(s^\alpha)=0$ then $m'(s^\alpha)=m'_n(s^\alpha)-F'(s^\alpha,t^\mu)+F'(t^\mu,s^\alpha)=0$.
 — If $m'_n(s^\alpha)=1$ then by inductive hypothesis we have $\alpha=m_n(s)=\mu(s)$, hence $F'(s^\alpha,t^\mu)=1$ and $m'(s^\alpha)=m'_n(s^\alpha)-F'(s^\alpha,t^\mu)+F'(t^\mu,s^\alpha)=0$.

∘ $\alpha=\omega$: by inductive hypothesis and condition (1) we have that $m'_n(s^\alpha)=0$; by condition (1) we have that $m_{n+1}(s)\leqslant EL(s)$, hence by (2) $\mu(s)-F(s,t)+F(t,s)\leqslant EL(s)$; by definition of F' we obtain that $F'(t^\mu,s^\alpha)=0$. We have $m'(s^\alpha)=m'_n(s^\alpha)-F'(s^\alpha,t^\mu)+F'(t^\mu,s^\alpha)=0$; by condition (1) we obtain that $m'_{n+1}(s^\alpha)=0$.

- $\forall j\ (0\leqslant j\leqslant n \Rightarrow m_j(s)\leqslant EL(s))\wedge m_{n+1}(s)>EL(s).$ (3)

By (3) and definition of $\mu(s)$ we have that $\mu(s)=m_n(s)$, hence $\mu(s)-F(s,t)+F(t,s)=m_n(s)-F(s,t)+F(t,s)=m_{n+1}(s)>EL(s)$, i.e.

$$\mu(s)-F(s,t)+F(t,s)>EL(s). \tag{4}$$

Three cases can occur:

- $\alpha = m_n(s)$: by condition (3) and inductive hypothesis we have that $m'_n(s^\alpha) = 1$; as $\alpha = m_n(s) = \mu(s)$, by definition of F' we have that $F'(s^\alpha, t^\mu) = 1$. By condition (4) we have $\mu(s) - F(s,t) + F(t,s) > EL(s)$; as $\alpha = m_n(s)$ and, by condition (3), $m_n(s) \leqslant EL(s)$, we have $\alpha \leqslant EL(s)$, hence $\alpha \neq \mu(s) - F(s,t) + F(t,s)$; hence, by definition of F', we have that $F'(t^\mu, s^\alpha) = 0$. We have $m'(s^\alpha) = m'_n(s^\alpha) - F'(s^\alpha, t^\mu) + F'(t^\mu, s^\alpha) = 0$; by condition (3) we have that $m'_{n+1}(s^\alpha) = 0$.
- $\alpha < \omega$ and $\alpha \neq m_n(s)$: by inductive hypothesis we have that $m'_n(s^\alpha) = 0$; by (4) and definition of F' we have that $F'(t^\mu, s^\alpha) = 0$, hence $m'(s^\alpha) = 0$; by (3) we have that $m'_{n+1}(s^\alpha) = 0$.
- $\alpha = \omega$: by condition (3) and inductive hypothesis we have that $m'_n(s^\alpha) = 0$; we have $\mu(s) = m_n(s)$, hence $\mu(s) < \omega$ and $F'(s^\alpha, t^\mu) = 0$; moreover by (4) $\mu(s) - F(s,t) + F(t,s) > EL(s)$, hence by definition of F' we have that $F'(t^\mu, s^\alpha) = \mu(s) - F(s,t) + F(t,s) + 1 = m_n(s) - F(s,t) + F(t,s) + 1 = m_{n+1}(s) + 1$; hence $m'(s^\alpha) = m_{n+1}(s) + 1$. By condition (3) we have that $m'_{n+1}(s^\alpha) = m_{n+1}(s) + 1$.

- $$\exists j \; (0 \leqslant j \leqslant n \wedge m_j(s) > EL(s)). \tag{5}$$

By condition (5) we have that $\mu(s) = \omega$.

Two cases can occur:

- $\alpha < \omega$: by condition (5) and inductive hypothesis we have that $m'_n(s^\alpha) = 0$; as $\mu(s) = \omega$, by definition of F' we have that $F'(t^\mu, s^\alpha) = 0$; hence $m'(s^\alpha) = 0$ and $F'(s^\alpha, t^\mu) = 0$. By condition (5) we have that $m'_{n+1}(s^\alpha) = 0$.
- $\alpha = \omega$: by condition (5) and inductive hypothesis we have that $m'_n(s^\alpha) = m_n(s) + 1$; as $\mu(s) = \omega$, by definition of F' we have that $F'(s^\alpha, t^\mu) = F(s,t) + 1$ and $F'(t^\mu, s^\alpha) = F(t,s) + 1$; hence $m'(s^\alpha) = m'_n(s^\alpha) - F'(s^\alpha, t^\mu) + F'(t^\mu, s^\alpha) = m_n(s) + 1 - F(s,t) - 1 + F(t,s) + 1 = m_{n+1}(s) + 1$.
  By condition (5) we have that $m'_{n+1}(s^\alpha) = m_{n+1}(s) + 1$.

Thus, we have obtained a firing sequence $m'_0[t_1^{\mu_1}\rangle m'_1 \ldots m'_n[t^\mu\rangle m'$ which satisfies the conditions required by the lemma. □

We show that markings in $Norm(N)$ faithfully represent markings in N. As the presence of exactly i tokens in the inhibiting place s of the net N is represented either as one token in the place s^i or as $i+1$ tokens in the place s^ω of the net $Norm(N)$, we expect that in each marking reachable in $Norm(N)$ the following holds for each $s \in Inib(N)$: either a single token is contained in exactly one place s^i and the other places s^α with $\alpha \neq i$ are empty, or at least one token is contained in place s^ω and the other places s^α with $\alpha \neq \omega$ are empty.

Lemma 5.3. *Let $m' \in [m'_0\rangle$. Then, for each $s \in Inib(N)$:*
(1) $m'(s^k) \leqslant 1$ for $k = 0, \ldots, EL(s)$.
(2) there exists a unique $\alpha \in \{0, \ldots, EL(s), \omega\}$ such that $m'(s^\alpha) > 0$.

Proof. By induction on the length of the firing sequence leading to m'.

The base of the induction is when $m' = m'_0$. Then,

(1) Obvious from definition of m'_0.

(2) Two cases can occur:

- if $m'_0(s) \leqslant EL(s)$ then $m'_0(s^\omega) = 0$, $m'_0(s^{m_0(s)}) = 1$ and $m'_0(s^k) = 0$ for $k \neq m_0(s)$.
- If $m'_0(s) > EL(s)$ then $m'_0(s^k) = 0$ and $m'_0(s^\omega) = m_0(s) + 1$.

Let $m'_0 \dots [t_n^{\mu_n}\rangle m'_n [t^\mu\rangle m'$ be a firing sequence leading to m'. By inductive hypothesis, m'_n satisfies the conditions of the lemma.

If $s \notin {}^\circ t \cup dom({}^\bullet t) \cup dom(t^\bullet)$ then $m'(s^\alpha) = m'_n(s^\alpha)$ for each α and the condition is true by inductive hypothesis.

Let $s \in {}^\circ t \cup dom({}^\bullet t) \cup dom(t^\bullet)$:

(1) We show that $m'(s^k) \leqslant 1$ for $k = 0, \dots, EL(s)$.

By inductive hypothesis we have that $m'_n(s^k) \leqslant 1$, hence two cases can occur:

- $m'_n(s^k) = 0$: as $k < \omega$, by definition of F' we have that $F'(t^\mu, s^k) \leqslant 1$, hence $m'(s^k) = m'_n(s^k) - F'(s^k, t^\mu) + F'(t^\mu, s^k) \leqslant 1$.
- $m'_n(s^k) = 1$: three cases can happen, according to the value of $\mu(s)$:
  - $\mu(s) = k$: we have that $F'(s^k, t^\mu) = 1$; as $k < \omega$, $F'(t^\mu, s^k) \leqslant 1$, hence $m'(s^k) = m'_n(s^k) - F'(s^k, t^\mu) + F'(s^k, t^\mu) \leqslant 1$.
  - $\mu(s) \neq k$ and $\mu(s) < \omega$: we have $F'(s^{\mu(s)}, t^\mu) = 1$; as t^μ is enabled at m'_n we have $m'_n(s^{\mu(s)}) \geqslant 1$; as $m'_n(s^k) = 1$ for $k \neq \mu(s)$, this contradicts condition (2) of the lemma, that is true for inductive hypothesis.
  - $\mu(s) = \omega$: we have $F'(s^\omega, t^\mu) = F(s,t) + 1$; as t^μ is enabled at m'_n we have $m'_n(s^\omega) \geqslant 1$; as $m'_n(s^k) = 1$, this contradicts condition (2) of the lemma, that is true for inductive hypothesis.

