ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

International Journal of Solids and Structures 45 (2008) 2160–2172

INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES

www.elsevier.com/locate/ijsolstr

On the permeability of fibre-reinforced porous materials

Salvatore Federico, Walter Herzog *

Human Performance Laboratory, Faculty of Kinesiology, The University of Calgary, 2500 University Drive NW, Calgary, Alta., Canada T2N 1N4

Received 11 May 2007; received in revised form 4 September 2007
Available online 4 December 2007

Abstract

Biological tissues can be considered as composite materials comprised of a porous matrix filled with interstitial fluid and reinforced by impermeable collagen fibres. Motivated by studies on fluid flow in articular cartilage, we would like to quantify the undeformed configuration permeability of fibre-reinforced composite materials. If there is a sufficient scale separation between the internal structure of the porous matrix and the arrangement of the fibres, the matrix can be taken as a porous continuum at the fibre scale. In this case, the fibres can be treated as inclusions in a porous continuum, and the overall permeability of the composite can be evaluated using homogenisation procedures. For an isotropic homogeneous matrix, the symmetry of the system is governed by the orientation of the fibres. Here, we propose to retrieve the overall permeability through geometrical considerations and directional averaging methods. The special case of transverse isotropy is discussed in detail, with particular attention to the sub-cases of aligned fibres and fibres lying on a plane.

Keywords: Composite material; Fibre-reinforced; Permeability; Fluid flow; Anisotropy; Homogenisation

1. Introduction

Biological soft tissues are often regarded as biphasic continua, with a porous solid phase saturated by an interstitial fluid. The solid phase is comprised of collagen fibres and other macromolecules, while the fluid contains ions and other chemical agents (see, e.g., Fung, 1993).

The arrangement of the collagen fibres is thought to relate closely to their function in the tissue, and it has been thought that they constitute the main source of anisotropy in the elastic properties (see, e.g., Farquhar et al., 1990; Soulhat et al., 2000; Holzapfel et al., 2000; Ogden, 2003; Federico et al., 2005; Gasser et al., 2006). However, the effect of the arrangement of collagen fibres on permeability has not been studied theoretically, although experimental evidence suggests that its effect on soft tissue fluid flow might be significant (Han et al., 2000; Wellen et al., 2004; Maroudas and Bullough, 1968). The purpose of this study was to develop a mathematical model of the effect of fibre arrangement on the permeability of a porous fibre-reinforced composite

* Corresponding author. Tel.: +1 403 220 8525; fax: +1 403 220 2070.
E-mail address: walter@kin.ucalgary.ca (W. Herzog).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2007.11.014

material in the undeformed configuration. Therefore, the change in permeability due the fact that the size of the pores changes considerably under large deformations (e.g., Holmes and Mow, 1990) is not taken into account.

We assumed that the spacing between the fibres is at least one order of magnitude larger than the size of the pores of the matrix. This is the case for many biological tissues, such as articular cartilage, for which the spacing between collagen fibres is approximately 10–40 nm (an estimate derived from Hedlund et al. (1993) and Långsjö et al. (1999)), and the spacing between the matrix macromolecules is approximately 2–4 nm (Quinn et al., 2001).

The assumption of *scale separation* implies that we deal with three distinct scales: the macroscale (whole system) the mesoscale (fibres), and the microscale (pores of the matrix). This can be seen from two equivalent points of view:

(a) At the mesoscale, the matrix is a porous continuum, and it can be assumed that the fluid filtrates through the porous matrix according to Darcy's law. The fibres are *impermeable* inclusions that break the continuity of the matrix and cause an *anisotropic* obstacle to fluid filtration.
(b) At the macroscale (whole system scale), the interstices between the fibres constitute a system of "mesoscopic anisotropic pores" filled by the matrix, the "microscopic pores" of which are in turn saturated by the fluid phase. Therefore, at the mesoscale, the *anisotropic* arrangement of the fibres makes the filtration velocity deviate from the direction of the pressure gradient, because of the *tortuosity* of the filtration velocity flow lines (see, e.g., Nicholson, 2001; Dormieux et al., 2006).

According to the first interpretation, we shall show that, for non-random orientations of the fibres, the overall permeability is anisotropic. The homogenisation procedure by which we evaluate the overall permeability of a composite with a given arrangement of fibres is constructed in steps.

We first calculate the permeability of a composite with a low volumetric fraction of aligned fibres, by exploiting the perfect formal analogy between fluid filtration in a porous medium, and electric induction in a dielectric (Podzniakov and Tsang, 2004), and Landau and Lifshitz' (1960) solution for an infinite matrix in which a single infinite fibre is embedded. Then, we extend this solution to a composite with a high volumetric fraction of aligned fibres, by means of differential methods previously used for the evaluation of the elastic properties of composites (McLaughlin, 1977; Norris, 1985; Zimmerman, 1991). This result constitutes the homogenised solution for an elementary volume, comprised of a fibre and its cylindrical neighbourhood, filled with matrix and fluid (Fig. 1), representative of the whole system of aligned fibres. Finally, assuming that the orientation of the fibres obeys a given statistical law, we average the solution for the elementary volume over all directions in space, and obtain the overall permeability.

Aside from the cases of arbitrary fibre orientation and perfectly aligned fibres, we also discuss the cases of transversely isotropic distributed fibres and fibres lying in a plane, which are of interest for biological tissues, and fibre-reinforced porous resins.

2. Theoretical background

In this section, we recall some basic aspects of the theory of transversely isotropic second order tensors (Walpole, 1981), and the generalisation needed to treat statistical averages of these objects. In the following, we denote $\mathbb{E} = \mathbb{R}^3$ the physical three-dimensional Euclidean space, $\mathbb{E}_2$ the space of second order tensors on $\mathbb{E}$, $\boldsymbol{I}$ and $\boldsymbol{O}$ the identity and null second order tensors, respectively. The tensor product is indicated by $\otimes$ and the scalar product in $\mathbb{E}_2$ is indicated by $\boldsymbol{A}: \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}}) = A_{ij}B_{ij}$. The set of all possible directions in $\mathbb{E}$ is described by the unit sphere $\mathbb{S}^2 = \{\boldsymbol{w} \in \mathbb{E} : \|\boldsymbol{w}\| = 1\}$.

Isotropic second order tensors are all tensors, and only tensors, invariant for rotations, i.e., proportional to the identity tensor: $\boldsymbol{T} = T\boldsymbol{I} \Rightarrow T_{ij} = T\delta_{ij}$.

