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Discrete Galerkin method for Fredholm integro-differential equations with weakly singular kernels☆

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Abstract

Approximations to a solution and its derivatives of a boundary value problem of an nth order linear Fredholm integro-differential equation with weakly singular or other nonsmooth kernels are determined. These approximations are piecewise polynomial functions on special graded grids. For their finding a discrete Galerkin method and an integral equation reformulation of the boundary value problem are used. Optimal global convergence estimates are derived and an improvement of the convergence rate of the method for a special choice of parameters is obtained. To illustrate the theoretical results a collection of numerical results of a test problem is presented.

MSC: 65R20; 45J05

Keywords: Weakly singular integro-differential equation; Discrete Galerkin method; Graded grid

1. Introduction

Let $\mathbb{R}=(-\infty,\infty)$ and $\mathbb{N}=\{1,2,\ldots\}$. In the present paper we study the convergence behaviour of a discrete Galerkin method for the numerical solution of boundary value problems of the form

$$u^{(n)}(t)=\sum_{i=0}^{n_0} a_i(t)u^{(i)}(t)+\sum_{i=0}^{n_0}\int_0^b K_i(t,s)u^{(i)}(s)\,\mathrm{d}s+f(t),\quad 0\leqslant t\leqslant b,\quad b>0, \tag{1.1}$$

$$\sum_{i=0}^{n-1}[\alpha_{ij}u^{(i)}(0)+\beta_{ij}u^{(i)}(b)]=0,\quad j=1,\ldots,n, \tag{1.2}$$

where $n\in\mathbb{N}$, $0\leqslant n_0\leqslant n-1$, $\alpha_{ij},\beta_{ij}\in\mathbb{R}$, $i=0,1,\ldots,n-1$; $j=1,\ldots,n$. We assume that $a_i, f\in C^{m,\nu}(0,b)$, $K_i\in W^{m,\nu}(\Delta)$, $i=0,\ldots,n_0$, $m\in\mathbb{N}$, $\nu\in\mathbb{R}$, $-\infty<\nu<1$.

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The set $C^{m,\nu}(0,b)$, with $m \in \mathbb{N}$, $-\infty < \nu < 1$, is defined as the collection of all continuous functions $u : [0,b] \to \mathbb{R}$ which are m times continuously differentiable in $(0,b)$ and such that for all $t \in (0,b)$ and $i = 1,\dots,m$ the following estimate holds:

$$|u^{(i)}(t)| \leqslant c \begin{cases} 1 & \text{if } i < 1-\nu, \\ 1 + |\log \varrho(t)| & \text{if } i = 1-\nu, \\ \varrho(t)^{1-\nu-i} & \text{if } i > 1-\nu. \end{cases} \tag{1.3}$$

Here $c = c(u)$ is a positive constant and

$$\varrho(t) = \min\{t, b-t\} \quad (0 < t < b)$$

is the distance from $t \in (0,b)$ to the boundary of the interval $(0,b)$.

Note that $C^m[0,b]$, the set of m times ($m \geqslant 1$) continuously differentiable functions $u : [0,b] \to \mathbb{R}$, belongs to $C^{m,\nu}(0,b)$ for arbitrary $\nu < 1$. Conversely, if $u \in C^{m,\nu}(0,b)$ and $\nu < 1-k$, $k = 1,\dots,m$, then the derivative $u^{(k)}$ is bounded on $(0,b)$ and the derivatives $u',\dots,u^{(k-1)}$ of u can be extended so that $u \in C^{k-1}[0,b]$. Here and below by $C^0[0,b] \equiv C[0,b]$ we denote the Banach space of continuous functions $u : [0,b] \to \mathbb{R}$ equipped with the usual norm $\|u\|_\infty = \max_{0 \leqslant t \leqslant b} |u(t)|$.

The set $W^{m,\nu}(\Delta)$, with $m \in \mathbb{N}$, $-\infty < \nu < 1$,

$$\Delta = \{(t,s) : 0 \leqslant t \leqslant b, 0 \leqslant s \leqslant b,\ t \neq s\}, \tag{1.4}$$

consists of all m times continuously differentiable functions $K : \Delta \to \mathbb{R}$ satisfying for all $(t,s) \in \Delta$ and all nonnegative integers i and j such that $i + j \leqslant m$ the condition

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s) \right| \leqslant c \begin{cases} 1 & \text{if } \nu + i < 0, \\ 1 + |\log|t-s|| & \text{if } \nu + i = 0, \\ |t-s|^{-\nu-i} & \text{if } \nu + i > 0, \end{cases} \tag{1.5}$$

where $c = c(K)$ is a positive constant.

It follows from (1.5) that if $K \in W^{m,\nu}(\Delta)$ with some $0 \leqslant \nu < 1$, then $K(t,s)$ may possess a weak singularity at $t = s$; if $\nu < 0$, then $K(t,s)$ is bounded on Δ but its derivatives may be singular as $s \to t$. Most important examples of weakly singular kernels that belongs to $W^{m,\nu}(\Delta)$ are given by the formula

$$K_{\alpha,\beta}(t,s) = K_1(t,s)|t-s|^{-\alpha}(\log|t-s|)^\beta + K_2(t,s),$$

where K_1 and K_2 are some m times continuously differentiable functions on $[0,b] \times [0,b]$, $0 \leqslant \alpha < 1$, $0 \leqslant \beta < \infty$, $\alpha + \beta \neq 0$. Clearly, $K_{\alpha,0} \in W^{m,\alpha}(\Delta)$ and $K_{\alpha,\beta} \in W^{m,\alpha+\varepsilon}(\Delta)$ for some $0 < \varepsilon < 1-\alpha$.

The existence and regularity of the solution of problem (1.1), (1.2) is described in the following lemma proved in [12].

Lemma 1.1. *Let $n \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $i = 0,\dots,n-1$; $j = 1,\dots,n$. Assume that $f, a_i \in C^{m,\nu}(0,b)$, $K_i \in W^{m,\nu}(\Delta)$, $i = 0,\dots,n_0$, $0 \leqslant n_0 \leqslant n-1$, $m \in \mathbb{N}$, $-\infty < \nu < 1$. Moreover, assume that problem* (1.1), (1.2) *with $f = 0$ has only the trivial solution $u = 0$ and from all solutions of the equation $u^{(n)}(t) = 0$, $0 \leqslant t \leqslant b$, only $u = 0$ satisfies conditions* (1.2).

Then problem (1.1), (1.2) *possesses a unique solution $u \in C^{m+n,\nu-n}(0,b)$ and for its derivatives $u', u'', \dots, u^{(n)}$ we have that $u^{(i)} \in C^{m+n-i,\nu-n+i}(0,b)$, $i = 1,\dots,n$.*

Thus, under the conditions of Lemma 1.1 the higher-order derivatives of the solution of problem (1.1), (1.2) may be unbounded near the boundary of the interval $(0,b)$. This complicates the construction of high-order methods for the numerical solution of problem (1.1), (1.2). We refer here to [11,12] where a discussion of the optimal (global and local) order of convergence of piecewise polynomial collocation methods on graded grids for solving (1.1), (1.2) in case of nonsmooth input data is given. Similar results may also be found in [3–5,13–15].

In the present paper we will use for the solution of problem (1.1), (1.2) a discretized version of the Galerkin method that in case of smooth solutions is studied in [7–9], see also [1,2,6]. However, if we allow weakly singular kernels $K_i \in W^{m,\nu}(\Delta)$, $i = 0,\dots,n_0$, then the resulting solution to (1.1), (1.2) is typically nonsmooth on the closed interval $[0, b]$ (even if $f, a_i \in C^\infty[0, b]$, $i = 0, \ldots, n_0$) and the results of [7–9] cannot be applied. In order to construct a high-order discrete Galerkin method for solving (1.1), (1.2) we shall use as in [11,12] a special graded grid reflecting the possible singular behaviour of the exact solution. Actually, this method exploits a discretized version of an inner product which is formed from the standard inner product of the space $L^2(0, b)$ by dividing $[0, b]$ into subintervals and by substituting on each subinterval the integral with a sum on the basis of a fixed quadrature rule.