(2) We show that there exists a unique $\alpha \in \{0, \dots, EL(s), \omega\}$ such that $m'(s^\alpha) > 0$.

Two cases can occur, according to the value of $\mu(s)$:

- $\mu(s) < \omega$: by definition of F' we have $F'(s^{\mu(s)}, t^\mu) = 1$. as t^μ is enabled at m'_n, we have $m'_n(s^{\mu(s)}) \geqslant 1$; by inductive hypothesis we obtain $m'_n(s^{\mu(s)}) = 1$ and $m'_n(s^\alpha) = 0$ for $\alpha \neq \mu(s)$. Hence $m'_n(s^\alpha) - F'(s^\alpha, t^\mu) = 0$ for each α.
  Let $\beta = \mu(s) - F(s,t) + F(t,s)$.
  Two cases can occur, according to the value of β:
  - $\beta \leqslant EL(s)$: by definition of F' we have $F'(t^\mu, s^\beta) = 1$ and $F'(t^\mu, s^\alpha) = 0$ for $\alpha \neq \beta$. Hence we obtain $m'(s^\beta) = 1$ and $m'(s^\alpha) = 0$ for $\alpha \neq \beta$.
  - $\beta > EL(s)$: by definition of F' we have $F'(t^\mu, s^\omega) = \mu(s) - F(s,t) + F(t,s) + 1 = \beta + 1$ and $F'(t^\mu, s^\alpha) = 0$ for $\alpha < \omega$. We obtain $m'(s^\omega) \geqslant 1$ and $m'(s^\alpha) = 0$ for $\alpha < \omega$.
- $\mu(s) = \omega$: by definition of F' we have $F'(s^\omega, t^\mu) = F(s,t) + 1$; as t^μ is enabled at m'_n, we have $m'_n(s^\omega) \geqslant 1$; hence, by inductive hypothesis, $m'_n(s^\alpha) = 0$ for $\alpha < \omega$. By definition of F' we have $F'(t^\mu, s^\omega) = F(t,s) + 1$ and $F'(t^\mu, s^\alpha) = 0$ for $\alpha < \omega$, hence $m'(s^\omega) \geqslant 1$ and $m'(s^\alpha) = 0$ for $\alpha < \omega$. □

From this lemma it easily follows that at most one t^μ is enabled at each reachable marking.

Corollary 5.4. *Let $m' \in [m'_0\rangle$. If $m'[t^{\mu}\rangle$ and $m'[t^{\mu'}\rangle$ then $\mu = \mu'$.*

Proof. Suppose there exists s such that $\mu(s) \neq \mu'(s)$; as $dom(\mu) \subseteq dom({}^{\bullet}t) \cup dom(t^{\bullet})$ and $dom(\mu') \subseteq dom({}^{\bullet}t) \cup dom(t^{\bullet})$, we have $\mu(s) = 0 = \mu'(s)$ for $s \notin dom({}^{\bullet}t) \cup dom(t^{\bullet})$, hence $s \in dom({}^{\bullet}t) \cup dom(t^{\bullet})$.

Three cases can occur:

- $\mu(s) < \omega$ and $\mu'(s) < \omega$: by definition of F' we have $F'(s^{\mu(s)}, t^{\mu}) = 1$ and $F'(s^{\mu'(s)}, t^{\mu'}) = 1$; as both t^{μ} and $t^{\mu'}$ are enabled at m', we have $m'(s^{\mu(s)}) > 0$ and $m'(s^{\mu'(s)}) > 0$ with $\mu(s) \neq \mu'(s)$, in contradiction with item (2) of Lemma 5.3.
- $\mu(s) = \omega$ and $\mu'(s) < \omega$: by definition of F' we have $F'(s^{\mu(s)}, t^{\mu}) = F(s,t) + 1$ and $F'(s^{\mu'(s)}, t^{\mu'}) = 1$; hence $m'(s^{\mu(s)}) > 0$ and $m'(s^{\mu'(s)}) > 0$ with $\mu(s) \neq \mu'(s)$, a contradiction.
- $\mu(s) < \omega$ and $\mu'(s) = \omega$: similar to the item above. □

Corollary 5.5. *If $m'_0[t_1^{\mu_1}\rangle m'_1 \ldots [t_n^{\mu_n}\rangle m'_n$ and $m'_0[t_1^{\bar{\mu}_1}\rangle \bar{m}'_1 \ldots [t_n^{\bar{\mu}_n}\rangle \bar{m}'_n$ then $\mu_i = \bar{\mu}_i$ and $m'_i = \bar{m}'_i$ for $i = 1, \ldots, n$.*

Lemma 5.6. *Let $m'_0[t_1^{\mu_1}\rangle m'_1 \ldots [t_n^{\mu_n}\rangle m'_n$ be a firing sequence of $Norm(N)$. Then there exists a firing sequence $m_0[t_1\rangle m_1 \ldots [t_n\rangle m_n$ of N such that, for $i = 0, \ldots, n$, and for all $s \in S$,*

- *if $s \in S \backslash Inib(N)$ then $m_i(s) = m'_i(s)$,*
- *if $s \in Inib(N)$ then*

$$m_i(s) = \begin{cases} k & \textit{if } m'_i(s^k) = 1, \\ m'_i(s^{\omega}) - 1 & \textit{otherwise}. \end{cases}$$

Proof. Note that, by Lemma 5.3, we have that m_i is well defined: at most one of $m'_i(s^k)$ is equal to 1; if $m'_i(s^k) = 0$ for each k, then $m'_i(s^{\omega}) > 0$, hence $m_i(s) = m'_i(s^{\omega}) \geqslant 0$.

The proof proceeds by induction on the length of the firing sequence.

If the firing sequence is m'_0, then:

- If $s \in S \backslash Inib(N)$ then $m'_0(s) = m_0(s)$.
- If $s \in Inib(N)$ two cases can occur:
  - $m_0(s) \leqslant EL(s)$: we have $m'_0(s^{m_0(s)}) = 1$, hence the condition is satisfied.
  - $m_0(s) > EL(s)$: we have $m'_0(s^{\omega}) = m_0(s) + 1$; by Lemma 5.3 we have $m'_0(s^k) = 0$ for each k, and $m'_0(s^{\omega}) - 1 = m_0(s) + 1 - 1 = m_0(s)$.

Let $m'_0 \ldots m'_n[t^{\mu}\rangle m'_{n+1}$ be a firing sequence of $Norm(N)$.

By inductive hypothesis there exists a firing sequence $m_0 \ldots m_n$ of N satisfying the condition of the lemma.

We show that t is enabled at m_n:

- Let $s \in {}^{\circ}t$; we show that $m_n(s) = 0$. By definition of F', $F'(s^0, t^{\mu}) = 1$; as t^{μ} is enabled at m'_n, we have $m'_n(s^0) > 0$, and by Lemma 5.3, $m'_n(s^0) = 1$; hence, by inductive hypothesis, $m_n(s) = 0$.
- Let $s \in {}^{\bullet}t$; we show that $F(s,t) \leqslant m_n(s)$;

∘ If $s \in S \setminus Inib(N)$, then $F(s,t) = F'(s,t^\mu)$; as t^μ is enabled at m'_n, we have $F'(s,t^\mu) \leqslant m'_n(s)$; by inductive hypothesis, $m_n(s) = m'_n(s)$, hence $F(s,t) = F'(s,t^\mu) \leqslant m'_n(s) = m_n(s)$.

∘ If $s \in Inib(N)$ we distinguish the following two cases:

— If $\mu(s) < \omega$, then $F'(s^{\mu(s)}, t^\mu) = 1$; as t^μ is enabled at m'_n, and by Lemma 5.3, we have $m'_n(s^{\mu(s)}) = 1$; by inductive hypothesis we have $m_n(s) = \mu(s)$. by definition of the set of transitions T' of the net $Norm(N)$ we have ${}^\bullet t(s) \leqslant \mu(s)$, hence $F(s,t) = {}^\bullet t(s) \leq \mu(s) = m_n(s)$.