Transverse isotropy is defined as the symmetry (invariance) under rotations about a given direction $\boldsymbol{w} \in \mathbb{S}^2$. The tensors with transverse isotropy with respect to $\boldsymbol{w}$ constitute a two-dimensional subspace of $\mathbb{E}_2$, denoted $\mathbb{E}_2(\boldsymbol{w})$. Walpole (1981) obtained a basis for $\mathbb{E}_2(\boldsymbol{w})$ by decomposing the identity as

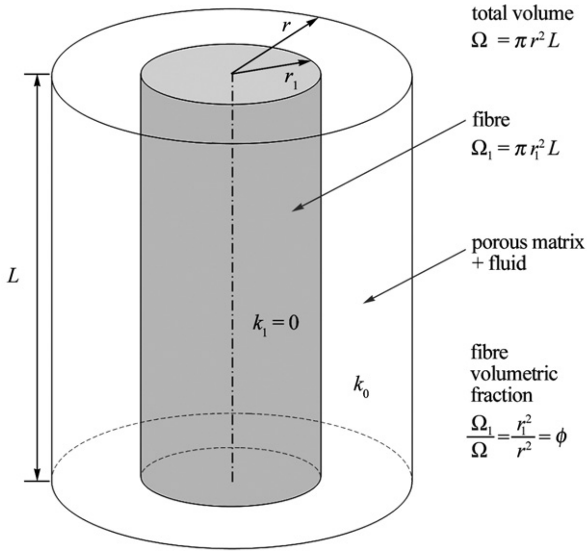

Fig. 1. Elementary volume, representative of a system comprised of aligned impermeable fibres ($k_1 = 0$) and a porous matrix of permeability, k_0. The fibre has a radius r_1, and the cylinder has a radius r, such that the ratio of the volumes equals the fibre volumetric fraction: $\Omega_1/\Omega = r_1^2/r^2 = \phi$.

$$\boldsymbol{I} = \boldsymbol{a} + \boldsymbol{b} \Rightarrow \delta_{ij} = a_{ij} + b_{ij}, \tag{2.1}$$

where the symmetric tensors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined as

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{w} \otimes \boldsymbol{w} \Rightarrow a_{ij} = w_i w_j, \\ \boldsymbol{b} &= \boldsymbol{I} - \boldsymbol{a} \Rightarrow b_{ij} = \delta_{ij} - w_i w_j. \end{aligned} \tag{2.2}$$

Tensor $\boldsymbol{a}$ is often referred to as the *fabric tensor*. It is straightforward to prove that tensors $\boldsymbol{a}$ and $\boldsymbol{b}$ constitute an "orthogonal basis" in the sense that their matrix product $\boldsymbol{ab}$ (and therefore also the *properly defined* scalar product, $\boldsymbol{a}{:}\boldsymbol{b} = \mathrm{tr}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b})$) vanishes identically, and they are idempotent (i.e., $\boldsymbol{a}^2 = \boldsymbol{aa} = \boldsymbol{a}$, $\boldsymbol{b}^2 = \boldsymbol{bb} = \boldsymbol{b}$). A given tensor $\boldsymbol{T} \in \mathbb{E}_2(\boldsymbol{w})$ it admits the unique decomposition

$$\boldsymbol{T} = T_{\parallel}\boldsymbol{a} + T_{\perp}\boldsymbol{b}, \tag{2.3}$$

where $T_{\parallel}$ and $T_{\perp}$ are the components of $\boldsymbol{T}$ parallel and orthogonal to $\boldsymbol{w}$, and are obtained by means of the contractions

$$\begin{aligned} T_{\parallel} &= \boldsymbol{T} : \boldsymbol{a} = \boldsymbol{T} : (\boldsymbol{w} \otimes \boldsymbol{w}) = \boldsymbol{w} \cdot (\boldsymbol{T}\boldsymbol{w}), \\ T_{\perp} &= \tfrac{1}{2}\boldsymbol{T} : \boldsymbol{b} = \tfrac{1}{2}\boldsymbol{T} : (\boldsymbol{I} - \boldsymbol{a}) = \tfrac{1}{2}[tr(\boldsymbol{T}) - \boldsymbol{T} : \boldsymbol{a}]. \end{aligned} \tag{2.4}$$

The tensors of the basis $\{\boldsymbol{a}, \boldsymbol{b}\}$ for the subspace $\mathbb{E}_2(\boldsymbol{w})$ can be redefined as an explicit function of the direction, $\boldsymbol{w}$ (Federico et al., 2004), by treating them as functions defined on $\mathbb{S}^2$, which allows for a simple formalisation of averaging integrals:

$$\begin{aligned} \boldsymbol{a} &: \mathbb{S}^2 \to \mathbb{E}_2 : \boldsymbol{w} \mapsto \boldsymbol{a}(\boldsymbol{w}), \\ \boldsymbol{b} &: \mathbb{S}^2 \to \mathbb{E}_2 : \boldsymbol{w} \mapsto \boldsymbol{b}(\boldsymbol{w}), \end{aligned} \tag{2.5}$$

In this way, a tensor $\boldsymbol{T}$ with explicit dependence on the direction $\boldsymbol{w}$ is written as

$$\boldsymbol{T}(\boldsymbol{w}) = T_{\parallel}\boldsymbol{a}(\boldsymbol{w}) + T_{\perp}\boldsymbol{b}(\boldsymbol{w}), \tag{2.6}$$

i.e., the components, $T_{\parallel}$ and $T_{\perp}$, are independent of $\boldsymbol{w}$, as this dependence is already accounted for by the basis $\{\boldsymbol{a}(\boldsymbol{w}), \boldsymbol{b}(\boldsymbol{w})\}$.

The basis $\{\boldsymbol{a}(\boldsymbol{e}_1), \boldsymbol{b}(\boldsymbol{e}_1)\}$, generating $\mathbb{E}_2(\boldsymbol{e}_1)$, is obtained for $\boldsymbol{w} = \boldsymbol{e}_1 = (1,0,0)$ and denoted

$$\begin{aligned} \boldsymbol{\alpha} &= \boldsymbol{a}(\boldsymbol{e}_1) = \mathrm{diag}(1,0,0), \\ \boldsymbol{\beta} &= \boldsymbol{b}(\boldsymbol{e}_1) = \mathrm{diag}(0,1,1). \end{aligned} \tag{2.7}$$

3. Methods: Aligned fibres

In this section, we evaluate the homogenised permeability for a composite comprised of a porous matrix and a phase of fibre inclusions all aligned in the same direction.

We assume that, at the fibre level, the internal structure of the porous matrix can be treated as a continuum, i.e., the spacing between the molecules constituting the matrix is at least one order of magnitude smaller than the spacing between the reinforcing fibres. Moreover, we assume that the fibres are very long compared to their diameter and to the typical inter-fibre spacing distance, so that they can be treated as infinitely long. Lastly, we assume that the spatial distribution of the aligned fibres is random, so that a suitably chosen neighbourhood of each fibre is representative of the whole system.