Note also that such a discrete Galerkin method is close related to the collocation method considered in [12] and in a special case these methods coincide. These two methods have nearly the same cost. However, the main advantage of the discrete Galerkin method over the collocation method is that in the first case we find an approximation to the derivative $u^{(n)}$ of the solution u of (1.1), (1.2) as a continuous spline but in the second case usually as a discontinuous spline. Therefore, for the calculation of approximate solutions with the same accuracy in the case of the discrete Galerkin method it is necessary to solve smaller resulting systems of algebraic equations as in the case of the collocation method.

Section 2 below provides necessary background material. In particular, in Lemma 2.1 some error estimates for a discrete projection of a function in $C^{m,\nu}(0, b)$ on a graded grid are given. With the help of these estimates in Sections 3 and 4 the convergence behaviour of the discrete Galerkin method is analysed. The main results are formulated in Theorems 3.1 and 4.1. In Section 5 these results are verified by some numerical examples.

2. Discrete orthogonal projection

For $N \in \mathbb{N}$, let

$$\Pi_N = \{t_0, \ldots, t_{2N} : 0 = t_0 < t_1 < \cdots < t_{2N} = b\}$$

be a partition (a graded grid) of the interval $[0, b]$ with the grid points

$$\begin{aligned} t_j &= \frac{b}{2}\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \ldots, N, \\ t_{N+j} &= b - t_{N-j}, \quad j = 1, \ldots, N, \end{aligned} \tag{2.1}$$

where $r \in \mathbb{R}$, $r \geqslant 1$. If $r = 1$, then the grid points (2.1) are distributed uniformly; for $r > 1$ the points (2.1) are more densely clustered near the endpoints of the interval $[0, b]$.

For given integers $m \geqslant 2$ and $0 \leqslant d \leqslant m - 2$, let $S_m^{(d)}(\Pi_N)$ be the spline space of piecewise polynomial functions on the grid Π_N:

$$S_m^{(d)}(\Pi_N) = \{v \in C^d[0, b] : v|_{[t_{j-1}, t_j]} \in \pi_{m-1},\ j = 1, \ldots, 2N\}.$$

Here $v|_{[t_{j-1}, t_j]}$ is the restriction of v onto the subinterval $[t_{j-1}, t_j]$ and π_{m-1} denotes the set of polynomials of degree not exceeding $m - 1$.

Let

$$\int_0^1 g(x)\, \mathrm{d}x \sim \sum_{k=1}^{M} w_k g(\eta_k) \tag{2.2}$$

be a basic quadrature rule with weights $w_k > 0$, $k = 1, \ldots, M$, and nodes

$$0 \leqslant \eta_1 < \cdots < \eta_M \leqslant 1. \tag{2.3}$$

Denote

$$(\psi, \varphi)_N = \sum_{j=1}^{2N} h_j \sum_{k=1}^{M} w_k \psi(t_{jk}) \varphi(t_{jk}), \quad \psi, \varphi \in C[0, b], \tag{2.4}$$

where $h_j = t_j - t_{j-1} > 0$ and

$$t_{jk} = t_{j-1} + \eta_k h_j, \quad k = 1, \ldots, M; \quad j = 1, \ldots, 2N. \tag{2.5}$$

If $M \geqslant m \geqslant 2$, then $(\psi, \varphi)_N$ is an inner product on $S_m^{(0)}(\Pi_N)$, see [10]. We can regard it as an approximation to the standard inner product of the space $L^2(0, b)$.

Further, for any $N \in \mathbb{N}$ the discrete inner product (2.4) induces a discrete orthogonal projection operator $\mathscr{P}_N : C[0, b] \to S_m^{(0)}(\Pi_N)$ defined by

$$\mathscr{P}_N v \in S_m^{(0)}(\Pi_N), \quad (\mathscr{P}_N v, \varphi)_N = (v, \varphi)_N, \quad v \in C[0, b], \ \forall \varphi \in S_m^{(0)}(\Pi_N). \tag{2.6}$$

From [10] we get the following result about the uniform boundness of $\mathscr{P}_N$.

Lemma 2.1. *Let* (2.2) *be a quadrature rule with some weights* $w_k > 0$, $k = 1, \ldots, M$, *and nodes* (2.3). *Let* $\mathscr{P}_N : C[0, b] \to S_m^{(0)}(\Pi_N)$ *be defined by the settings* (2.6) *where* $M \geqslant m \geqslant 2$. *Finally, let one of the following three conditions* (i), (ii) *or* (iii) *be fulfilled*:

(i) the quadrature rule (2.2) *is symmetric*, *i.e.*, $\eta_k = 1 - \eta_{M-k+1}$ *and* $w_k = w_{M-k+1}$, $k = 1, \ldots, M$;
(ii) the quadrature rule (2.2) *is exact for all polynomials of degree* $2m - 2$;
(iii) $M = m$ *and* $\eta_1 = 0$, $\eta_M = 1$.

Then

$$\|\mathscr{P}_N v\|_p \leqslant c\|v\|_{N,p}, \quad v \in C[0, b], \quad 1 \leqslant p \leqslant \infty, \tag{2.7}$$

with a constant c which is independent of $N \in \mathbb{N}$ *and* $v \in C[0, b]$. *Here*

$$\|v\|_\infty = \max_{0 \leqslant t \leqslant b} |v(t)|, \quad \|v\|_{N,\infty} = \max_{1 \leqslant j \leqslant 2N} \ \max_{1 \leqslant k \leqslant M} |v(t_{jk})|,$$

$$\|v\|_p = \left(\int_0^b |v(s)|^p \, \mathrm{d}s \right)^{1/p}, \quad \|v\|_{N,p} = \left(\sum_{j=1}^{2N} h_j \sum_{k=1}^{M} w_k |v(t_{jk})|^p \right)^{1/p}, \quad 1 \leqslant p < \infty.$$

Corollary 2.1. *Let the conditions of Lemma* 2.1 *be fulfilled*. *Then*

$$\|\mathscr{P}_N v\|_\infty \leqslant c\|v\|_\infty,$$

where c is a positive constant not depending on $N \in \mathbb{N}$ *and* $v \in C[0, b]$.

Lemma 2.2. *Let the conditions of Lemma* 2.1 *be fulfilled and let the nodes* (2.5) *with grid points* (2.1) *be used*. *If* $v \in C^{m,v}(0, b)$, $m \geqslant 2$, $v < 1$, *then the following estimates for the error* $v - \mathscr{P}_N v$ *hold*:

$$\|v - \mathscr{P}_N v\|_\infty \leqslant c \begin{cases} N^{-r(1-v)} & \text{for } 1 \leqslant r < \dfrac{m}{1-v}, \\[2ex] N^{-m}(1 + \log N) & \text{for } r = \dfrac{m}{1-v} = 1, \\[2ex] N^{-m} & \text{for } r = \dfrac{m}{1-v} > 1 \text{ or } r > \dfrac{m}{1-v}, \quad r \geqslant 1, \end{cases} \tag{2.8}$$

$$\|v - \mathscr{P}_N v\|_p \leqslant c \, \theta_N(m, v, r, p), \tag{2.9}$$

with

$$\theta_N(m, v, r, p) = \begin{cases} N^{-r(1-v+1/p)} & \text{for } 1 \leqslant r < \dfrac{m}{1-v+1/p}, \\[2ex] N^{-m}(1 + (\log N)^{1/p}) & \text{for } r = \dfrac{m}{1-v+1/p} \geqslant 1, \\[2ex] N^{-m} & \text{for } r > \dfrac{m}{1-v+1/p}, \quad r \geqslant 1. \end{cases}$$

Here $1 \leqslant p < \infty$ *and c is a positive constant not depending on* $N \in \mathbb{N}$.