— If $\mu(s) = \omega$, then $F'(s^\omega, t^\mu) = F(s,t) + 1$. as t^μ is enabled at m'_n, we have $m'_n(s^\omega) \geqslant F'(s^\omega, t^\mu) = F(s,t) + 1 > 0$; by Lemma 5.3 we have $m'_n(s^k) = 0$ for each k, hence by inductive hypothesis we obtain $m_n(s) = m'_n(s^\omega) - 1$. We have $F(s,t) = F'(s^\omega, t^\mu) - 1 \leqslant m'_n(s^\omega) - 1 = m_n(s)$.

Let $m = m_n \setminus {}^\bullet t \oplus t^\bullet$. Let $m_{n+1}(s) = m'_{n+1}(s)$ if $s \in S \setminus Inib(N)$ and

$$m_{n+1}(s) = \begin{cases} k & \text{if } m'_{n+1}(s^k) = 1, \\ m'_{n+1}(s^\omega) - 1 & \text{otherwise} \end{cases}$$

if $s \in Inib(N)$.

We show that $m = m_{n+1}$.

- If $s \in S \setminus Inib(N)$ then $F(s,t) = F'(s,t^\mu)$ and $F(t,s) = F'(t^\mu, s)$. By inductive hypothesis we have $m_n(s) = m'_n(s)$, hence $m(s) = m_n(s) - F(s,t) + F(t,s) = m'_n(s) - F'(s,t^\mu) + F(t^\mu, s) = m'_{n+1}(s) = m_{n+1}(s)$.
- If $s \in Inib(N)$ two cases can happen:

∘ $\mu(s) < \omega$: we have $F'(s^{\mu(s)}, t^\mu) = 1$; as t^μ is enabled at m'_n, $m'_n(s^{\mu(s)}) = 1$, and by inductive hypothesis we have (1) $m_n(s) = \mu(s)$.

Two cases can occur:

— $\mu(s) - F(s,t) + F(t,s) \leqslant EL(s)$: Let $h = \mu(s) - F(s,t) + F(t,s)$; by definition of F' we have $F'(t^\mu, s^h) = 1$; as $m'_{n+1}(s^h) = m'_n(s^h) - F'(s^h, t^\mu) + F'(t^\mu, s^h) \geqslant 1$, by Lemma 5.3 we obtain $m'_{n+1}(s^h) = 1$; hence $m_{n+1}(s) = h$. We have $m'(s) = m_n(s) - F(s,t) + F(t,s)$; by condition (1) we obtain $m'(s) = \mu(s) - F(s,t) + F(t,s) = h = m_{n+1}(s)$.

— $\mu(s) - F(s,t) + F(t,s) > EL(s)$: by definition of F' we have $F'(s^{\mu(s)}, t^\mu) = 1$; as t^μ is enabled at m'_n, and by Lemma 5.3, we have $m'_n(s^{\mu(s)}) = 1$; by Lemma 5.3 we obtain $m'_n(s^\omega) = 0$. By definition of F' we have $F'(t^\mu, s^\omega) = \mu(s) - F(s,t) + F(t,s) + 1$; hence $m'_{n+1}(s^\omega) = m'_n(s^\omega) - F'(s^\omega, t^\mu) + F'(t^\mu, s^\omega) = F'(t^\mu, s^\omega) = \mu(s) - F(s,t) + F(t,s) + 1$. As $m'_{n+1}(s^\omega) > 0$, by Lemma 5.3 we have $m'_{n+1}(s^k) = 0$ for each k, hence $m_{n+1}(s) = m'_{n+1}(s^\omega) - 1 = \mu(s) - F(s,t) + F(t,s)$.

We have $m(s) = m_n(s) - F(s,t) + F(t,s)$; by (1) we obtain $m(s) = \mu(s) - F(s,t) + F(t,s) = m_{n+1}(s)$.

∘ $\mu(s) = \omega$: by definition of F' we have $F'(s^\omega, t^\mu) = F(s,t) + 1$ and $F'(t^\mu, s^\omega) = F(t,s) + 1$. As t^μ is enabled at m'_n, we have $m'_n(s^\omega) > 0$; by Lemma 5.3 we obtain $m'_n(s^k) = 0$ for each k, hence by inductive hypothesis $m_n(s) = m'_n(s^\omega) - 1$;

thus we obtain $m'_{n+1}(s^\omega) = m'_n(s^\omega) - F'(s^\omega, t^\mu) + F'(t^\mu, s^\omega) = m_n(s) + 1 - F(s,t) - 1 + F(t,s) + 1 = m_n(s) - F(s,t) + F(t,s) + 1$.

Thus $m'_{n+1}(s^\omega) > 0$; by Lemma 5.3 we have $m'_{n+1}(s^k) = 0$ for each k, hence $m_{n+1}(s) = m'_{n+1}(s^\omega) - 1 = m_n(s) - F(s,t) + F(t,s) = m(s)$. $\square$

5.1. Reachability

We define the set $Equiv(m)$ of markings of $Norm(N)$ corresponding to a marking m of the primitive system N.

$$\begin{aligned} Equiv(m) = \{m' : S' \to \omega \mid & \forall s \in S \backslash (Inib(N))(m'(s) = m(s)) \wedge \\ & \forall s \in Inib(N)((m(s) \leqslant EL(s) \wedge m'(s^{m(s)}) = 1) \vee \\ & (m'(s^\omega) = m(s) + 1))\}. \end{aligned}$$

Corollary 5.7. *$m \in [m_0\rangle$ iff there exists $m' \in Equiv(m)$ such that $m' \in [m'_0\rangle$.*

Corollary 5.8. *Reachability is decidable for primitive nets.*

Proof. The statements easily follows from decidability of reachability for P/T nets and the fact that $|Equiv(m)| = 2^{|Inib(N)|}$, i.e. is finite; hence, we need to check at most $|Equiv(m)|$ markings for reachability in the system $Norm(N)$. $\square$

A similar reasoning shows that also the submarking reachability problem for N can be reduced to decide a finite set of instances of the submarking reachability problem for $Norm(N)$.

5.2. Deadlock

We show that deadlock is decidable for primitive nets, by reduction to the deadlock problem for P/T nets.

Theorem 5.9. *Deadlock is decidable for primitive nets.*

Proof. Suppose that N has a deadlock; hence there exists a firing sequence $m_0[t_1\rangle \dots [t_n\rangle m_n$ such that $\neg(m_n[t\rangle)$ for each $t \in T$.

By Lemma 5.2 there exists a firing sequence $m'_0[t_1^{\mu_1}\rangle \dots [t_n^{\mu_n}\rangle m'_n$ of $Norm(N)$.

We show that m'_n is dead.

Suppose it is not dead; then there exists t^μ such that $m'_n[t^\mu\rangle m'_{n+1}$; hence $m'_0 \dots m'_n[t^\mu\rangle m'_{n+1}$ is a firing sequence of $Norm(N)$; by Lemma 5.6 there exists a firing sequence $m_0[t_1\rangle \dots [t_n\rangle \bar{m}_n[t\rangle \bar{m}_{n+1}$ of N. As $m_0[t_1\rangle \dots [t_n\rangle m_n$ and $m_0[t_1\rangle \dots [t_n\rangle \bar{m}_n$, we have $\bar{m}_n = m_n$, hence $m_n[t\rangle$, contradicting the hypothesis.

Suppose that $Norm(N)$ has a deadlock; hence there exists a firing sequence $m'_0[t_1^{\mu_1}\rangle \dots [t_n^{\mu_n}\rangle m'_n$ such that $\neg(m'_n[t^\mu\rangle)$ for each $t^\mu \in T'$.

By Lemma 5.6 there exists a firing sequence $m_0[t_1\rangle \dots [t_n\rangle m_n$ of N.

We show that m_n is dead.

Suppose it is not dead; then there exists a transition t such that $m_n[t\rangle m_{n+1}$; so $m_0 \dots m_n[t\rangle m_{n+1}$ is a firing sequence of N; by Lemma 5.2 there exists a firing sequence $m'_0[t_1^{\bar{\mu}_1}\rangle \dots [t_n^{\bar{\mu}_n}\rangle \bar{m}'_n[t^{\mu}\rangle \bar{m}'_{n+1}$ of $Norm(N)$.