Under these assumptions, our material is constituted by the fluid, the solid phase of the matrix, and the fibres, with volumetric fractions equal to ϕ_f, ϕ_0, and ϕ_1, respectively. The void ratio, ratio of the fluid to the solid fraction, is given by

$$e = \frac{\phi_f}{\phi_0 + \phi_1} = \frac{1 - (\phi_0 + \phi_1)}{\phi_0 + \phi_1}. \tag{3.1}$$

In the following, the volumetric fraction of the fibres is indicated simply by ϕ, so that the fraction of matrix-fluid is $1 - \phi$. The matrix has an isotropic intrinsic permeability tensor $\boldsymbol{k}_0 = k_0\boldsymbol{I}$, while the fibres are assumed to be totally impermeable. However, for the following calculations, it is convenient to assign the fibres an intrinsic permeability, $\boldsymbol{k}_1 = k_1\boldsymbol{I}$, which is eventually set to be equal to the null tensor: $\boldsymbol{k}_1 = \boldsymbol{O}$, i.e., $k_1 = 0$.

We make use of Darcy's equation for the description of the motion of a fluid in a porous medium. For the general anisotropic case, Darcy's equation reads

$$\boldsymbol{V} = -\boldsymbol{k}\ \mathrm{grad}\ p \Rightarrow V_i = -k_{ij}p_{,j}, \tag{3.2}$$

where $\boldsymbol{V}$ is the filtration velocity, p is the fluid pressure, $p_{,j}$ is the derivative of p with respect to the j-th space variable, and $\boldsymbol{k}$ is the second order tensor for permeability. Eq. (3.2) can be written in the equivalent form

$$\boldsymbol{V} = \boldsymbol{k}\boldsymbol{H} \Rightarrow V_i = k_{ij}H_j, \tag{3.3}$$

where $\boldsymbol{H} = -\mathrm{grad}\ p$ is the hydraulic head. Isotropy is obtained when $\boldsymbol{k}$ is spherical.

Darcy's law (Eqs. (3.2) and (3.3)) describes a linear response to an external action, and belongs to a large and well-known class of equations in Physics. In this paper, we shall exploit a known result (Landau and Lifshitz, 1960) for the electric induction in a dielectric material, which is governed by the equation

$$\boldsymbol{D} = \boldsymbol{\varepsilon}\boldsymbol{E} = -\boldsymbol{\varepsilon}\ \mathrm{grad}\ \Phi, \tag{3.4}$$

where $\boldsymbol{D}$ is the electric induction field, $\boldsymbol{\varepsilon}$ is the dielectric permeability tensor of the medium, Φ is the scalar potential, and $\boldsymbol{E} = -\mathrm{grad}\ \Phi$ is the electric field.

3.1. Aligned fibres: Low volumetric fraction

The purpose of this section is to calculate the effective permeability, $\boldsymbol{k}$, for a system constituted by the porous matrix described above, in which a small volumetric fraction of impermeable fibres, all aligned in the direction $\boldsymbol{w}$, is embedded.

Let us apply a hydraulic head $\boldsymbol{H}$ to the system. Following Shvidler (1985) and Podzniakov and Tsang (2004), the balance equations for the fluid flow and the hydraulic head read

$$\begin{aligned} \boldsymbol{k}\boldsymbol{H} &= (1-\phi)\boldsymbol{k}_0\boldsymbol{H}_0 + \phi\boldsymbol{k}_1\boldsymbol{H}_1, \\ \boldsymbol{H} &= (1-\phi)\boldsymbol{H}_0 + \phi\boldsymbol{H}_1. \end{aligned} \tag{3.5}$$

By considering that the permeability of the fibres, $\boldsymbol{k}_1$, is zero, and by eliminating $(1-\phi)\boldsymbol{H}_0$, we obtain

$$\boldsymbol{k}\boldsymbol{H} = \boldsymbol{k}_0\boldsymbol{H} - \phi\boldsymbol{k}_0\boldsymbol{H}_1. \tag{3.6}$$

The effective permeability is obtained if an expression of $\boldsymbol{H}_1$ can be found as a function of $\boldsymbol{H}$. By exploiting the dielectric analogy ($\boldsymbol{D} \to \boldsymbol{V}$, $\boldsymbol{\varepsilon} \to \boldsymbol{k}$, $\boldsymbol{E} \to \boldsymbol{H}$, $\Phi \to p$) between Eqs. (3.3) and (3.4), we can use the solution reported by Landau and Lifshitz (1960) for the electric field in a dielectric cylinder embedded in an infinite dielectric medium. The hydraulic head in the cylinder, $\boldsymbol{H}_1$, can be expressed in the form:

$$\boldsymbol{H}_1 = \boldsymbol{M}\boldsymbol{H}, \tag{3.7}$$

where $\boldsymbol{M}$ is the tensor of the influence coefficients, which is transversely isotropic in the direction $\boldsymbol{w}$ of the fibres, and can thus be written as a linear combination of the basis tensors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Eq. (2.2), relative to $\boldsymbol{w}$:

$$\boldsymbol{M} = M_{\parallel}\boldsymbol{a} + M_{\perp}\boldsymbol{b} = \boldsymbol{a} + \frac{2k_0}{k_1 + k_0}\boldsymbol{b}. \tag{3.8}$$

For an impermeable fibre, $k_1 = 0$, and tensor $\boldsymbol{M}$ becomes

$$\boldsymbol{M} = M_{\parallel}\boldsymbol{a} + M_{\perp}\boldsymbol{b} = \boldsymbol{a} + 2\boldsymbol{b}. \tag{3.9}$$

By use of Eq. (3.7), Eq. (3.6) can be written as

$$\boldsymbol{k}\boldsymbol{H} = \boldsymbol{k}_0\boldsymbol{H} - \phi\boldsymbol{k}_0\boldsymbol{M}\boldsymbol{H}. \tag{3.10}$$

Since $\boldsymbol{k}_0$ is spherical, $\boldsymbol{M}$ and $\boldsymbol{k}_0$ commute, and then it follows that

$$\boldsymbol{k} = (\boldsymbol{I} - \phi\boldsymbol{M})\boldsymbol{k}_0. \tag{3.11}$$

Eq. (3.11) was obtained by using the solution of Landau and Lifshitz (1960) for a cylinder in an infinite medium. Therefore, it is valid only for small values of the volumetric fraction, ϕ.

3.2. Aligned fibres: High volumetric fraction

The case of high fibre volumetric fraction can be approached by means of the differential methods described by McLaughlin (1977) and Norris (1985), in the form reported by Zimmerman (1991).