Proof. For arbitrary $v_N \in S_m^{(0)}(\Pi_N)$ we have $\mathscr{P}_N v_N = v_N$ and

$$v - \mathscr{P}_N v = v - v_N + \mathscr{P}_N(v_N - v).$$

This together with Lemma 2.1 yields

$$\|v - \mathscr{P}_N v\|_p \leqslant \|v - v_N\|_p + c\|v - v_N\|_{N,p}, \quad 1 \leqslant p \leqslant \infty. \tag{2.10}$$

Further, we choose m parameters $\xi_1, \dots, \xi_m$ such that $0 = \xi_1 < \cdots < \xi_m = 1$ and determine $v_N \in S_m^{(0)}(\Pi_N)$ from the interpolation conditions

$$v_N(x_{jk}) = v(x_{jk}), \quad k = 1, \dots, m; \;\; j = 1, \dots, 2N,$$

where $x_{jk} = t_{j-1} + \xi_k h_j$, $k = 1, \dots, m$; $j = 1, \dots, 2N$. Since $\|v - v_N\|_{N,\infty} \leqslant \|v - v_N\|_\infty$, estimate (2.8) for $p = \infty$ follows from (2.10) and Lemma 7.2 in [15]. If $1 \leqslant p < \infty$, then due to Lemma 7.2 in [15], $\|v - v_N\|_p \leqslant c\theta_N(m, v, r, p)$. Moreover, in a similar way as in [15] we can show that $\|v - v_N\|_{N,p} \leqslant c\theta_N(m, v, r, p)$, $1 \leqslant p < \infty$. Now (2.9) follows from (2.10). □

Remark 2.1. If $M = m$, $\eta_1 = 0$, $\eta_M = 1$, then $\mathscr{P}_N v \in S_m^{(0)}(\Pi_N)$ is uniquely determined by the collocation conditions $(\mathscr{P}_N v)(t_{jk}) = v(t_{jk})$, $k = 1, \dots, m$; $j = 1, \dots, 2N$, and Lemma 2.2 is an immediate inference from Lemma 7.2 in [15].

3. Discrete Galerkin method

First of all we consider a reformulation of problem (1.1), (1.2) based on introducing a new unknown function $v = u^{(n)}$. If from all solutions of the linear homogeneous differential equation $u^{(n)} = 0$ only $u = 0$ satisfies conditions (1.2), then the nonhomogeneous equation

$$u^{(n)}(t) = v(t), \quad t \in [0, b], \;\; v \in C[0, b], \tag{3.1}$$

with boundary conditions (1.2), has a unique solution

$$u(t) = \int_0^b G(t, s)v(s)\,\mathrm{d}s, \quad t \in [0, b], \tag{3.2}$$

where $G(t, s)$ is the Green's function of problem (3.1), (1.2). The derivatives of the function u given by (3.2) can be expressed in the form

$$u^{(i)}(t) = (J_i v)(t), \quad t \in [0, b], \;\; i = 0, \dots, n-1, \tag{3.3}$$

where

$$(J_i v)(t) = \int_0^b \frac{\partial^i G(t, s)}{\partial t^i} v(s)\,\mathrm{d}s, \quad t \in [0, b], \;\; i = 0, \dots, n-1. \tag{3.4}$$

Since the general solution of equation $u^{(n)}(t) = 0$ is an arbitrary polynomial of degree $n - 1$, the Green's function $G(t, s)$ for (3.1), (1.2) can be expressed both for $t < s$ and for $t > s$ as the polynomial at most of degree $n - 1$ with respect to t and s. Moreover, $\partial^i G(t, s)/\partial t^i$, $i = 0, \dots, n-2$, the derivatives of $G(t, s)$ with respect to t up to the order $n - 2$, are continuous on $\bar{\Delta} = [0, b] \times [0, b]$. Also $\partial^{n-1} G(t, s)/\partial t^{n-1}$ is continuous and bounded in the region Δ (see (1.4)), but it has a discontinuity at $t = s$. From this it follows that the operators J_i, $i = 0, \dots, n-1$, defined by (3.4), are linear and compact as operators from $C[0, b]$ into $C[0, b]$.

Using $u^{(n)} = v$ and (3.3), problem (1.1), (1.2) may be rewritten as a linear operator equation of the second kind with respect to v:

$$v = Tv + f, \tag{3.5}$$

where

$$T=\sum_{i=0}^{n_0}(A_iJ_i+T_iJ_i), \tag{3.6}$$

$$(A_iv)(t)=a_i(t)v(t), \quad (T_iv)(t)=\int_0^b K_i(t,s)v(s)\,\mathrm{d}s, \quad t\in[0,b], \; i=0,\dots,n_0. \tag{3.7}$$

Further, we look for an approximation v_N to the solution v of Eq. (3.5) in $S_m^{(0)}(\Pi_N)$, $m, N\in\mathbb{N}$, $m\geqslant 2$. We determine v_N by the discrete Galerkin method (see, e.g., [2,7]) as follows:

$$\text{find } v_N\in S_m^{(0)}(\Pi_N) \text{ such that } (v_N-Tv_N-f,\varphi)_N=0 \quad \forall\varphi\in S_m^{(0)}(\Pi_N). \tag{3.8}$$

Method (3.8) can be presented equivalently in the following form: find v_N such that

$$v_N=\mathscr{P}_N Tv_N+\mathscr{P}_N f, \tag{3.9}$$

where $\mathscr{P}_N$ is defined by (2.6).

Having determined the approximation v_N for $v=u^{(n)}$, we determine the approximation $u_N^{(0)}$ for the solution $u=u^{(0)}$ of problem (1.1), (1.2) and the approximations $u_N^{(1)},\dots,u_N^{(n-1)}$ for the corresponding derivatives $u^{(1)},\dots,u^{(n-1)}$ of u by the formulas

$$u_N^{(i)}=J_iv_N, \quad i=0,\dots,n-1. \tag{3.10}$$

We will call both (3.8), (3.10) and (3.9), (3.10) as a discrete Galerkin method for the numerical solution of problem (1.1), (1.2).

Remark 3.1. If $v_N\in S_m^{(0)}(\Pi_N)$ then (see Remark 4.1 in [12])

$$u_N^{(i)}=J_iv_N\in S_{m+n-i}^{(n-i)}(\Pi_N)\subset C^{n-i}[0,b], \quad i=0,\dots,n-1.$$

Remark 3.2. If $M=m$, $\eta_1=0$, $\eta_M=1$ then the discrete Galerkin method (3.8), (3.10) or (3.9), (3.10) coincides with the collocation method considered in [12].

In the sequel, for given Banach spaces E and F we denote by $\mathscr{L}(E,F)$ the Banach space of linear bounded operators $A:E\to F$ with the norm $\|A\|_{\mathscr{L}(E,F)}=\sup\{\|Au\|_F : u\in E, \|x\|_E\leqslant 1\}$. By c, c_1 and c_2 we will denote positive constants that are independent of N and may have different values in different occurrences.

For the convergence of the discrete Galerkin method the following result is valid.