We have $m'_0[t_1^{\mu_1}\rangle \dots [t_n^{\mu_n}\rangle m'_n$ and $m'_0[t_1^{\bar{\mu}_1}\rangle \dots [t_n^{\bar{\mu}_n}\rangle \bar{m}'_n$; by Corollary 5.5 we have $m'_n = \bar{m}'_n$, hence $m'_n[t^{\mu}\rangle$, contradiction. $\square$

5.3. Liveness

Now we show that liveness is decidable for primitive nets.

Definition 5.10. A transition t is *live* in the net $N = (S, T, F, I, m_0)$ if for each marking $m \in [m_0\rangle$ there exists a marking $m' \in [m\rangle$ such that t is enabled at m'.

A net N is live if all its transitions are live.

To reduce the liveness problem of N to an analogous problem on $Norm(N)$ we need the following:

Definition 5.11. A finite set of transitions $U \subseteq T$ is *group live* in N iff for each $m \in [m_0\rangle$ there exist a transition $t \in U$ and a marking $m' \in [m\rangle$ such that t is enabled at m'.

Theorem 5.12. *Group-liveness reduces to single transition liveness in P/T systems.*

Proof. Let $N = (S, T, F, m_0)$ and $U \subseteq T$.

Take $p \notin S$ and $l \notin T$. We construct $N' = (S \cup \{p\}, T \cup \{l\}, F', m'_0)$, where

$$F'(x,y) = \begin{cases} 1 & \text{if } (x \in U \wedge y = p), \\ & \text{or } (x = p \wedge y = l), \\ F(x,y) & \text{otherwise,} \end{cases}$$

$$m'_0(s) = \begin{cases} 0 & \text{if } s = p, \\ m_0(s) & \text{otherwise.} \end{cases}$$

This construction is depicted in Fig. 7

Let $\sigma = t_1, \dots, t_n$. It is easy to see that if $m_0[\sigma\rangle m$ then $m'_0[\sigma\rangle m'$, with

$$m'(s) = \begin{cases} |\{i \mid 1 \leqslant i \leqslant n \wedge t_i \in U\}| & \text{if } s = p, \\ m(s) & \text{otherwise} \end{cases}$$

if $m'_0[\sigma\rangle m'$ then $m_0[\sigma'\rangle m$, where σ' is obtained from σ by eliminating all the occurrences of l and $m = m'|_S$.

We show that U is group live in N iff l is live in N'.

Suppose U group live in N; let $m' \in [m'_0\rangle$ in N'; take $m = m'|_S$; then $m \in [m_0\rangle$ in N; by group liveness, there exist $t \in U$ and $m_1 \in [m\rangle$ such that t is enabled at m_1; take the

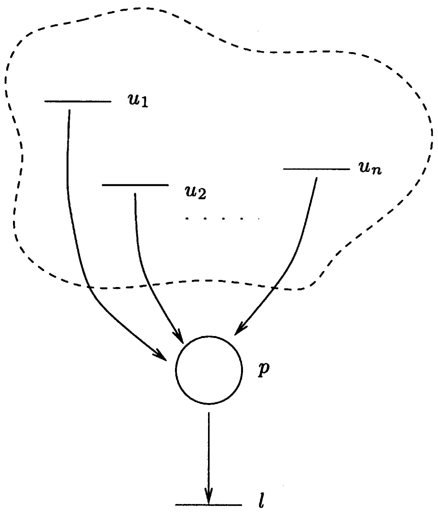

Fig. 7. The transformation used to produce system N' from system N, used to show that group liveness reduces to liveness.

m_1' corresponding to m_1; we have that $m_1' \in [m'\rangle$ and t is enabled at m_1'; thus $m_1'[t\rangle m_2'$; moreover $m_2'(p) \geqslant 1$, hence l is enabled at m_2'.

Now suppose l live in N'. let $m \in [m_0\rangle$ in N; take the corresponding m' in N': if $m'(p) = k$, let σ be a sequence of k occurrences of l; σ can be fired at m', and $m'[\sigma\rangle m_1'$, with $m_1'(p) = 0$; by liveness of l, there exists m_2' and τ such that $m_1'[\tau\rangle m_2'$ and l is enabled at m_2'; hence, $m_2'(p) \geqslant 1$; the only transitions producing tokens in p are those in U, so there exists a transition $t \in U$ occurring in τ, i.e. $\tau = \tau_1 u \tau_2$, with $u \in U$; let τ_1' be the sequence obtained by removing all occurrences of l from τ; then there exists a marking m' such that $m[\tau_1'\rangle m'[u\rangle$. $\square$

Corollary 5.13. *Group liveness is decidable for P/T systems.*

Proof. An easy consequence of Theorems 3.15 and 5.12. $\square$

Each transition t of N is split in a set of corresponding transitions $\{t^\mu\}$ in $Norm(N)$, and we can show that t is live in N iff $\{t^\mu\}$ is group live in $Norm(N)$.

Theorem 5.14. *Transition t is live in N if and only if the set of transitions $\{t^\mu \mid t^\mu \in T'\}$ is group live in $Norm(N)$.*

Proof. Suppose t live in N. We show that $U_t = \{t^\mu \mid t^\mu \in T'\}$ is group live in $Norm(N)$. Let $m' \in [m_0'\rangle$; we want to show that there exists a marking reachable from m' and a transition in U_t that is enabled in such marking.

As $m' \in [m_0'\rangle$, there exists a firing sequence $m_0'[t_1^{\mu_1}\rangle \ldots [t_n^{\mu_n}\rangle m_n'$ in $Norm(N)$ such that $m_n' = m'$. By Lemma 5.6 $m_0[t_1\rangle \ldots [t_n\rangle m_n$ is a firing sequence of N, and m_n, m_n' are linked by the relation described in that lemma. Thus $m_n \in [m_0\rangle$. As t is live in N, there exists $\bar{m} \in [m_n\rangle$ such that $\bar{m}[t\rangle$; hence we have the following firing sequence: $m_0[t_1\rangle \dots [t_n\rangle m_n \dots [t_{n+k}\rangle m_{n+k}[t\rangle$, with $m_{n+k} = \bar{m}$. By Lemma 5.2 there exists a firing sequence of $Norm(N)$ with the form $m'_0[t_1^{\bar{\mu}_1}\rangle \dots [t_n^{\bar{\mu}_n}\rangle \bar{m}'_n \dots [t_{n+k}^{\bar{\mu}_{n+k}}\rangle \bar{m}'_{n+k}[t^{\mu}\rangle$.

As $m'_0[t_1^{\mu_1}\rangle \dots [t_n^{\mu_n}\rangle m'_n$, by Lemma 5.5 we have that $\bar{m}'_n = m'_n$; hence there exist a marking, $\bar{m}'_{n+k}$, such that $\bar{m}'_{n+k} \in [m'_n\rangle$, and a transition $t^{\mu} \in U_t$ such that $\bar{m}'_{n+k}[t^{\mu}\rangle$.

Suppose U_t group live. We show that t is live. Let $m \in m_0$; we want to show that there exists a marking reachable from m such that t is enabled in that marking.

As $m \in [m_0\rangle$, there exists a firing sequence $m_0[t_1\rangle \dots [t_n\rangle m_n$ such that $m_n = m$. By Lemma 5.2 we have a firing sequence with the following form in $Norm(N)$: $m'_0[t_1^{\mu_1}\rangle \dots [t_n^{\mu_n}\rangle m'_n$. Hence $m'_n \in [m'_0\rangle$; as U_t is group live in $Norm(N)$, there exist a marking $\bar{m}' \in [m'_n\rangle$ and a transition $t^{\mu} \in U_t$ such that $\bar{m}'[t^{\mu}\rangle$. Hence we have a firing sequence $m'_0[t_1^{\mu_1}\rangle \dots [t_n^{\mu_n}\rangle m'_n \dots [t_{n+k}^{\mu_{n+k}}\rangle m'_{n+k}[t^{\mu}\rangle$ such that $m'_{n+k} = \bar{m}'$.