Let us imagine that we construct a fibre-reinforced composite by adding fibres to a homogeneous matrix in steps. In the initial state, the fibre volumetric fraction is $\phi^{(0)} = 0$, i.e., the composite is made by pure homogeneous matrix. In the first step, we replace a small fraction $\Delta\Gamma$ of the volume by fibres, and obtain a composite with fibre fraction $\phi^{(1)} = \Delta\Gamma$. In the second step, we remove another portion $\Delta\Gamma$ of the total volume, thus removing a fraction $(1-\phi^{(1)})\Delta\Gamma$ of matrix, and a fraction $\phi^{(1)}\Delta\Gamma$ of fibres, and we add another fraction $\Delta\Gamma$of fibres. At the end of the second step, the fibre fraction is then $\phi^{(2)} = \phi^{(1)} - \phi^{(1)}\Delta\,\Gamma + \Delta\Gamma$. By induction, we can say that, after the $n+1$-th step, the fibre fraction is

$$\phi^{(n+1)} = \phi^{(n)} - \phi^{(n)}\Delta\Gamma + \Delta\Gamma. \tag{3.12}$$

If we redefine ϕ as a continuous function of the parameter $\Gamma = n\Delta\Gamma$ by imposing $\phi(\Gamma) = \phi(n\Delta\,\Gamma) = \phi^{(n)}$, and substitute in Eq. (3.12), the latter can be rearranged as

$$\frac{\phi(\Gamma + \Delta\Gamma) - \phi(\Gamma)}{\Delta\Gamma} = 1 - \phi(\Gamma). \tag{3.13}$$

By passing to the limit $\Delta\Gamma \to 0$ on the left-hand side, we obtain the differential equation $\phi'(\Gamma) = 1 - \phi\,(\Gamma)$, which, with the initial condition $\phi(0) = \phi^{(0)} = 0$, can be integrated into

$$\phi = 1 - \mathrm{e}^{-\Gamma}. \tag{3.14}$$

Eq. (3.14) describes the incremental process with which fibres are added. Note that $\Gamma = 0$ and $\Gamma \to \infty$ represent the states $\phi = 0$ and $\phi = 1$, respectively, and that Γ approximates ϕ for small fractions: $\phi = 1 - \mathrm{e}^{-\Gamma} \cong \Gamma$.

Since Eq. (3.11) is written for small values of the volumetric fraction, ϕ can be replaced by Γ and Eq. (3.11) reads

$$\boldsymbol{k} = (\boldsymbol{I} - \Gamma\boldsymbol{M})\boldsymbol{k}_0, \tag{3.15}$$

which can be written in the incremental form

$$\frac{\boldsymbol{k}-\boldsymbol{k}_0}{\Gamma}=-\boldsymbol{M}\boldsymbol{k}_0. \tag{3.16}$$

Eq. (3.16) can be extended to high fractions by means of the transformation

$$\begin{aligned}&\boldsymbol{k}\to\boldsymbol{k}(\Gamma+\Delta\Gamma),\\&\boldsymbol{k}_0\to\boldsymbol{k}(\Gamma),\\&\Gamma\to\Delta\Gamma.\end{aligned} \tag{3.17}$$

In practice, we represent a composite with fibre fraction $\Gamma+\Delta\Gamma$, as a homogenised material (equivalent to a composite with fibre fraction Γ) to which an incremental fraction of fibres, $\Delta\Gamma$, is added:

$$\frac{\boldsymbol{k}(\Gamma+\Delta\Gamma)-\boldsymbol{k}(\Gamma)}{\Delta\Gamma}=-\boldsymbol{M}\boldsymbol{k}(\Gamma). \tag{3.18}$$

By passing to the limit $\Delta\Gamma\to 0$ on the left hand side, we obtain the separable differential equation

$$\boldsymbol{k}'(\Gamma)=-\boldsymbol{M}\boldsymbol{k}(\Gamma). \tag{3.19}$$

Since all tensors in (3.19) are transversely isotropic in the direction, $\boldsymbol{w}$, of the fibres, by virtue of the orthogonality of the basis tensors $\boldsymbol{a}$ and $\boldsymbol{b}$, Eq. (3.19) can be decomposed into its parallel and orthogonal components, as:

$$k'_{\parallel}(\Gamma)=-k_{\parallel}(\Gamma),\quad k'_{\perp}(\Gamma)=-2k_{\perp}(\Gamma). \tag{3.20}$$

The initial conditions for the system are that, at zero fraction, the parallel permeability, $k_{\parallel}$, and the orthogonal permeability, $k_{\perp}$, are equal to the permeability of the matrix, k_0:

$$k_{\parallel}(0)=k_0,\quad k_{\perp}(0)=k_0. \tag{3.21}$$

The solution of the differential equations (3.20) with the initial conditions (3.21) is then:

$$k_{\parallel}(\Gamma)=k_0e^{-\Gamma},\quad k_{\perp}(\Gamma)=k_0e^{-2\Gamma}, \tag{3.22}$$

which, by transforming back to the volumetric fraction ϕ, becomes:

$$k_{\parallel}=(1-\phi)k_0,\quad k_{\perp}=(1-\phi)^2k_0. \tag{3.23}$$

Eq. (3.23) can be written in tensor form as:

$$\boldsymbol{k}=k_{\parallel}\boldsymbol{a}+k_{\perp}\boldsymbol{b}=(1-\phi)k_0\boldsymbol{a}+(1-\phi)^2k_0\boldsymbol{b}. \tag{3.24}$$

3.3. Elementary volume: The local anisotropy factor

Here we study an elementary volume representative of a system with an arbitrary volumetric fraction, ϕ, of aligned impermeable fibres ($k_1=0$), embedded in a porous matrix of permeability k_0. The elementary volume is constituted by a fibre of radius r_1, embedded in a concentric cylinder of radius r, filled with the mixture of matrix and fluid (Fig. 1). For the elementary volume to be representative of the whole system, the ratio of the volume Ω_1 of the inner cylinder (fibre) to the volume Ω of the embedding cylinder must equal the fibre fraction, ϕ:

$$\frac{\Omega_1}{\Omega}=\frac{\pi r_1^2L}{\pi r^2L}=\frac{r_1^2}{r^2}=\phi. \tag{3.25}$$

The effective permeability of this elementary volume, i.e., the permeability of the equivalent homogenised material, is given by Eq. (3.23) or (3.24). The anisotropy factor, χ, of the permeability of the elementary volume (and of the equivalent homogenised material) is defined as the ratio between the permeability in the direction orthogonal to the fibre, and that in the direction parallel to the fibre. By virtue of Eq. (3.23), the anisotropy factor for the elementary volume is given by

$$\chi = \frac{k_\perp}{k_\parallel} = 1 - \phi, \tag{3.26}$$

i.e., it is linearly related to the volumetric fraction of fibres, ϕ. Eq. (3.26) shows that the permeability can be isotropic *only* in the absence of fibres, and that permeability is always greater in the direction of the fibre, and allows for writing Eqs. (3.23) and (3.24) in the form

$$k_\parallel = \chi k_0, \quad k_\perp = \chi^2 k_0. \tag{3.27}$$

$$\boldsymbol{k} = k_\parallel \boldsymbol{a} + k_\perp \boldsymbol{b} = \chi k_0 \boldsymbol{a} + \chi^2 k_0 \boldsymbol{b}. \tag{3.28}$$

We also note that, in a system with fibres all aligned in one direction, the elementary volume described above and the anisotropy factor, χ, represent the whole system.