Theorem 3.1. *Let the conditions of Lemmas* 1.1 *and* 2.1 *be fulfilled and let nodes* (2.5) *with grid points* (2.1) *be used. Then there exists an integer $N_0\in\mathbb{N}$ such that, for $N\geqslant N_0$, Eq.* (3.9) *possesses a unique solution $v_N\in S_m^{(0)}(\Pi_N)$ and the following error estimates hold*:

$$\|u^{(n)}-v_N\|_\infty\leqslant c\begin{cases} N^{-r(1-\nu)} & \text{for } 1\leqslant r<\dfrac{m}{1-\nu},\\ N^{-m}(1+\log N) & \text{for } r=\dfrac{m}{1-\nu}=1,\\ N^{-m} & \text{for } r=\dfrac{m}{1-\nu}>1 \text{ or } r>\dfrac{m}{1-\nu},\ r\geqslant 1,\end{cases} \tag{3.11}$$

$$\max_{0\leqslant i\leqslant n-1}\|u^{(i)}-J_iv_N\|_\infty\leqslant c\begin{cases} N^{-r(2-\nu)} & \text{for } 1\leqslant r<\dfrac{m}{2-\nu},\\ N^{-m}(1+\log N) & \text{for } r=\dfrac{m}{2-\nu}\geqslant 1,\\ N^{-m} & \text{for } r>\dfrac{m}{2-\nu},\quad r\geqslant 1.\end{cases} \tag{3.12}$$

Here c is a positive constant which is independent of N, $u = u^{(0)}$ *is the solution of problem* (1.1), (1.2) *and* J_i *is defined by formula* (3.4).

Proof. Due to the assumptions of Lemma 1.1, the operators A_i and T_i, $i = 0, \ldots, n_0$, defined by (3.7), are linear and bounded as operators from $C[0, b]$ into $C[0, b]$. Now it follows from (3.6) that T is linear and compact as an operator from $C[0, b]$ into $C[0, b]$. On the basis of Lemmas 2.1 and 2.2 we obtain that (cf. [5])

$$\|T - \mathscr{P}_N T\|_{\mathscr{L}(C[0,b],C[0,b])} \to 0 \quad \text{as } N \to \infty. \tag{3.13}$$

Since equation $v = Tv$ has in $C[0, b]$ only the trivial solution $v = 0$, there exists an inverse operator $(I - T)^{-1} \in \mathscr{L}(C[0, b], C[0, b])$ where I is the identity mapping. This together with (3.13) yields that there exists a number $N_0 \in \mathbb{N}$ such that for $N \geqslant N_0$ the operator $(I - \mathscr{P}_N T)$ is invertible in $C[0, b]$ and

$$\|(I - \mathscr{P}_N T)^{-1}\|_{\mathscr{L}(C[0,b],C[0,b])} \leqslant c, \quad N \geqslant N_0. \tag{3.14}$$

Thus, since $f \in C[0, b]$, Eq. (3.9) possesses a unique solution $v_N \in C[0, b]$ for $N \geqslant N_0$. Actually, $v_N \in S_m^{(0)}(\Pi_N)$. It follows from (3.5) and (3.9) that

$$(I - \mathscr{P}_N T)(v - v_N) = v - \mathscr{P}_N v. \tag{3.15}$$

On the basis of (3.14) we obtain from (3.15) that

$$\|v - v_N\|_\infty \leqslant c\|v - \mathscr{P}_N v\|_\infty, \qquad N \geqslant N_0. \tag{3.16}$$

Due to Lemma 1.1, $v = u^{(n)} \in C^{m,\nu}(0, b)$. Now (3.16) and (2.8) yield estimate (3.11).

Further, since

$$(I - \mathscr{P}_N T)^{-1} = I + (I - \mathscr{P}_N T)^{-1}\mathscr{P}_N T, \quad N \geqslant N_0,$$

we get from (3.15) and (3.3) that, for $N \geqslant N_0$,

$$u^{(i)} - J_i v_N = J_i(v - \mathscr{P}_N v) + J_i(I - \mathscr{P}_N T)^{-1}\mathscr{P}_N T(v - \mathscr{P}_N v), \quad i = 0, \ldots, n-1. \tag{3.17}$$

Using (3.4), (3.6), (3.14) and Corollary 2.1, we obtain from (3.17) that

$$\|u^{(i)} - J_i v_N\|_\infty \leqslant c\|v - \mathscr{P}_N v\|_1, \quad N \geqslant N_0, \quad i = 0, \ldots, n-1.$$

This together with $v \in C^{m,\nu}(0, b)$ and (2.9) yields estimate (3.12). □

4. Higher-order estimates

It follows from Theorem 3.1 that for method (3.8), (3.10) a convergence of order $O(N^{-m})$ can be expected, using sufficiently large values of the grid parameter r. With respect to the underlying quadrature rule (2.2) it is sufficient to require only the symmetricity of it (even it is not necessary that the corresponding quadrature sum is an approximation of the integral). Actually, under stronger conditions on the quadrature rule (2.2) it is possible to improve the covergence rate of method (3.8), (3.10). In particular, if we suppose that the quadrature rule (2.2) is exact for all polynomials of degree $2m - 1$ and the derivatives up to a certain order of the solution u of problem (1.1), (1.2) are bounded on the interval $[0, b]$, then it follows from the results of the paper [7] that for all values of $r \geqslant 1$ (thus also for the uniform grid) and for sufficiently large N we can get the following error estimate:

$$\max_{0 \leqslant j \leqslant i} \|u^{(j)} - J_j v_N\|_\infty \leqslant cN^{-m-\min\{m,n-i\}}, \quad i = n_0, \ldots, n-1. \tag{4.1}$$

In our case the derivatives of the solution of problem (1.1), (1.2) are typically unbounded on the interval $[0, b]$ (see Lemma 1.1). Therefore, in general, the results of [7] cannot be applied. Below we show (see Theorem 4.1) that estimate (4.1) is valid also in case of integro-differential equations with weakly singular kernels for sufficiently large values of r. This result we will obtain from (3.17) on the basis of an estimate for $\| J_i(v - \mathscr{P}_N v\|_\infty$, $i = 0, \ldots, n-1$, proved in Lemma 4.2. First of all we present in a suitable for us form an estimate for the error of a polynomial interpolation.

Lemma 4.1. *Let* $[t_{j-1}, t_j] \subset [0, b]$, $h_j = t_j - t_{j-1} > 0$, $\gamma \in C^{p-1}[t_{j-1}, t_j]$, $\gamma^{(p)} \in L^\infty(t_{j-1}, t_j)$, $p \in \mathbb{N}$. *Then there exists a polynomial* φ *of degree at most* $q-1$ *such that*

$$\varphi(t_{j-1}) = \gamma(t_{j-1}), \quad \varphi(t_j) = \gamma(t_j), \tag{4.2}$$

$$\max_{t_{j-1} \leqslant s \leqslant t_j} |\gamma^{(\mu)}(s) - \varphi^{(\mu)}(s)| \leqslant c h_j^{p-\mu} \sup_{t_{j-1} < s < t_j} |\gamma^{(p)}(s)|, \quad \mu = 0, \ldots, q-1, \tag{4.3}$$

where $q = \max\{2, p\}$ *and the constant* c *is independent of* h_j *and* t_j.

Proof. Let the function $\gamma(s)$ be given. Let $s = t_{j-1} + h_j x \in [t_{j-1}, t_j]$ and denote

$$g(x) = \gamma(t_{j-1} + h_j x) = \gamma(s), \quad x \in [0, 1].$$

Taking $q = \max\{2, p\}$ parameters $0 = \xi_1 < \xi_2 < \cdots < \xi_q = 1$, we construct the Lagrange's interpolation polynomial $\sum_{k=1}^q \lambda_k(x) g(\xi_k)$ for $g(x)$, with $\lambda_k(x)$, $k = 1, \ldots, q$, the polynomials of degree $q-1$ such that $\lambda_k(\xi_k) = 1$ and $\lambda_k(\xi_i) = 0$ if $i \neq k$, $i = 1, \ldots, q$. With the help of the Taylor formula

$$g(x) = \sum_{i=0}^{p-1} \frac{1}{i!} g^{(i)}(0) x^i + \frac{1}{(p-1)!} \int_0^x (x-\xi)^{p-1} g^{(p)}(\xi)\, \mathrm{d}\xi, \quad x \in [0, 1]$$

we observe that

$$\begin{aligned} g(x) - \sum_{k=1}^q \lambda_k(x) g(\xi_k) = {} & \sum_{i=0}^{p-1} \frac{1}{i!} g^{(i)}(0) \left[x^i - \sum_{k=1}^q \lambda_k(x) \xi_k^i \right] \\ & + \frac{1}{(p-1)!} \left[\int_0^x (x-\xi)^{p-1} g^{(p)}(\xi)\, \mathrm{d}\xi \right. \\ & \left. - \sum_{k=1}^q \int_0^{\xi_k} \lambda_k(x) (\xi_k - \xi)^{p-1} g^{(p)}(\xi)\, \mathrm{d}\xi \right], \quad x \in [0, 1]. \end{aligned}$$