By Lemma 5.6 there exists a firing sequence of N with the form $m_0[t_1\rangle \dots [t_n\rangle \bar{m}_n \dots [t_{n+k}\rangle \bar{m}_{n+k}[t\rangle$. As $m_0[t_1\rangle \dots [t_n\rangle m_n$ we have $\bar{m}_n = m_n$, hence there exists a marking $\bar{m}_{n+k} \in [m_n\rangle$ such that $\bar{m}_{n+k}[t\rangle$. $\square$

Corollary 5.15. *Liveness is decidable for primitive nets.*

Proof. An easy consequence of Theorem 5.14 and Corollary 5.13. $\square$

5.4. Simulation

We introduce a notion of simulation of a primitive system by a P/T system: we define a labelling of each transition of the P/T system with a transition of the primitive system, and require the firing sequences of the P/T system to simulate the firing sequences of the primitive nets via the labelling; in other words, for any transition sequence $t_1 \dots t_n$ of the primitive system there is a transition sequence of the P/T system that is mapped by the labelling on $t_1 \dots t_n$; moreover, the sequence obtained applying the labelling to any transition sequence of the P/T system is a transition sequence of the primitive system.

Definition 5.16. Let $N = (S, T, F, I, m_0)$ be a P/T system with inhibitor arcs and $N' = (S', T', F', m'_0)$ be a P/T system. We say that N' *simulates* the transition sequences of N iff there exists a mapping $\eta : T' \to T$ such that

- If $t_1 \dots t_n$ is a transition sequence of N then there exists a transition sequence $t'_1 \dots t'_n$ of N' such that $\eta(t'_i) = t_i$ for $i = 1, \dots, n$;
- If $t'_1 \dots t'_n$ is a transition sequence of N' then $\eta(t'_1) \dots \eta(t'_n)$ is a transition sequence of N.

We show that the class of primitive nets is the largest one for which there exists a P/T net simulating its transition sequences.

Theorem 5.17. *The class of primitive nets is the largest one that can be simulated by a P/T net.*

Proof. Let $N=(S,T,F,I,m_0)$ be a PTI system.

If N is primitive, take the P/T system $Norm(N)$ and the mapping $\eta : t^{\mu} \rightarrow t$. We show that $Norm(N)$ simulates N.

Let $t_1^{\mu_1} \ldots t_n^{\mu_n}$ be a transition sequence of $Norm(N)$; hence there exist markings $m_1', \ldots, m_n'$ such that $m_0'[t_1^{\mu_1}\rangle \ldots [t_n^{\mu_n}\rangle m_n'$; by Lemma 5.6 we have that $m_0[t_1\rangle \ldots [t_n\rangle m_n$ in N, hence $t_1 \ldots t_n$ is a transition sequence of N, moreover $\eta(t_1^{\mu_1} \ldots t_n^{\mu_n}) = t_1 \ldots t_n$.

Let $t_1 \ldots t_n$ be a transition sequence of N; hence there exist $m_1, \ldots, m_n$ such that $m_0[t_1\rangle \ldots [t_n\rangle m_n$; by Lemma 5.2 we have $m_0'[t_1^{\mu_1}\rangle \ldots [t_n^{\mu_n}\rangle m_n'$ in $Norm(N)$, hence $t_1^{\mu_1} \ldots t_n^{\mu_n}$ is a transition sequence of $Norm(N)$, moreover $\eta(t_1^{\mu_1} \ldots t_n^{\mu_n}) = t_1 \ldots t_n$.

Hence $Norm(N)$ simulates the transition sequences of N.

We show that if N is not primitive then there exist no P/T system simulating N. If N is not primitive, then there exists a place $s \in Inib(N)$ such that, for all $k \geqslant 0$, exists $m_k' \in [m_0\rangle$ such that $m_k'(s) > k$ and exist $m_k'' \in [m_k'\rangle$ and $t_k \in T$ such that $m_k''[t_k\rangle$ and $s \in {}^{\circ}t_k$.

In other words, for any $k \geqslant 0$ there exist m_k', m_k'', σ_k, τ_k and t_k such that

$$m_0[\sigma_k\rangle m_k'[\tau_k\rangle m_k''[t_k\rangle,$$

$$m_k'(s) > k,$$

$$s \in {}^{\circ}t_k.$$

Suppose there exists a P/T net $\bar{N} = (\bar{S}, \bar{T}, \bar{F}, \bar{m}_0)$ simulating N by a mapping $\eta : \bar{T} \rightarrow T$.

Hence, for any $k \geqslant 0$, there exist $\bar{m}_k'$, $\bar{m}_k''$, $\bar{\sigma}_k$, $\bar{\tau}_k$ and $\bar{t}_k$ such that

$$\bar{m}_0[\bar{\sigma}_k\rangle \bar{m}_k'[\bar{\tau}_k\rangle \bar{m}_k''[\bar{t}_k\rangle,$$

$$\eta(\bar{\sigma}_k) = \sigma_k,$$

$$\eta(\bar{\tau}_k) = \tau_k,$$

$$\eta(\bar{t}_k) = t_k.$$

Consider the sequence of markings, of $\bar{N}$, $\bar{m}_0', \bar{m}_1', \ldots, \bar{m}_i', \ldots$; by Lemma 4.9, this sequence contains a nondecreasing subsequence, i.e. there exists a sequence of indexes $k_1 < k_2 < \cdots < k_i < \cdots$ such that $\bar{m}_{k_1}' \subseteq \bar{m}_{k_2}' \subseteq \cdots \subseteq \bar{m}_{k_i}' \subseteq \cdots$.

We choose an index k_j such that $k_j > m_{k_1}'(s)$ (as $m_{k_1}'(s) > k_1$, we have $k_j > k_1$, hence $j > 1$).

We know that $\bar{m}_0[\bar{\sigma}_{k_1}\rangle \bar{m}_{k_1}'[\bar{\tau}_{k_1}\rangle \bar{m}_{k_1}''[\bar{\sigma}_{k_1}\rangle$; we know also that $\bar{m}_0[\bar{\sigma}_{k_j}\rangle \bar{m}_{k_j}'$. As $\bar{m}_{k_1}' \subseteq \bar{m}_{k_j}'$, the transition sequence $\bar{\tau}_{k_1}\bar{t}_{k_1}$ is firable also at $\bar{m}_{k_j}'$, i.e.

$$\bar{m}_0[\bar{\sigma}_{k_j}\rangle \bar{m}_{k_j}'[\bar{\tau}_{k_1}\bar{t}_{k_1}\rangle.$$

Hence $\bar{\sigma}_{k_j}\bar{\tau}_{k_1}\bar{t}_{k_1}$ is a transition sequence of $\bar{N}$. We have that $\eta(\bar{\sigma}_{k_j}\bar{\tau}_{k_1}\bar{t}_{k_1}) = \sigma_{k_j}\tau_{k_1}t_{k_1}$.

Now we show that $\sigma_{k_j}\tau_{k_1}t_{k_1}$ is not a transition sequence of N. We know that $m_0[\sigma_{k_j}\rangle m'_{k_j}$.

If $\neg m'_{k_j}[\tau_{k_1}\rangle$ we have done.

Suppose $m'_{k_j}[\tau_{k_1}\rangle m'''$; we show that $\neg m'''[t_{k_1}\rangle$.

We know that $m'_{k_1}[\tau_{k_1}\rangle m''_{k_1}$, $m'_{k_1}(s) > k_1$, $m''_{k_1}[t_{k_1}\rangle$ and $s \in {}^\circ t_{k_1}$. From $m''_{k_1}[t_{k_1}\rangle$ and $s \in {}^\circ t_{k_1}$ we get $m''_{k_1}(s) = 0$. Let $\tau_{k_1} = u_1 \dots u_n$, with $u_i \in T$ for $i = 1, \dots, n$. It is easy to see that

$$m''_{k_1}(s) = m'_{k_1}(s) - \sum_{i=1}^{n} {}^\bullet u_i(s) + \sum_{i=1}^{n} u_i^\bullet(s).$$

As $m''_{k_1}(s) = 0$, we obtain

$$m'_{k_1}(s) = \sum_{i=1}^{n} {}^\bullet u_i(s) - \sum_{i=1}^{n} u_i^\bullet(s).$$

From $m'_{k_j}[\tau_{k_1}\rangle m'''$ we have

$$\begin{aligned} m'''(s) &= m'_{k_j}(s) - \sum_{i=1}^{n} {}^\bullet u_i(s) + \sum_{i=1}^{n} u_i^\bullet(s) \\ &= m'_{k_j}(s) - m'_{k_1}(s). \end{aligned}$$

We have $m'_{k_j}(s) > k_j$ and we have chosen k_j in such a way that $k_j > m'_{k_1}(s)$.