4. Methods: Fibres with statistical orientation

In order to evaluate the overall permeability of a material in which the orientation of the fibres obeys a given probability distribution, we need a statistical averaging procedure. Let us imagine an arbitrary point $\boldsymbol{x}$ on the axis of a fibre, which may be straight or curved. The linearisation of the fibre around $\boldsymbol{x}$ is accomplished with a straight cylindrical fibre that has its axis along the tangent vector $\boldsymbol{w}$ of the real fibre at $\boldsymbol{x}$ (Fig. 2). The elementary volume at point $\boldsymbol{x}$ is then comprised of the linearised fibre, embedded in a cylinder filled with a mixture of matrix and fluid, such that the ratio of the linearised fibre and the embedding cylinder equals the fibre volumetric fraction, ϕ, as previously shown (Fig. 1).

The permeability tensor, $\boldsymbol{k}$, of the elementary volume relative to the direction $\boldsymbol{w}$ (Eq. (3.24)) can be represented in the tensor basis $\{\boldsymbol{a}(\boldsymbol{w}), \boldsymbol{b}(\boldsymbol{w})\}$ (Eq. (2.6)):

$$\boldsymbol{k}(\boldsymbol{w}) = k_\parallel \boldsymbol{a}(\boldsymbol{w}) + k_\perp \boldsymbol{b}(\boldsymbol{w}) = (1-\phi)k_0 \boldsymbol{a}(\boldsymbol{w}) + (1-\phi)^2 k_0 \boldsymbol{b}(\boldsymbol{w}). \tag{4.1}$$

The parallel and orthogonal components $k_\parallel$ and $k_\perp$ are independent of the direction $\boldsymbol{w}$, and are given by Eq. (3.23), as the dependence on $\boldsymbol{w}$ is carried by the basis $\{\boldsymbol{a}(\boldsymbol{w}), \boldsymbol{b}(\boldsymbol{w})\}$.

At every point of a globally homogeneous composite, the probability to find a fibre with orientation $\boldsymbol{w}$ is given by the probability density function (sometimes referred to as an *orientation density function* (Lanir et al., 1996))

$$\psi : \mathbb{S}^2 \to \mathbb{R} : \boldsymbol{w} \mapsto \psi(\boldsymbol{w}) \tag{4.2}$$

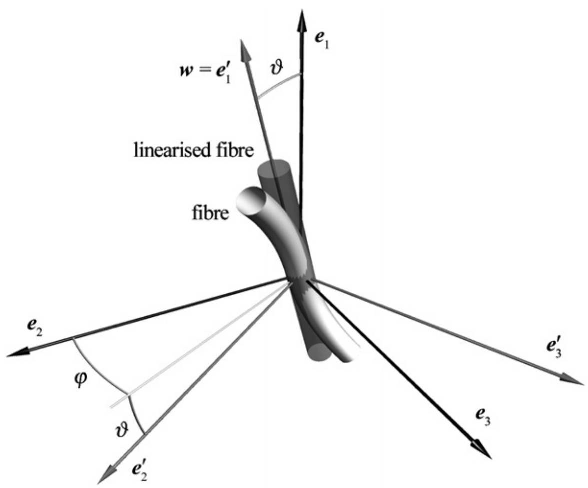

Fig. 2. Global and local reference frames for identification of an arbitrary fibre, and related Euler's angles (co-latitude, ϑ, and longitude, φ).

which must be such that its integral over the unit sphere, $\mathbb{S}^2$, equals one, in order to satisfy the normalisation condition:

$$\int_{\mathbb{S}^2} \psi \mathrm{d}S = \int_{\mathbb{S}^2} \psi(\boldsymbol{w}) \mathrm{d}S(\boldsymbol{w}) = 1. \tag{4.3}$$

Remark. We always linearise fibres, as depicted in Fig. 2. In this framework, curved fibres are represented by means of a probability distribution. For example, a composite in which the fibres are aligned *on average*, but are individually not straight, can be thought of as a composite with straight fibres that have an angular dispersion around the average direction. Furthermore, for a globally inhomogeneous composite, the probability density ψ must be defined as an explicit function of location. Such a probability density can describe a composite in which the main direction of fibres varies with location, as for example in articular cartilage (Federico et al., 2005).

4.1. General solution

The general expression for the overall permeability tensor, $\boldsymbol{K}$, of a fibre reinforced composite with probability distribution ψ is given by

$$\boldsymbol{K} = \int_{\mathbb{S}^2} \psi \boldsymbol{k} \mathrm{d}S = \int_{\mathbb{S}^2} \psi(\boldsymbol{w}) \boldsymbol{k}(\boldsymbol{w}) \mathrm{d}S(\boldsymbol{w}). \tag{4.4}$$

Since $\boldsymbol{k}(\boldsymbol{w})$ is transversely isotropic in the direction $\boldsymbol{w}$, Eq. (4.4) can be split into two terms, by decomposition (4.1):

$$\boldsymbol{K} = k_{\parallel} \int_{\mathbb{S}^2} \psi(\boldsymbol{w}) \boldsymbol{a}(\boldsymbol{w}) \mathrm{d}S(\boldsymbol{w}) + k_{\perp} \int_{\mathbb{S}^2} \psi(\boldsymbol{w}) \boldsymbol{b}(\boldsymbol{w}) \mathrm{d}S(\boldsymbol{w}). \tag{4.5}$$

For the overall elastic properties of a composite with a statistical distribution of inclusions, we can calculate the directional average of the six fourth-order tensors of the basis for transverse isotropy, with suitable projection operators that reduce the number of integrations (Federico et al., 2004). Here, we deal with a second-order tensor basis $\{\boldsymbol{a}, \boldsymbol{b}\}$, and we show that we only need to evaluate the directional average of $\boldsymbol{a}$. Indeed, substituting $\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{I} - \boldsymbol{a}(\boldsymbol{w})$ (Eq. (2.2)) and the normalisation condition (4.3) into Eq. (4.5), we have:

$$\boldsymbol{K} = (k_{\parallel} - k_{\perp}) \boldsymbol{Q} + k_{\perp} \boldsymbol{I}, \tag{4.6}$$

where tensor $\boldsymbol{Q}$ is the directional average of the basis tensor $\boldsymbol{a}(\boldsymbol{w}) = \boldsymbol{w} \otimes \boldsymbol{w}$:

$$\boldsymbol{Q} = \int_{\mathbb{S}^2} \psi(\boldsymbol{w}) \boldsymbol{a}(\boldsymbol{w}) \mathrm{d}S(\boldsymbol{w}) = \int_{\mathbb{S}^2} \psi(\boldsymbol{w}) \boldsymbol{w} \otimes \boldsymbol{w} \mathrm{d}S(\boldsymbol{w}). \tag{4.7}$$

Therefore, the evaluation of the overall permeability, $\boldsymbol{K}$, reduces to the calculation of the integral defining $\boldsymbol{Q}$, which is symmetric (as $\boldsymbol{a}(\boldsymbol{w}) = \boldsymbol{w} \otimes \boldsymbol{w}$) and fully anisotropic, and depends only on the probability distribution, ψ.