Since the polynomial $\sum_{k=1}^q \lambda_k(x) g(\xi_k)$ coincides with the function $g(x)$ if g is a polynomial of degree at most $q-1$, we get

$$g(x) - \sum_{k=1}^q \lambda_k(x) g(\xi_k) = \int_0^1 \Gamma(x, \xi) g^{(p)}(\xi)\, \mathrm{d}\xi, \quad x \in [0, 1], \tag{4.4}$$

where $\Gamma(x, \xi)$ is the Peano kernel (see, e.g., [3]):

$$\Gamma(x, \xi) = \frac{1}{(p-1)!} \left[(x-\xi)_+^{p-1} - \sum_{k=1}^q \lambda_k(x) (\xi_k - \xi)_+^{p-1} \right],$$

$$(x-\xi)_+^{p-1} = \begin{cases} (x-\xi)^{p-1} & \text{if } x \geqslant \xi, \\ 0 & \text{if } x < \xi. \end{cases}$$

We take

$$\varphi(s) = \sum_{k=1}^q \lambda_k \left(\frac{s - t_{j-1}}{h_j} \right) \gamma(t_{j-1} + h_j \xi_k), \quad s \in [t_{j-1}, t_j].$$

Clearly, $\varphi(s)$ is a polynomial of degree at most $q-1$ satisfying the conditions (4.2). Because $g^{(p)}(x) = h_j^p \gamma^{(p)}(s)$ we obtain from (4.4) that

$$\gamma(s) - \varphi(s) = h_j^p \int_0^1 \Gamma\left(\frac{s - t_{j-1}}{h_j}, \xi \right) \gamma^{(p)}(t_{j-1} + h_j \xi)\, \mathrm{d}\xi, \quad s \in [t_{j-1}, t_j].$$

This yields (4.3) for $p \geqslant 2$ and for $p = 1$, $\mu = 0$, since

$$\sup_{(x,\,\xi)\in(0,1)\times(0,1)} \left| \frac{\partial^\mu \Gamma(x,\xi)}{\partial x^\mu} \right| \leqslant c, \quad \mu = 0, \ldots, p-1.$$

If $p = 1$ and $\mu = 1$ then we can immediately to check that (4.3) also holds. □

Lemma 4.2. *Let the conditions of Lemma* 2.1 *be fulfilled and let* $v \in C^{q_i,\nu}(0, b)$, *where* $q_i = m + \min\{m, n - i\}$ *with some* $i \in \{0, \ldots, n-1\}$ *and* $\nu < 1$. *Moreover, assume that from all polynomials u of degree* $n - 1$ *only* $u = 0$ *satisfies conditions* (1.2), *quadrature rule* (2.2) *is exact for all polynomials of degree* $q_i - 1$ *and nodes* (2.5) *with grid points* (2.1) *are used where*

$$r > \max\left\{\frac{m}{1-\nu}, \frac{q_i}{2-\nu}\right\}, \quad r \geqslant 1.$$

Then

$$\| J_i(v - \mathscr{P}_N v)\|_\infty \leqslant c N^{-q_i}, \tag{4.5}$$

where J_i *and* $\mathscr{P}_N$ *are defined, respectively, by* (3.4) *and* (2.6) *and c is a positive constant not depending on N.*

Proof. Suppose that $v \in C^{q_i,\nu}(0, b)$. Then

$$[J_i(v - \mathscr{P}_N v)](t) = \int_0^b \left[\frac{\partial^i G(t,s)}{\partial t^i} - \varphi(s)\right](v - \mathscr{P}_N v)(s)\,\mathrm{d}s + \int_0^b \varphi(s)(v - \mathscr{P}_N v)(s)\,\mathrm{d}s, \quad t \in [0, b], \ \ i \in \{0, \ldots, n-1\}, \tag{4.6}$$

where $G(t, s)$ is the Green's function of problem (3.1), (1.2) and $\varphi \in S_m^{(0)}(\Pi_N)$ is generated on the basis of Lemma 4.1 in the following way. Recall that $G(t, s)$ is both for $t < s$ and for $t > s$ a polynomial at most of degree $n-1$ with respect to t and s, the derivatives $\partial^{i+j} G(t, s)/\partial t^i \partial s^j$ are continuous for $(t, s) \in [0, b] \times [0, b]$ if $i + j \leqslant n - 2$ and they have a bounded discontinuity at $t = s$ if $i + j = n - 1$.

Let us fix $t \in (0, b)$ and $i \in \{0, \ldots, n-1\}$. We denote

$$\gamma(s) = \frac{\partial^i G(t,s)}{\partial t^i}, \quad p = \min\{m, n-i\}.$$

Note that, for simplicity of the presentation, we do not show the dependence of γ (below also φ) on t and i.

If $t \in (t_{l-1}, t_l]$ for an $l \in \{1, \ldots, 2N\}$ and $p \geqslant 2$ (i.e., $i \in \{0, \ldots, n-2\}$) then $\gamma \in C^{p-2}[t_{l-1}, t_l]$ and $\gamma^{(p-1)} \in L^\infty(t_{l-1}, t_l)$. Due to Lemma 4.1, there exist a polynomial φ at most of degree $q-2$, $q = \max\{3, p\}$, such that

$$\varphi(t_{l-1}) = \gamma(t_{l-1}), \quad \varphi(t_l) = \gamma(t_l),$$

$$\sup_{t_{l-1}<s<t_l} |\gamma^{(\mu)}(s) - \varphi^{(\mu)}(s)| \leqslant c h_l^{p-1-\mu} \sup_{t_{l-1}<s<t_l} |\gamma^{(p-1)}(s)|, \quad \mu = 0, \ldots, q-2. \tag{4.7}$$

Here and below c is a positive constant not depending on $t \in (0, b)$.

If $t \in (t_{l-1}, t_l]$ and $p = 1$ (i.e., $i = n - 1$) then $\gamma(s)$ has a finite jump discontinuity at $s = t$ and $\gamma \in L^\infty(t_{l-1}, t_l)$. The value $\gamma(s)$ at $s = t$ we determine as a right-hand limit of $\gamma(s)$ at t. Now we can define

$$\varphi(s) = \gamma(t_{l-1}) + \frac{1}{h_l}[\gamma(t_l) - \gamma(t_{l-1})](s - t_{l-1}), \quad s \in [t_{l-1}, t_l].$$

This polynomial satisfies (4.7) by $\mu = 0$ and the condition

$$\sup_{t_{l-1}<s<t_l} |\varphi'(s)| \leqslant \frac{c}{h_l}, \quad i = n-1. \tag{4.8}$$

If $j \neq l$, $j = 1, \ldots, 2N$, then $t \notin (t_{j-1}, t_j]$, $\gamma \in C^p[t_{j-1}, t_j]$ and there exists a polynomial φ of degree at most $q-1$, $q = \max\{2, p\}$, which satisfies the conditions (4.2) and (4.3). In particular, it follows from (4.3) for $\mu = 0$ that

$$\max_{t_{j-1} \leqslant s \leqslant t_j} |\gamma(s) - \varphi(s)| \leqslant c h_j^p \sup_{t_{j-1} < s < t_j} |\gamma^{(p)}(s)|. \tag{4.9}$$