Hence we have $m'_{k_j}(s) > m'_{k_1}(s)$, thus $m'''(s) = m'_{k_j}(s) - m'_{k_1}(s) > 0$. As $s \in {}^\circ t_{k_1}$, t_{k_1} is not enabled at m'''.

Hence $\sigma_{k_j}\tau_{k_1}t_{k_1}$ is not a transition sequence of N, i.e. $\bar{N}$ does not simulate N. □

We have constructed a mapping from primitive nets to P/T nets which preserves the firing sequences, but in general step firing sequences are not preserved: take for example two transitions t_1 and t_2 with disjoint preset but with the same inhibitor set. If the two transitions do not produce tokens in the inhibitor set, then they are concurrently enabled in the primitive net, and can fire together in the same step, but this does not happen in the corresponding P/T net. Now we show that we cannot do better, that is, there exists a primitive net for which no P/T net can exhibit the same step behaviour.

Consider the system in Fig. 8; it is easy to see that it is a primitive net, because the unique inhibiting place is bounded. We will show that there exists no P/T system with the same step transition sequences, in the following sense:

Definition 5.18. Let $N = (S, T, F, I, m_0)$ and $N' = (S', T', F', m'_0)$ be a PTI system and P/T system, respectively. We say that N' *simulates* the step transition sequences of N iff there exists a mapping $\eta : T' \to T$ such that

- if $G_1 \dots G_n$ is a step transition sequence of N then there exists a step transition sequence $G'_1 \dots G'_n$ of N' such that $\eta(G'_i) = G_i$ for $i = 1, \dots, n$;
- if $G'_1 \dots G'_n$ is a step transition sequence of N' then $\eta(G'_1) \dots \eta(G'_n)$ is step a transition sequence of N.

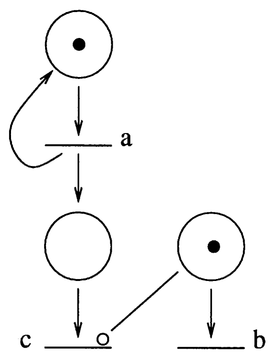

Fig. 8. A primitive system for which there exists no P/T system simulating its step transition sequences.

To prove the impossibility for a P/T system to simulate the step transition sequences of the system in Fig. 8 we need the following auxiliary result: given a sequence of multisets, over a finite set and with increasing cardinality, there exists an element whose number of occurrences in the multisets can exceed any bound, provided we take a sufficiently big multiset.

Lemma 5.19. *Let $\{C_i\}_{i \in \omega}$ be a sequence of multisets over the finite set T with increasing cardinality, i.e. $|C_i| < |C_{i+1}|$ for all $i \in \omega$. Then there exists $t \in T$ such that, for each natural number k, there exists i_k such that $C_{i_k}(t) > k$.*

Proof. Suppose that for each $t \in T$ there exist a limit k_t such that, for each i, $C_i(t) \leqslant k_t$. Let $max = \sum_{t \in T} k_t$; we have that $|C_i| \leqslant max$ for each i, hence $|C_{max}| = |C_{max+1}|$, a contradiction. □

Theorem 5.20. *Let N be the primitive system in Fig. 8. There exists no (finite) P/T system simulating its step transition sequences.*

Proof. Suppose there exists a P/T system N' and a mapping $\eta : T' \to T$ simulating the step transition sequences of N.

It is easy to see that, for each n, the step sequence formed by n occurrences of the step containing a single occurrence of transition a, followed by the step with a single occurrence of b, followed by the step containing n occurrences of c, i.e.

$$\underbrace{a \ldots a}_{n} b \{\underbrace{c, \ldots, c}_{n}\}$$

is a legal step transition sequence of N.

As N' simulates the step transition sequences of N, there exists a sequence $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ of step transition sequences of N' of the form

$$\tau_n = a_{n1} \ldots a_{nn} b_n \{c_{n1}, \ldots, c_{nn}\}$$

such that $\eta(a_{ij}) = a$, $\eta(b_i) = b$ and $\eta(c_{ij}) = c$.

As the set $\{b_i \mid \eta(b_i) = b\}$ is finite, there exists an index k such that b_k occurs infinitely often in the elements of the sequence $\{\tau_n\}_{n\in\omega}$; consider the subsequence $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_j}\ldots$ where b_k occurs, i.e. such that $b_{i_j} = b_k$.

The final step of each step sequence τ_n has cardinality n, hence the final steps of the subsequence $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_j}\ldots$ have increasing cardinality. Hence for Lemma 5.19 there exists c_h whose number of occurrences in the final step can exceed any limit, provided we take a sufficiently large i_j.

Now we show that the net N' admits a transition sequence of the form $a_{l1}\ldots a_{ll}c_h$, for a sufficiently large l; it's easy to see that $\eta(a_{l1}\ldots a_{ll}c_h) = \underbrace{a\ldots a}_{l}c$ is not a transition sequence of N.

Two cases can occur:

- $b_k^\bullet \cap {}^\bullet c_h = \emptyset$: take the first sequence τ_{i_j} such that c_h occurs at least one time in the last step (this sequence exists by Lemma 5.19). We have that the number of tokens in the places in the preset of c_h is not increased by the firing of b_k, hence a step formed by a single occurrence c_h can fire also before the firing of b_k, producing a firing sequence of the form $a_{i_j1}\ldots a_{i_ji_j}c_h$.
- $b_k^\bullet \cap {}^\bullet c_h \neq \emptyset$: let $maxpo_b = Max\{b_k^\bullet(s) \mid s \in S'\}$ be the maximum weight of the arcs exiting from b_k and $maxpre_c = Max\{{}^\bullet c_h(s) | s \in S'\}$ be the maximum weight of the arcs entering in c_h. Take the first sequence τ_{i_j} such that c_h occurs at least $maxpo_b + maxpre_c$ times in the last step (this sequence exists by Lemma 5.19). Let m_1' be the marking reached after the firing of the a_{pq} transitions and m_2' be the marking reached after the firing of b_k, i.e. $m_0'[a_{i_j1}\rangle\ldots[a_{i_ji_j}\rangle m_1'[b_k\rangle m_2'$. Now we show that c_h is enabled at m_1': let $s \in {}^\bullet c_h$; we have $m_2'(s) = m_1'(s) - {}^\bullet b_k(s) + b_k^\bullet(s)$, hence $m_1'(s) = m_2'(s) - b_k^\bullet(s) + {}^\bullet b_k(s)$; as $maxpo_b + maxpre_c$ occurrences of c_h are enabled at m_2', each place in the preset of c_h contains at least $maxpo_b + maxpre_c$ tokens in m_2'. Moreover $b_k^\bullet(s) \leqslant maxpo_b$, hence $m_1'(s) = m_2'(s) - b_k^\bullet(s) + {}^\bullet b_k(s) \geqslant (maxpo_b + maxpre_c) - maxpo_b + {}^\bullet b_k(s) \geqslant maxpre_c + {}^\bullet b_k(s) \geqslant maxpre_c \geqslant {}^\bullet c_h(s)$. Thus c_h is enabled at m_1', and $a_{i_j1}\ldots a_{i_ji_j}c_h$ is a firing sequence. $\square$

5.5. Linear time μ-calculus and bisimulation

We investigate some properties of labelled primitive systems: we show that the linear time μ-calculus is decidable and that the encoding on P/T systems preserves the interleaving behaviour.

The mapping of primitive systems on P/T systems, defined for unlabelled nets, can be easily extended to labelled nets by defining $l'(t^\mu) = l(t)$.

5.5.1. Linear time μ-calculus

We show that the linear time μ-calculus is decidable for primitive systems, by reduction to the decidability of the calculus for P/T systems.

Note that Definitions 3.17 and 3.18 can be applied also to contextual P/T systems.