The integral can be calculated in polar coordinates by expressing $\boldsymbol{w}$ in terms of the polar parameterisation of the unit sphere, $\mathbb{S}^2$,

$$\boldsymbol{w} = \boldsymbol{s}(\vartheta, \varphi) = (\cos\vartheta, \sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi), \tag{4.8}$$

where ϑ and φ are the co-latitude and longitude angle. In a global reference frame, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, ϑ and φ represent the Euler angles of the reference frame of the fibre, $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ (where $\boldsymbol{e}'_1$ coincides with $\boldsymbol{w}$) (Fig. 2). In this way, the probability distribution is redefined as an explicit function of ϑ and φ,

$$\Psi(\vartheta, \varphi) = \psi(\boldsymbol{s}(\vartheta, \varphi)), \tag{4.9}$$

and the integral in Eq. (4.7) can be written as:

$$\boldsymbol{Q} = \int_0^{2\pi} \left[\int_0^{\pi} \Psi(\vartheta, \varphi) \boldsymbol{s}(\vartheta, \varphi) \otimes \boldsymbol{s}(\vartheta, \varphi) \sin\vartheta \, \mathrm{d}\vartheta \right] \mathrm{d}\varphi. \tag{4.10}$$

4.2. Solution for transverse isotropy

When the probability distribution function, ψ, is transversely isotropic in the direction $\boldsymbol{e}_1 = (1,0,0)$, then it is invariant for rotation about the $\boldsymbol{e}_1$ axis (and its counterpart in polar coordinates, Ψ, is independent of the longitude angle, φ), and the overall permeability of the composite is transversely isotropic with respect to $\boldsymbol{e}_1$, with the plane $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$ representing the transverse plane. Rather than solving the integral in (4.7) by means of projection operators, we solved the integral in (4.10) by calculating tensor $\boldsymbol{Q}$ component by component in polar coordinates (Gasser et al., 2006):

$$\begin{aligned}
Q_{11} &= \int_0^{2\pi}\left[\int_0^{\pi}\Psi(\vartheta)\cos^2\vartheta\sin\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}\varphi,\\
Q_{22} &= \int_0^{2\pi}\left[\int_0^{\pi}\Psi(\vartheta)\cos^2\varphi\sin^3\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}\varphi,\\
Q_{33} &= \int_0^{2\pi}\left[\int_0^{\pi}\Psi(\vartheta)\sin^2\varphi\sin^3\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}\varphi,\\
Q_{12} = Q_{21} &= \int_0^{2\pi}\left[\int_0^{\pi}\Psi(\vartheta)\cos\varphi\cos\vartheta\sin^2\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}\varphi,\\
Q_{13} = Q_{31} &= \int_0^{2\pi}\left[\int_0^{\pi}\Psi(\vartheta)\sin\varphi\cos\vartheta\sin^2\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}\varphi,\\
Q_{23} = Q_{32} &= \int_0^{2\pi}\left[\int_0^{\pi}\Psi(\vartheta)\sin\varphi\cos\varphi\sin^3\vartheta\,\mathrm{d}\vartheta\right]\mathrm{d}\varphi.
\end{aligned} \tag{4.11}$$

Since Ψ does not depend on the longitude angle, φ, all integrals defining the non-diagonal components of $\boldsymbol{Q}$ vanish, and the diagonal components of $\boldsymbol{Q}$ are given by (Gasser et al., 2006)

$$Q_{11} = 1 - 2q, \quad Q_{22} = Q_{33} = q, \quad q = \pi\int_0^{\pi}\Psi(\vartheta)\sin^3\vartheta\,\mathrm{d}\vartheta. \tag{4.12}$$

Tensor $\boldsymbol{Q}$ can be conveniently written in the basis $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}$ of $\mathbb{E}_2(\boldsymbol{e}_1)$ (Eq. (2.7)):

$$\boldsymbol{Q} = (1-2q)\boldsymbol{\alpha} + q\boldsymbol{\beta}. \tag{4.13}$$

By substituting Eq. (4.13) and the identity $\boldsymbol{I} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ into Eq. (4.6), the overall permeability tensor, $\boldsymbol{K}$, becomes

$$\boldsymbol{K} = K_{\parallel}\boldsymbol{\alpha} + K_{\perp}\boldsymbol{\beta} = [k_{\parallel} - 2q(k_{\parallel} - k_{\perp})]\boldsymbol{\alpha} + [k_{\perp} + q(k_{\parallel} - k_{\perp})]\boldsymbol{\beta}, \tag{4.14}$$

where $K_{\parallel}$ and $K_{\perp}$ are the overall *axial* and *transverse* permeabilities, respectively.

In the following subsections, we study specific cases of all fibres lying in the transverse plane $\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, and the isotropic case of randomly oriented fibres. Note that the case of fibres aligned along the symmetry axis $\boldsymbol{e}_1$ might be studied by defining a suitable probability density Ψ^{aligned} and by a limit operation in the sense of distributions (see, e.g., Kolmogorov and Fomin, 1999). We have used this approach previously for elastic properties (Federico et al., 2004). However, it has already been shown (Gasser et al., 2006) that the factor q in 4.12 reduces to q^{aligned}=0, and, if we use this result in our Eq. (4.14), we find that the overall permeability tensor $\boldsymbol{K}$ reduces to the local permeability tensor $\boldsymbol{k}$.

Finally, we study the overall anisotropy factor $X = K_{\perp}/K_{\parallel}$, for the above mentioned cases.

4.3. Fibres lying in the transverse plane

The case of fibres lying on a plane can be of particular interest for laminate composites, and biological tissues, such as skin, or the superficial zone of articular cartilage. For such arrangement of fibres, the normalised probability is (Federico et al., 2004)

$$\Psi^{\text{plane}}(\vartheta) = \frac{\chi_{B(\pi/2,\varepsilon)}(\vartheta)}{2\pi\int_0^{\pi}\chi_{B(\pi/2,\varepsilon)}(\vartheta')\sin\vartheta'\,d\vartheta'}, \tag{4.15}$$

where $\chi_{B(\pi/2,\varepsilon)}$ is the characteristic function of the symmetric interval $B(\pi/2,\varepsilon) = [(\pi/2) - \varepsilon, (\pi/2) + \varepsilon]$. By solving the integral, Eq. (4.15) becomes

$$\Psi^{\text{plane}}(\vartheta) = \frac{\chi_{B(\pi/2,\varepsilon)}(\vartheta)}{4\pi \sin \varepsilon}. \tag{4.16}$$

In the limit $\varepsilon \to 0$, this function converges to the Dirac delta centred at $\pi/2$, in the sense of distributions. Indeed, the parameter q in 4.12, given by

$$q^{\text{plane}} = \lim_{\varepsilon \to 0} \pi \int_0^\pi \Psi^{\text{plane}}(\vartheta) \sin^3 \vartheta \, d\vartheta = \lim_{\varepsilon \to 0} \pi \int_0^\pi \frac{\chi_{B(\pi/2,\varepsilon)}(\vartheta)}{4\pi \sin \varepsilon} \sin^3 \vartheta \, d\vartheta = \frac{5 + \cos(2\varepsilon)}{12}, \tag{4.17}$$

converges to $q^{\text{plane}} = \frac{1}{2}$ in the limit $\varepsilon \to 0$. The permeability tensor (4.14) becomes:

$$\boldsymbol{K} = K_{\parallel} \boldsymbol{\alpha} + K_{\perp} \boldsymbol{\beta} = k_{\perp} \boldsymbol{\alpha} + \frac{1}{2}(k_{\parallel} + k_{\perp}) \boldsymbol{\beta}, \tag{4.18}$$

i.e., the overall axial permeability is equal to the local orthogonal permeability, and the overall transverse permeability is the average of the parallel and orthogonal permeability.