For a function $\varphi \in S_q^{(0)}(\Pi_N) \subset S_m^{(0)}(\Pi_N)$ ($q = \max\{2, p\} \leqslant m$), determined in such a way as above, we get from (4.3) and (4.7) the following estimates:

$$\max_{\mu=0,\ldots,m-1} \max_{j=1,\ldots,2N} \sup_{t_{j-1} < s < t_j} |\varphi^{(\mu)}(s)| \leqslant c, \quad i \in \{0, \ldots, n-2\}, \tag{4.10}$$

$$\max_{0 \leqslant s \leqslant b} |\varphi(s)| \leqslant c, \qquad \max_{j=1,\ldots,2N, j \neq l} \sup_{t_{j-1} < s < t_j} |\varphi'(s)| \leqslant c, \quad i = n-1. \tag{4.11}$$

Further, it follows from (4.7) and (4.9) that for the first integral on right-hand side of (4.6) we have

$$\begin{aligned} &\left| \int_0^b \left[\frac{\partial^i G(t,s)}{\partial t^i} - \varphi(s) \right] (v - \mathscr{P}_N v)(s)\, \mathrm{d}s \right| \\ &= \left| \sum_{j=1}^{2N} \int_{t_{j-1}}^{t_j} [\gamma(s) - \varphi(s)](v - \mathscr{P}_N v)(s)\, \mathrm{d}s \right| \\ &\leqslant c \left(h_l^p + \sum_{j=1, j \neq l}^{2N} h_j^{p+1} \right) \|v - \mathscr{P}_N v\|_\infty, \quad t \in (t_{l-1}, t_l], \quad l = 1, \ldots, 2N. \end{aligned}$$

This together with $0 < h_j = t_j - t_{j-1} \leqslant br/(2N)$, $j = 1, \ldots, 2N$, $v \in C^{q_i, \nu}(0, b)$ and Lemma 2.2 yields

$$\sup_{0<t<b} \left| \int_0^b \left[\frac{\partial^i G(t,s)}{\partial t^i} - \varphi(s) \right] (v - \mathscr{P}_N v)(s)\, \mathrm{d}s \right| \leqslant c N^{-q_i} \quad \text{if } r > \frac{m}{1-\nu}, \ \ r \geqslant 1, \tag{4.12}$$

where $q_i = m + p = m + \min\{m, n-i\}$, $i \in \{0, \ldots, n-1\}$.

Now we restrict our attention to the last integral in (4.6). Since $\mathscr{P}_N$ is determined by (2.6), $\varphi \in S_m^{(0)}(\Pi_N)$, the quadrature rule (2.2) is exact for all polynomials of degree q_i and $\varphi\, \mathscr{P}_N v$ is in each interval $[t_{j-1}, t_j]$ ($j = 1, \ldots, 2N$) a polynomial of degree at most $q_i - 1$, we have

$$\int_0^b \varphi(s)(\mathscr{P}_N v)(s)\, \mathrm{d}s = (\mathscr{P}_N v, \varphi)_N = (v, \varphi)_N.$$

Thus,

$$\int_0^b \varphi(s)(v - \mathscr{P}_N v)(s)\, \mathrm{d}s = \sum_{j=1}^{2N} \left[\int_{t_{j-1}}^{t_j} v(s)\varphi(s)\, \mathrm{d}s - h_j \sum_{k=1}^{M} w_k v(t_{jk}) \varphi(t_{jk}) \right]$$

or

$$\int_0^b \varphi(s)(v - \mathscr{P}_N v)(s)\, \mathrm{d}s = \sum_{j=1}^{2N} E_j, \tag{4.13}$$

where

$$E_j = \int_{t_{j-1}}^{t_j} \psi(s)\, \mathrm{d}s - h_j \sum_{k=1}^{M} w_k \psi(t_{jk}), \quad j = 1, \ldots, 2N, \ \ \psi = v\varphi. \tag{4.14}$$

Next we will estimate E_j, $j = 1, \ldots, 2N$. First we consider the case where $i \in \{0, \ldots, n-2\}$.

Since $v \in C^{q_i,\nu}(0,b)$, the function $\psi(s)=v(s)\varphi(s)$ is q_i times continuously differentiable in each interval (t_{j-1},t_j), $j=1,\dots,2N$, and

$$\psi^{(q_i)}(s)=\sum_{\mu=0}^{m-1}\binom{q_i}{\mu}v^{(q_i-\mu)}(s)\varphi^{(\mu)}(s),\quad s\in(t_{j-1},t_j),\quad j=1,\dots,2N.$$

This together with (1.3) and (4.10) yields

$$|\psi^{(q_i)}(s)|\leqslant c\begin{cases}1 & \text{if } q_i<1-\nu,\\ 1+|\log\varrho(s)| & \text{if } q_i=1-\nu,\\ \varrho(s)^{1-\nu-q_i} & \text{if } q_i>1-\nu,\end{cases}\tag{4.15}$$

where $s\in(t_{j-1},t_j)$, $j=1,\dots,2N$, $\varrho(s)=\min\{s,b-s\}$ and c is a positive constant not depending on j and $t\in(0,b)$.

Using in (4.14) a change of variables $s=t_{j-1}+h_jx$ we get

$$E_j=h_j\left[\int_0^1 g(x)\,\mathrm{d}x-\sum_{k=1}^{M}w_kg(\eta_k)\right],\quad j=1,\dots,2N,$$

where $g(x)=\psi(t_{j-1}+h_jx)=\psi(s)$. For $j=2,\dots,2N-1$, we have $g\in C^{q_i}[0,1]$ and by a Taylor's expansion

$$g(x)=\sum_{\mu=0}^{q_i-1}\frac{1}{\mu!}g^{(\mu)}(0)x^\mu+\frac{1}{(q_i-1)!}\int_0^x(x-\xi)^{q_i-1}g^{(q_i)}(\xi)\,\mathrm{d}\xi,\quad x\in[0,1],$$

we obtain

$$E_j=h_j\sum_{\mu=0}^{q_i-1}\frac{1}{\mu!}g^{(\mu)}(0)\left[\int_0^1x^\mu\,\mathrm{d}x-\sum_{k=1}^{M}w_k\eta_k^\mu\right]$$
$$+\frac{h_j}{(q_i-1)!}\left[\int_0^1\int_0^x(x-\xi)^{q_i-1}g^{(q_i)}(\xi)\,\mathrm{d}\xi\,\mathrm{d}x-\sum_{k=1}^{M}w_k\int_0^{\eta_k}(\eta_k-\xi)^{q_i-1}g^{(q_i)}(\xi)\,\mathrm{d}\xi\right].$$

Since the quadrature rule (2.2) is exact for polynomials of degree at most q_i-1 and $g^{(q_i)}(x)=h_j^{q_i}\psi^{(q_i)}(s)$, we can present E_j in the form

$$E_j=h_j\int_0^1\Gamma(\xi)g^{(q_i)}(\xi)\,\mathrm{d}\xi=h_j^{q_i+1}\int_0^1\Gamma(\xi)\psi^{(q_i)}(t_{j-1}+h_j\xi)\,\mathrm{d}\xi,\quad j=2,\dots,2N-1,\tag{4.16}$$

where

$$\Gamma(\xi)=\frac{1}{(q_i-1)!}\left[\int_0^1(x-\xi)_+^{q_i-1}\,\mathrm{d}x-\sum_{k=1}^{M}w_k(\eta_k-\xi)_+^{q_i-1}\right].$$

For $j=2,\dots,N$ we have $\varrho(t_j)=t_j$ and $t_{j-1}<t_j\leqslant 2^rt_{j-1}$. As $\max_{0\leqslant\xi\leqslant1}|\Gamma(\xi)|\leqslant c$, then it follows from (4.16) and (4.15) that, for $j=2,\dots,N$,

$$|E_j|\leqslant ch_j^{q_i+1}\begin{cases}1 & \text{if } q_i<1-\nu,\\ 1+|\log t_j| & \text{if } q_i=1-\nu,\\ t_j^{1-\nu-q_i} & \text{if } q_i>1-\nu,\end{cases}\tag{4.17}$$

with a constant c which is independent of j and $t\in(0,b)$. Actually, (4.17) is valid for $j=1$, also.