Lemma 5.21. *Let N be a primitive system. Then $L^\infty(N) = L^\infty(Norm(N))$.*

Proof. We prove that $L^\infty(N) \subseteq L^\infty(Norm(N))$. Take $\sigma \in L^\infty(N)$. Two cases can occur:

- $\sigma \in L(N)$: let $n = |\sigma|$; there exists a firing sequence $m_0[t_1\rangle \ldots [t_n\rangle m_n$ of N such that $l(t_i) = \sigma(i)$ for $i = 1, \ldots, n$. By Lemma 5.2 the following is a firing sequence of $Norm(N)$: $m'_0[t_1^{\mu_1}\rangle \ldots [t_n^{\mu_n}\rangle m'_n$. We have $l'(t_i^{\mu_i}) = l(t_i) = \sigma(i)$ for $i = 1, \ldots, n$, hence $\sigma \in L(Norm(N))$.
- $\sigma \in L^\omega(N)$: then there exists an infinite firing sequence $m_0[t_1\rangle \ldots [t_i\rangle m_i \ldots$ of N such that $l(t_i) = \sigma(i)$ for $i \in \omega^+$. Hence for each i we have that $m_0[t_1\rangle \ldots [t_i\rangle m_i$ is a firing sequence of N. By Lemma 5.2 and Corollary 5.5 we have that, for each i, $m'_0[t_1^{\mu_1}\rangle \ldots [t_i^{\mu_i}\rangle m'_i$ is a firing sequence of $Norm(N)$. Hence $m'_0[t_1^{\mu_1}\rangle \ldots [t_i^{\mu_i}\rangle m'_i \ldots$ is an infinite firing sequence of $Norm(N)$; moreover $l'(t_i^{\mu_i} = l(t_i) = \sigma(i)$ for $i \in \omega^+$, hence $\sigma \in L^\omega(Norm(N))$.

As $L^\infty(Norm(N)) = L(Norm(N)) \cup L^\omega(Norm(N))$, we have obtained $\sigma \in L^\infty(Norm(N))$.

The other inclusion can be proved in a similar way, using Lemma 5.6. □

Theorem 5.22. *The model checking problem for the linear time μ-calculus and labelled primitive systems is decidable.*

Proof. To decide the model checking problem for a formula ϕ and a labelled primitive system N we have to check if $L^\infty(N) \subseteq [\![\phi]\!]$. By Lemma 5.21 we have $L^\infty(N) = L^\infty(Norm(N))$, hence we have reduced the problem to check if $L^\infty(Norm(N)) \subseteq [\![\phi]\!]$, which is decidable because $Norm(N)$ is a P/T system (cf. Theorem 3.19). □

5.5.2. Bisimulation

We show that our mapping preserves the interleaving behaviour, i.e. there exists a bisimulation between N and $Norm(N)$.

Theorem 5.23. *Let N be a primitive net. Then N and $Norm(N)$ are bisimilar.*

Proof. Let $N = (S, T, F, I, m_0, l)$, $Norm(N) = (S', T', F', m'_0, l')$ and $\eta(t^\mu) = t$ for all $t^\mu \in T'$.

Let $R = \{(m, m') \mid$ there exists a transition sequence σ of $Norm(N)$ such that $m_0[\eta(\sigma)\rangle m$ and $m'_0[\sigma\rangle m'\}$.

We show that R is a bisimulation:

- We have that $(m_0, m'_0) \in R$ (we take the empty sequence as σ).
- Suppose $(m_1, m'_1) \in R$ and $m_1 \xrightarrow{a} m_2$. We show that there exists m'_2 such that $m'_1 \xrightarrow{a} m'_2$ and $(m_2, m'_2) \in R$. As $(m_1, m'_1) \in R$, there exists σ such that $m_0[\eta(\sigma)\rangle m_1$ and $m'_0[\sigma\rangle m'_1$. If $m_1 \xrightarrow{a} m_2$, then there exists a transition $t \in T$ such that $l(t) = a$ and $m_1[t\rangle m_2$. Hence we have $m_0[\eta(\sigma)\rangle m_1[t\rangle m_2$. By Lemma 5.2 we have that $m'_0[\tau\rangle \bar{m}'_1 [t^\mu\rangle m'_2$, with $\eta(\tau) = \eta(\sigma)$. By Lemma 5.5 we have that $\tau = \sigma$ and $\bar{m}'_1 = m'_1$. As $l'(t^\mu) = l(t) = a$, we have $m'_1 \xrightarrow{a} m'_2$. Moreover, $\eta(\tau t^\mu) = \eta(\tau)t = \eta(\sigma)t$, hence $(m_2, m'_2) \in R$.

- Suppose $(m_1, m_1') \in R$ and $m_1' \xrightarrow{a} m_2'$. We show that there exists m_2 such that $m_1 \xrightarrow{a} m_2$ and $(m_2, m_2') \in R$. As $(m_1, m_1') \in R$, there exists σ such that $m_0[\eta(\sigma)\rangle m_1$ and $m_0'[\sigma\rangle m_1'$. If $m_1' \xrightarrow{a} m_2'$, then there exists a transition t^μ such that $l'(t^\mu) = a$ and $m_1'[t^\mu\rangle m_2'$. Hence we have $m_0'[\sigma\rangle m_1'[t^\mu\rangle m_2'$. By Lemma 5.6 we have that $m_0[\eta(\sigma)\rangle m_1[t\rangle m_2$. We have $l(t) = l'(t^\mu) = a$, hence $m_1 \xrightarrow{a} m_2$. Moreover, $\eta(\sigma t^\mu) = \eta(\sigma)t$, hence $(m_2, m_2') \in R$. □

6. Expressiveness of languages generated by PTI systems

In this section we make use of the results established in the previous sections to obtain information about the expressiveness of inhibitor arcs w.r.t. the class of generated languages.

The class of languages generated by labelled primitive systems coincides with the class of languages generated by standard P/T systems:

Corollary 6.1. *Let N be a labelled primitive system. Then there exists a labelled P/T system N' such that $L(N) = L(N')$.*

Proof. An easy consequence of Theorem 5.17. □

The result presented above does not hold for general PTI systems:

Theorem 6.2. *There exists a labelled P/T system with inhibitor arcs N such that $L(N)$ cannot be generated by any (finite) labelled P/T system.*

Proof. Consider the labelled PTI system N depicted in Fig. 3: it is easy to see that $a^k b^k c \in L(N)$, whereas $a^k b^h c \notin L(N)$ for $k \neq h$.

Suppose there exists a labelled P/T system $\bar{N}$ such that $L(\bar{N}) = L(N)$; note that, after the firing of k occurrences of transition a, the inhibiting place s contains k tokens; by instantiating the proof of Theorem 5.17 we obtain that there exist k_1 and k_j such that $k_1 < k_j$ and $a^{k_j} b^{k_1} c \in L(\bar{N})$, but $a^{k_j} b^{k_1} c \notin L(N)$, a contradiction. □

To find an expressiveness gap between primitive systems and P/T systems, we need to look at sequences of multisets of labels:

Definition 6.3. Let $N = (S, T, F, K, I, m_0, l)$ be a labelled PTI system. The step-language of N is defined as $SL(N) = \{A_1 \dots A_n \mid m_0[G_1\rangle \dots [G_n\rangle m_n$ is a step firing sequence of N and $A_i = l(G_i) \in \mathcal{M}_{fin}(Act)$ for $i = 1, \dots, n\}$.

The class of step-languages generated by labelled primitive systems is richer that the one generated by labelled P/T systems:

Corollary 6.4. *There exists a labelled primitive system N such that $SL(N)$ cannot be generated by any labelled P/T system.*

Proof. An easy consequence of Theorem 5.20. □

7. Conclusion

In this paper we have extended the analysis techniques developed for P/T systems to primitive systems, a subclass of P/T systems with inhibitor arcs. We have also found a characterization of primitive systems as the maximal class of PTI systems whose firing sequences can be simulated by a P/T system. An interesting research topic concerns the identification of classes for which, even if their firing sequences cannot be simulated by any P/T system, it is possible to construct a P/T system with an equivalent behaviour w.r.t. deadlock or reachable markings. Some preliminary study in this direction can be found in [4], where a deadlock preserving transformation on P/T systems is provided for a class of PTI systems, generated by the terms of a Linda-based process algebra, and in general not primitive. Actually, there exists a close correspondence between markings of the PTI system and of the corresponding P/T system, which could be used to decide marking reachability in the PTI system by reduction to submarking reachability in the corresponding P/T system. The characterization of a larger class for which such a transformation is feasible deserves further investigation.