4.4. Randomly oriented fibres

In the case of isotropy, the fibres are randomly oriented, and the probability distribution function Ψ^{iso} is a constant equal to the measure $\frac{1}{4\pi}$ of the solid angle. The parameter q in Eq. 4.12 becomes then $q = \frac{1}{3}$, and the permeability (4.14) becomes

$$\boldsymbol{K} = \left[\frac{1}{3} k_{\parallel} + \frac{2}{3} k_{\perp}\right] \boldsymbol{\alpha} + \left[\frac{1}{3} k_{\parallel} + \frac{2}{3} k_{\perp}\right] \boldsymbol{\beta} = \left[\frac{1}{3} k_{\parallel} + \frac{2}{3} k_{\perp}\right] \boldsymbol{I} = \frac{1}{3} (\text{tr } \boldsymbol{k}) \boldsymbol{I}. \tag{4.19}$$

4.5. Overall anisotropy factor for transverse isotropy

Similarly as the local anisotropy factor, χ (Eq. (3.26)), the overall anisotropy factor, X, is defined as the ratio of the overall transverse to the overall axial permeability:

$$X = \frac{K_{\perp}}{K_{\parallel}}. \tag{4.20}$$

For the case of aligned fibres, $K_{\parallel}$ coincides with $k_{\parallel} = (1 - \phi)k_0$, and $K_{\perp}$ coincides with $k_{\perp} = (1 - \phi)^2 k_0$. Therefore, the overall anisotropy factor is (Eq. (3.26))

$$X = \chi = 1 - \phi. \tag{4.21}$$

For the case of fibres all lying in the transverse plane, the overall anisotropy factor is (Eq. (4.18))

$$X = \frac{k_{\parallel} + k_{\perp}}{2k_{\perp}} = \frac{1}{2}\left(\frac{k_{\parallel}}{k_{\perp}} + 1\right) = \frac{1}{2}\left(\frac{1}{1 - \phi} + 1\right). \tag{4.22}$$

For the case of randomly oriented inclusions (isotropy), X is trivially equal to one, for every value of the fibre fraction, ϕ.

Fig. 3 shows the behaviour of the overall anisotropy factor as a function of the fibre volumetric fraction, ϕ. In all cases, the anisotropy factor is one at zero fibre volumetric fraction, a situation that represents the pure isotropic matrix. For aligned fibres, the anisotropy factor decreases linearly to zero with increasing fibre fraction: the higher the fibre fraction, the larger is the axial compared to the transverse permeability. For fibres lying on a plane, the anisotropy factor increases monotonically and diverges for fibre fractions approaching unity: the higher the fibre fraction, the lower the axial compared to the transverse permeability.

We note, however, that values of the fibre volumetric fraction close to unity have no physical meaning. This is because there is a limit on the volumetric fraction imposed by the packing of fibres. For example, in the case of fibres all with identical diameter, and all aligned in one direction, the maximum possible packing is hexag-

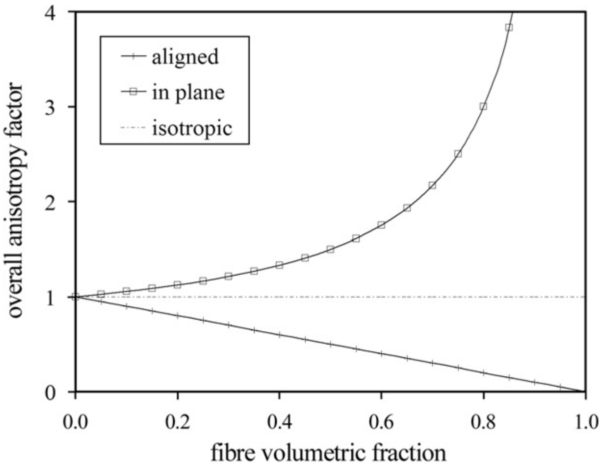

Fig. 3. Overall anisotropy factor, X, as a function of the fibre volumetric fraction, ϕ, for the cases of aligned inclusions (crosses), inclusions lying in a plane (squares), and randomly oriented fibres (dashed line). For the case of aligned fibres, X decreases with increasing ϕ, while for inclusions lying in a plane, X increases and diverges for ϕ going towards one.

onal, and corresponds to a maximum theoretical volumetric fraction equal to the ratio between the area of a circle to the area of its circumscribed hexagon ($\phi = \frac{\sqrt{3}}{6}\pi \cong 0.9069$). More importantly, such a high volumetric fraction would violate the hypothesis of scale separation, as the distance between fibres would go towards zero.

5. Discussion

We have addressed the effect of the presence of fibres on the permeability of a porous composite material, in the undeformed configuration. The system is comprised of an isotropic porous matrix fully saturated with fluid and reinforced by impermeable fibres, a description that well represents a large class of biological soft tissues (e.g., Fung, 1993). The presence of the fibres has been taken into account in terms of their volumetric fraction and orientation, which was assumed to be governed by a normalised probability distribution function. The overall permeability tensor of a composite with a given fibre arrangement was obtained as the directional average of the permeability tensor of an elementary volume constituted by a fibre and its cylindrical neighbourhood.

In a system with all fibres aligned in one direction, the elementary volume described in Section 3.3 and the anisotropy factor, χ (Eq. (3.26)), represent the whole system.

This is the case for tendons, in which fluid flows preferentially along the axial direction (Han et al., 2000; Wellen et al., 2004), and for which the anisotropy factor has been estimated experimentally in the range from 0.36 (fresh tendons) to 0.68 (phosphate-buffered-saline (PBS) stored tendons) (Han et al., 2000). A purely geometric evaluation made through Eq. (3.26) gives a range of volumetric fractions from 32% to 64%, which contains the value around 60% measured optically by Han et al. (2000). These considerations on tendons provide evidence in support of the model.

Furthermore, we have applied this method to articular cartilage, for which there were experimental results that could not be explained by considerations made on the tissue matrix only (Maroudas and Bullough, 1968). By means of a position-dependent probability distribution, describing the inhomogeneous arrangement of the collagen fibres in cartilage, the application of this model (Federico and Herzog, in press) gave good agreement with experiments (Maroudas and Bullough, 1968).