Indeed, using in (4.14) a Taylor expansion

$$\psi(s)=\sum_{\mu=0}^{q_i-1}\frac{1}{\mu!}\psi^{(\mu)}(t_1)(s-t_1)^{\mu}+\frac{1}{(q_i-1)!}\int_{t_1}^{s}(s-\tau)^{q_i-1}\psi^{(q_i)}(\tau)\,\mathrm{d}\tau,\quad s\in[0,t_1],$$

we obtain

$$E_1=\frac{1}{(q_i-1)!}\left[\int_0^{t_1}\int_{t_1}^{s}(s-\tau)^{q_i-1}\psi^{(q_i)}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s-h_1\sum_{k=1}^{M}w_k\int_{t_1}^{t_{1k}}(t_{1k}-\tau)^{q_i-1}\psi^{(q_i)}(\tau)\,\mathrm{d}\tau\right].\tag{4.18}$$

With the help of (4.15) we get

$$\sup_{0<s<t_1}\left|\int_{t_1}^{s}(s-\tau)^{q_i-1}\psi^{(q_i)}(\tau)\,\mathrm{d}\tau\right|\leqslant c\begin{cases}t_1^{q_i} & \text{if } q_i<1-\nu,\\ t_1^{q_i}(1+|\log t_1|) & \text{if } q_i=1-\nu,\\ t_1^{1-\nu} & \text{if } q_i>1-\nu.\end{cases}$$

This together with (4.18) and $t_1=h_1$ yields (4.17) for $j=1$.

Further, we have

$$t_j=\frac{b}{2}\left(\frac{j}{N}\right)^r,\quad 0<h_j=t_j-t_{j-1}\leqslant\frac{br}{2}j^{r-1}N^{-r},\quad j=1,\ldots,N.\tag{4.19}$$

For $r(2-\nu)>q_i>1-\nu$ it follows from (4.17) and (4.19) that

$$\sum_{j=1}^{N}|E_j|\leqslant c\sum_{j=1}^{N}h_j^{q_i+1}t_j^{1-\nu-q_i}\leqslant c_1N^{-r(2-\nu)}\sum_{j=1}^{N}j^{r(2-\nu)-q_i-1}\leqslant c_2N^{-q_i}.$$

For $q_i\leqslant 1-\nu$ we obtain that $\sum_{j=1}^{N}|E_j|\leqslant cN^{-q_i}$ if $r\geqslant 1$. In a similar way we get that $\sum_{j=N+1}^{2N}|E_j|\leqslant cN^{-q_i}$ for $r(2-\nu)>q_i>1-\nu$ and for $q_i\leqslant 1-\nu$, $r\geqslant 1$. Thus, due to (4.13),

$$\left|\int_0^b\varphi(s)(v-\mathscr{P}_Nv)(s)\,\mathrm{d}s\right|\leqslant\sum_{j=1}^{2N}|E_j|\leqslant cN^{-q_i}\quad\text{if } r>\frac{q_i}{2-\nu},\quad r\geqslant 1,\tag{4.20}$$

where $i\in\{0,\ldots,n-2\}$ and c is a constant not depending on $t\in(0,b)$. Actually, (4.20) holds for $i=n-1$ also.

Indeed, let $i=n-1$. Then $q_i=m+1$, $p=\min\{m,n-i\}=1$ and φ is a linear polynomial on every subinterval $s\in(t_{j-1},t_j)$, $j=1,\ldots,2N$. Therefore,

$$\psi^{(m+1)}(s)=v^{(m+1)}(s)\varphi(s)+(m+1)v^{(m)}(s)\varphi'(s),\quad s\in(t_{j-1},t_j),\quad j=1,\ldots,2N.$$

If $t\in(t_{l-1},t_l]$, then it follows from (4.8) and (4.11) that

$$|\psi^{(m+1)}(s)|\leqslant c[|v^{(m+1)}(s)|+h_l^{-1}|v^{(m)}(s)|],\quad s\in(t_{l-1},t_l),$$

$$|\psi^{(m+1)}(s)|\leqslant c[|v^{(m+1)}(s)|+|v^{(m)}(s)|],\quad s\in(t_{j-1},t_j),\quad j\neq l,\quad j=1,\ldots,2N.$$

Thus, if $j\neq l$, $j=1,\ldots,N$, then estimates (4.15) and (4.17) are valid with $i=n-1$ and $q_{n-1}=m+1$. If $j=l$, $2\leqslant l\leqslant N$ and $m+1>2-\nu$, then on the basis of (4.16) and (1.3) we obtain the estimate

$$|E_l|\leqslant c(h_l^{m+2}t_l^{-\nu-m}+h_l^{m+1}t_l^{1-\nu-m}).$$

In a similar way as above we can show that the last estimate is valid for $l=1$, also. Hence, due to (4.19), we get for $r(2-v)>m+1>2-v$ and $1\leqslant l\leqslant N$, that

$$\sum_{j=1}^{N}|E_j|\leqslant c\left[\sum_{j=1}^{N}h_j^{m+2}t_j^{-v-m}+h_l^{m+1}t_l^{1-v-m}\right]$$
$$\leqslant c_1\left[N^{-r(2-v)}\sum_{j=1}^{N}j^{r(2-v)-m-2}+N^{-r(2-v)}l^{r(2-v)-m-1}\right]\leqslant c_2N^{-m-1}.$$

If $\{m+1=2-v, r>1\}$ or $\{m+1<2-v, r\geqslant 1\}$, then we get also that $\sum_{j=1}^{N}|E_j|\leqslant cN^{-m-1}$. Due to symmetry of the grid (2.1), we obtain the estimate $\sum_{j=N+1}^{2N}|E_j|\leqslant cN^{-m-1}$ provided that $r>(m+1)/(2-v)$, $r\geqslant 1$. These observations yield (4.20) for $i=n-1$ ($q_{n-1}=m+1$).

This completes the proof of Lemma 4.2 since (4.5) is a consequence of (4.6), (4.12) and (4.20). □

Now it is easy to prove our main result about the convergence of the discrete Galerkin method.

Theorem 4.1. *Let the conditions of Lemmas* 1.1 *and* 2.1 *be fulfilled and let nodes* (2.5) *with grid points* (2.1) *be used. Assume in addition that* $f, a_i\in C^{q,v}(0,b)$, $K_i\in W^{q,v}(\Delta)$, $i=0,\dots,n_0$, *and that the quadrature rule* (2.2) *is exact for all polynomials of degree* $q-1$, *where* $q=m+\min\{m,n\}$ *and* $v<1$.