We have seen that the step transition sequences of primitive systems cannot, in general, be simulated by a P/T system; it should be worthwhile interesting to identify a subclass of primitive systems whose step transition sequences can be simulated by a P/T system.

To lighten the notation and to simplify the proofs, in this paper we have considered PTI systems with unweighted inhibitor arcs, i.e. arcs permitting to test for absence of tokens in a place. A generalization of PTI systems with unweighted inhibitor arcs consists in decorating each inhibitor arc with a positive number, called arc weight. If an inhibitor arc with weight k connects place s and transition t, then t can fire only if place s contains at most $k-1$ tokens. All the analysis and simulation techniques developed for primitive systems with unweighted arcs can be rephrased in the more general setting of PTI systems with weighted arcs.

Appendix A. Turing equivalence of P/T systems with inhibitor arcs

We recall the result about Turing equivalence of P/T systems with inhibitor arcs; this result was proved for the first time in [1]; here we show that P/T systems with inhibitor arcs are expressive enough to model Random Access Machines (RAM) [23]; similar proofs can be found e.g. in [10, 21].

A RAM is a computational model composed of a finite set of registers, that can hold arbitrary large natural numbers, and by a program, that is a sequence of simple numbered instructions, like arithmetical operations on the contents of registers or conditional jumps.

To perform a computation, the inputs are provided in registers $r_1,\dots,r_m$; if other registers $r_{m+1},\dots,r_n$ are used in the program, they are supposed to contain the value 0 at the beginning of the computation. The execution of the program begins with the first instruction and continues by executing the other instructions in sequence, unless a jump instruction is encountered. The execution stops when an instruction number higher than the length of the program is reached; this happens if the program was executing the last instruction of the program and this instruction does not require a jump, or if the current instruction requires a jump to an instruction number not appearing in the program. If the program terminates, the result of the computation is the contents of the registers specified as outputs.

In [19] it is shown that the following two instructions are sufficient to model every recursive function:

- $Succ(r_j)$: add 1 to the contents of register r_j;
- $DecJump(r_j,s)$: if the contents of register r_j is not zero, then decrease it by 1 and go to the next instruction, otherwise jump to instruction s.

For example, the following program computes the sum of registers r_1 and r_2, putting the result in register r_1 (note that the third instruction corresponds to an unconditional jump, because register r_3 contains the value 0 at the beginning of the computation and its contents is never modified by the program):

1: $DecJump(r_2,4)$,
2: $Succ(r_1)$,
3: $DecJump(r_3,1)$.

In the net representation of a RAM, we model the registers and the program counter by means of places: we represent the fact that the program counter points to instruction i by the presence of one token in the corresponding place p_i; the contents of register r_j is represented by an equal number of tokens in place r_j (see Fig. 9).

A *Succ* instruction on register r_j at position i is represented by a transition that consumes a token in program counter place p_i, adds one token in the register place r_j and updates the program counter by putting one token in place p_{i+1}. An instruction $DecJump(r_j,s)$ at position i is modeled by two transitions: both transitions consume the token in the program counter place p_i; the first one, managing the fact that the contents of register r_j is greater than zero, consumes one token from r_j and produces a token in place p_{i+1}; the second one, managing the fact that the contents of r_j is zero, tests place r_j for absence and produces a token in place p_s, corresponding to perform a jump to instruction s.

To perform the computation for input $n_1,\dots,n_k$, the initial marking is composed by one token in p_1 and n_j tokens in r_j, for $j=1,\dots,k$.

It is easy to see that the RAM program terminates iff the PTI system has a dead marking, and the contents of the registers at the end of the RAM program corresponds to the number of tokens in the corresponding register places in the PTI system.

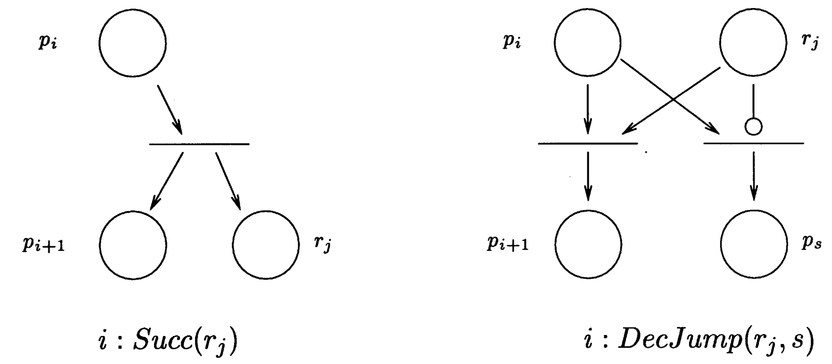

Fig. 9. Representing a RAM by a PT system with inhibitor arcs.

Moreover, note that the obtained PTI system is deterministic, hence it has only one possible execution.

References

[1] T. Agerwala, A complete model for representing the coordination of asynchronous processes, Hopkins Computer Research Report 32, John Hopkins University, 1974.

[2] N. Busi, Petri nets with inhibitor and read arcs: semantics, analysis and application to process calculi, Ph.D. Thesis, Department of Mathematics, University of Siena, Italy, 1998.

[3] N. Busi, R. Gorrieri, A Petri net semantics for π-calculus, Proc. Concur'95, Lecture Notes in Computer Science, Vol. 962, Springer, Berlin, 1995, pp. 145–159.

[4] N. Busi, R. Gorrieri, G. Zavattaro, On the expressiveness of Linda coordination primitives, Inform. Comput. 156 (1/2) (2000) 90–121.

[5] N. Busi, G.M. Pinna, Process semantics for place/transition nets with inhibitor and read arcs, Fund. Inform. 40 (1999) 165–197.

[6] A. Cheng, J. Esparza, J. Palsberg, Complexity results for 1-safe nets, Theoret. Comput. Sci. 147 (1995) 117–136.

[7] J. Desel, J. Esparza, Free Choice Petri Nets, Cambridge Tracts in Theoretical Computer Science, Vol. 40, Cambridge Univ. Press, Cambridge, 1995.

[8] J. Esparza, Decidability of model checking for infinite-state concurrent systems, Acta Inform. 34 (1997) 85–107.

[9] M. Flynn, T. Agerwala, Comments on capabilities, limitations and 'correctness' of Petri nets, Comput. Archit. News 4 2 (1973).

[10] M. Hack, The recursive equivalence of the reachability problem and the liveness problem for Petri nets and Vector Addition Systems, Proc. 15th Ann. Symp. on Switching and Automata Theory, IEEE, 1974.

[11] M. Hack, Petri net languages, Tech. Rep. 159, MIT, 1976.

[12] R.M. Karp, R.E. Miller, Parallel program schemata, J. Comput. System Sci. 3 (1969) 147–195.

[13] S.R. Kosaraju, Decidability of reachability in vector addition systems, Proc. 6th ACM Symp. on the Theory of Computing, 1982, pp. 267–281.

[14] R. Janicki, M. Koutny, Semantics of inhibitor nets, Inform. Comput. 123 (1995) 1–16.

[15] K. Lautenbach, Liveness in Petri nets, Internal Report GMD-ISF 72-02.1, 1972.

[16] R. Lipton, The reachability problem requires exponential space, Tech. Rep. 62, Yale University, 1976.

[17] E.W. Mayr, An algorithm for the general Petri net reachability problem, SIAM J. Comput. 13 (1984) 441–460.

[18] R. Milner, J. Parrow, D. Walker, A calculus of mobile processes, Inform. Comput. 100 (1992) 1–77.

[19] M.L. Minsky, Computation: Finite and Infinite Machines, Prentice-Hall, Englewood Cliffs, NJ, 1967.

[20] U. Montanari, F. Rossi, Contextual nets, Acta Inform. 32 (1995) 545–596.
[21] J.L. Peterson, Petri Net Theory and the Modeling of Systems, Prentice-Hall, Englewood Cliffs, NJ, 1981.
[22] C.A. Petri, Kommunikation mit Automaten, Ph.D. Thesis, Institut für Instrumentelle Mathematik, Bonn, Germany, 1962.
[23] J.C. Shepherdson, J.E. Sturgis, Computability of recursive functions, J. ACM 10 (1963) 217–255.
[24] W. Vogler, Partial order semantics and read arcs, Tech. Rep. 1997-01, Institut für Informatik, Universität Augsburg, 1997. (An extended abstract appeared in Proc. MFCS'97.)