The proposed model is based on purely mechanical-geometrical considerations. In biological tissues, other factors, related to the electrochemical interactions between the fluid and the polar regions of the fibres, may enhance the permeability in the direction of the fibre (see, e.g., Pollack, 2001).

Furthermore, Darcy's law with constant permeability ceases to be satisfactory in describing fluid flow when large deformations are taken into account (see., e.g., Rajagopal, 2003). Indeed, when the deformations are large, pore size decreases (e.g., Holmes and Mow, 1990), and strain-induced anisotropy may occur due to the reorientation of the microstructure (e.g., Quinn et al., 2001).

Nevertheless, this model may constitute a starting point, taking into account the effect of fibres on permeability, at the mechanical-geometrical level for composites in the undeformed configuration, or in the small deformations regime. The natural continuation of this work would be the incorporation of the dependence of permeability on deformation, and, in a further step, the inclusion of electrochemical interactions and large deformations.

Acknowledgments

The Authors are grateful to Dr. Alfio Grillo and Mr. Giandomenico Zingali, for crucial references and useful advice, and to Dr. Kumbakonam Rajagopal for the helpful discussions on the limits of validity of Darcy's law.

The Canada Research Chair Programme, the Canadian Institute of Health Research (CIHR); the Alberta Ingenuity Fund (AIF, Canada), the Alberta Heritage Foundation for Medical Research (AHFMR, Canada).

References

Dormieux, L., Kondo, D., Ulm, F.-J., 2006. Microporomechanics. Wiley & Sons, Wiley & Sons, Ltd., England, UK.

Farquhar, T., Dawson, P.R., Torzilli, P.A., 1990. A microstructural model for the anisotropic drained stiffness of articular cartilage. Journal of Biomechanical Engineering 112, 414–425.

Federico, S., Grillo, A., Herzog, W., 2004. A transversely isotropic composite with a statistical distribution of spheroidal inclusions: a geometrical approach to overall properties. Journal of the Mechanics and Physics of Solids 52 (10), 2309–2327.

Federico, S., Grillo, A., La Rosa, G., Giaquinta, G., Herzog, W., 2005. A transversely isotropic, transversely homogeneous microstructural-statistical model of articular cartilage. Journal of Biomechanics 38 (10), 2008–2018.

Federico, S., Herzog, W., in press. On the anisotropy and inhomogeneity of permeability in articular cartilage. Biomechanics and Modeling in Mechanobiology, http://dx.doi.org/10.1007/s10237-007-0091-0.

Fung, Y.C., 1993. Biomechanics—Mechanical Properties of Living Tissues. Springer-Verlag, New York, USA.

Gasser, T.C., Holzapfel, G.A., Ogden, R.W., 2006. Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. Journal of the Royal Society Interface 3, 15–35.

Han, S., Gemmell, S.J., Helmer, K.G., Grigg, P., Wellen, J.W., Hoffman, A.H., Sotak, C.H., 2000. Changes in ADC caused by tensile loading of rabbit achilles tendon: evidence for water transport. Journal of Magnetic Resonance 144 (2), 217–227.

Hedlund, H., Mengarelli-Widholm, S., Reinholt, F., Svensson, O., 1993. Stereological studies on collagen in bovine articular cartilage. Acta Pathologica, Microbiologica et Immunologica Scandinavica 101, 133–140.

Holmes, M.H., Mow, V.C., 1990. The non-linear characteristics of soft gels and hydrated connective tissues in ultrafiltration. Journal of Biomechanics 23, 1145–1156.

Holzapfel, G.A., Gasser, T.C., Ogden, R.W., 2000. A new constitutive framework for arterial wall mechanics and a comparative study of material models. Journal of Elasticity 61, 1–48.

Kolmogorov, A.N., Fomin, S.V., 1999. Elements of the Theory of Functions and Functional Analysis. Dover Publications, Mineola NY, USA, 105–109.

Landau, L.D., Lifshitz, E.M., 1960. Electrodynamics of Continuous Media. Pergamon Press, Oxford, UK.

Lanir, Y., Lichtenstein, O., Imanuel, O., 1996. Optimal design of biaxial tests for structural material characterization of flat tissues. Journal of Biomechanical Engineering 118, 41–47.

Långsjö, T.K., Hyttinen, M., Pelttari, A., Kiraly, K., Arokoski, J., Helminen, H.J., 1999. Electron microscopic stereological study of collagen fibrils in bovine articular cartilage: volume and surface densities are best obtained indirectly (from length densities and diameters) using isotropic uniform random sampling. Journal of Anatomy 195, 281–293.

Maroudas, A., Bullough, P., 1968. Permeability of articular cartilage. Nature 219, 1260–1261.

McLaughlin, R., 1977. A study of the differential scheme for composite materials. International Journal of Engineering Science 15, 237–244.

Nicholson, C., 2001. Diffusion and related transport mechanisms in brain tissue. Reports on Progress in Physics 64, 812–884.

Norris, A.N., 1985. A differential scheme for the effective moduli of composites. Mechanics of Materials 4, 1–16.

Ogden, R.W., 2003. Nonlinear elasticity, anisotropy, material stability and residual stresses in soft tissue. In: Holzapfel, G.A., Ogden, R.W. (Eds.), Biomechanics of Soft Tissue in Cardiovascular Systems, CISM Courses and Lectures Series No. 441. Springer, Wien, Austria, pp. 65–108.

Podzniakov, S., Tsang, C.-F., 2004. A self-consistent approach for calculating the effective hydraulic conductivity of a binary, heterogeneous medium. Water Resources Research 40, 1–13, W05105.

Quinn, T.M., Dierickx, P., Grodzinsky, A.J., 2001. Glycosaminoglycan network geometry may contribute to anisotropic hydraulic permeability in cartilage under compression. Journal of Biomechanics 34, 1483–1490.

Pollack, G.H., 2001. Cells, Gels and the Engines of Life. Ebner & Sons, Seattle, USA.
Rajagopal, K.R., 2003. Diffusion through polymeric solids undergoing large deformations. Materials Science and Technology 19, 1175–1180.
Shvidler, M.I., 1985. Stochastic Hydrodynamics of Porous Medium (in Russian). Nedra, Moscow, Russia.
Soulhat, J., Buschmann, M.D., Shirazi-Adl, A., 2000. A fibril-network-reinforced biphasic model of cartilage in unconfined compression. Journal of Biomechanical Engineering 121, 150–159.
Walpole, L.J., 1981. Elastic behavior of composite materials: theoretical foundations. Advances in Applied Mechanics 21, 169–242.
Wellen, J., Helmer, K.G., Grigg, P., Sotak, C.H., 2004. Application of porous-media theory to the investigation of water ADC changes in rabbit Achilles tendon caused by tensile loading. Journal of Magnetic Resonance 170, 49–55.
Zimmerman, R.W., 1991. Elastic moduli of a solid containing spherical inclusions. Mechanics of Materials 12, 17–24.