Then the statements of Theorem 3.1 *are valid and for* $N\geqslant N_0$ *the following error estimate holds:*

$$\max_{0\leqslant j\leqslant i}\|u^{(j)}-J_jv_N)\|_\infty\leqslant cN^{-q_i}\quad \text{if } r>\max\left\{\frac{m}{1-v},\frac{q}{2-v}\right\},\quad r\geqslant 1. \tag{4.21}$$

Here u is the solution of (1.1), (1.2), v_N *is the solution of* (3.9), J_j *is defined by* (3.4), $i=n_0,\dots,n-1$, $q_i=m+\min\{m,n-i\}$ *and c is a positive constant which is independent of N.*

Proof. From (3.17) and (3.6) we obtain for $j=0,\dots,n-1$ that

$$\|u^{(j)}-J_jv_N)\|_\infty\leqslant\|J_j(v-\mathscr{P}_Nv)\|_\infty+c\sum_{\mu=0}^{n_0}\|J_\mu(v-\mathscr{P}_Nv)\|_\infty,\qquad N\geqslant N_0.$$

Due to Lemma 1.1, $v\in C^{q,v}(0,b)$. Since $q_i\leqslant q$, $i=0,\dots,n-1$, estimate (4.21) follows from Lemma 4.2. □

5. Numerical experiments

Let us consider the following boundary value problem:

$$u''(t)=\int_0^1|t-s|^{-1/2}u(s)\,\mathrm{d}s+f(t),\qquad t\in[0,1], \tag{5.1}$$

$$u(0)=u(1)=0. \tag{5.2}$$

The forcing function f is selected so that

$$u(t)=t^{5/2}+(1-t)^{5/2}-1$$

is the exact solution. Actually, this is a problem of form (1.1), (1.2), where $n=2$, $n_0=0$, $b=1$, $a_0=0$, $K_0(t,s)=|t-s|^{-1/2}$ and $f(t)=-g(t)-g(1-t)$ with

$$g(t)=\frac{5}{16}\pi t^3+\frac{1}{24}(8+10t+15t^2)\sqrt{1-t}+\frac{5}{16}t^3\,\ln\frac{1}{t}(2-t+2\sqrt{1-t})-\frac{23}{4}\sqrt{t}.$$

It is easy to check that $K_0\in W^{m,v}(\Delta)$ and $f\in C^{m,v}(0,1)$ with $v=\frac{1}{2}$ and arbitrary $m\in\mathbb{N}$.

Table 1
Results in the case $\eta_1 = 0, \eta_2 = 1, w_1 = w_2 = \frac{1}{2}$

	$r = 1$		$r = 1.4$		$r = 2$		$r = 3$	
N	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$
4	1.9E − 3	4.37	1.7E − 3	4.43	2.5E − 3	3.89	4.4E − 3	3.43
8	4.5E − 4	4.26	4.1E − 4	4.17	6.2E − 4	3.95	1.2E − 3	3.78
16	1.1E − 4	4.19	1.0E − 4	4.07	1.6E − 4	3.99	3.0E − 4	3.93
32	2.6E − 5	4.14	2.5E − 5	4.02	3.9E − 5	4.00	7.5E − 5	3.98
64	6.4E − 6	4.10	6.3E − 6	4.01	9.8E − 6	4.00	1.9E − 5	4.00
	$r = 1$		$r = 1.4$		$r = 2$		$r = 3$	
N	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$
4	3.0E − 2	2.63	1.8E − 2	3.46	1.4E − 2	3.92	1.8E − 2	3.65
8	1.1E − 2	2.70	5.0E − 3	3.58	3.4E − 3	3.98	4.7E − 3	3.89
16	4.0E − 3	2.74	1.4E − 3	3.65	8.6E − 4	3.99	1.2E − 3	3.97
32	1.4E − 3	2.77	3.7E − 4	3.71	2.2E − 4	4.00	3.0E − 4	3.99
64	5.2E − 4	2.79	9.8E − 5	3.74	5.4E − 5	4.00	7.4E − 5	4.00
	$r = 1$		$r = 2$		$r = 4$		$r = 5$	
N	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$
4	0.33	1.40	0.161	1.97	9.2E−2	3.27	1.3E − 1	2.93
8	0.23	1.41	0.081	1.98	2.7E − 2	3.35	3.9E − 2	3.24
16	0.16	1.41	0.041	1.99	7.8E − 3	3.52	1.2E − 2	3.41
32	0.12	1.41	0.020	2.00	2.1E − 3	3.69	3.2E − 3	3.61
64	0.08	1.41	0.010	2.00	5.5E − 4	3.82	8.5E − 4	3.77

Problem (5.1), (5.2) is solved numerically by discrete Galerkin method (3.8), (3.10) in the case $m = 2$. An approximation $v_N \in S_2^{(0)}(\Pi_N)$ to $v = u''$ is presented in the form $v_N = \sum_{j=0}^{2N} c_j \varphi_j$ where $\varphi_0, \ldots, \varphi_{2N}$ are the linear basic splines on the grid Π_N with nodes (2.1) by $b=1$: $\varphi_j \in S_2^{(0)}(\Pi_N)$, $\varphi_j(t_j) = 1$ and $\varphi_j(t_i) = 0$ if $i \neq j$, $i, j = 0, \ldots, 2N$. We take in (3.8) $\varphi = \varphi_i$, $i = 0, \ldots, 2N$, and so we get for the coefficients c_j, $j = 0 \ldots, 2N$, a system of $2N + 1$ linear algebraic equations. Having determined the approximation $u_N^{(2)} = v_N$ to u'', the approximations $u_N^{(0)} = J_0 v_N$ to u and $u_N^{(1)} = J_1 v_N$ to u' are found by formula (3.4) where $b = 1$ and

$$G(t, s) = \begin{cases} t(s-1) & \text{for } t < s, \\ (t-1)s & \text{for } t > s. \end{cases}$$

Since v_N is a linear spline, $u_N^{(0)} \in S_4^{(2)}(\Pi_N)$ is a cubic spline and $u_N^{(1)} \in S_3^{(1)}(\Pi_N)$ is a quadratic spline.

In Tables 1 and 2 some results for different values of the parameters N and r are presented. The quantities $\varepsilon_N^{(i)}$ $(i = 0, 1, 2)$ are the approximate values of the norms $\|u^{(i)} - u_N^{(i)}\|_\infty$ $(i = 0, 1, 2)$, calculated as follows:

$$\varepsilon_N^{(i)} = \max_{j=1,\ldots,2N} \max_{k=0,\ldots,10} |u^{(i)}(\tau_{jk}) - u_N^{(i)}(\tau_{jk})|,$$

where $i = 0, 1, 2$ and

$$\tau_{jk} = t_{j-1} + \frac{k}{10}(t_j - t_{j-1}), \quad k = 0, \ldots, 10; \quad j = 1, \ldots, 2N,$$

with the grid points $\{t_j\}$, defined by the formula (2.1) for $b = 1$. In Tables the ratios

$$\varrho_N^{(i)} = \frac{\varepsilon_{N/2}^{(i)}}{\varepsilon_N^{(i)}}, \quad i = 0, 1, 2,$$

characterizing the actual convergence rate, are also presented.

Table 2
Results in the case $\eta_1 = (3-\sqrt{3})/6$, $\eta_2 = 1-\eta_1$, $w_1 = w_2 = \frac{1}{2}$

	$r=1$		$r=1.4$		$r=2$		$r=3$	
N	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$
4	6.2E − 5	5.50	1.6E − 5	10.8	3.7E − 5	10.8	1.2E − 4	7.0
8	1.1E − 5	5.59	1.3E − 6	12.2	2.9E − 6	12.9	1.1E − 5	10.6
16	2.0E − 6	5.62	1.1E − 7	11.8	2.0E − 7	14.5	8.6E − 7	13.1
32	3.5E − 7	5.64	1.0E − 8	11.1	1.3E − 8	15.3	5.9E − 8	14.6
64	6.2E − 8	5.68	9.0E − 10	11.2	8.3E − 10	15.7	3.8E − 9	15.7
	$r=1$		$r=1.4$		$r=2$		$r=3$	
N	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$
4	1.6E − 3	2.81	6.1E − 4	4.20	5.4E − 4	6.93	1.4E − 3	4.88
8	5.6E − 4	2.82	1.4E − 4	4.25	7.4E − 5	7.24	2.2E − 4	6.42
16	2.0E − 4	2.82	3.4E − 5	4.27	9.8E − 6	7.58	3.1E − 5	7.15
32	7.0E − 5	2.82	7.9E − 6	4.28	1.3E − 6	7.79	4.1E − 6	7.57
64	2.5E − 5	2.83	1.8E − 6	4.28	1.6E − 7	7.90	5.3E − 7	7.79
	$r=1$		$r=2$		$r=4$		$r=5$	
N	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$
4	0.41	1.42	0.193	1.98	7.2E − 2	4.46	1.1E − 1	4.12
8	0.29	1.42	0.097	2.00	2.1E − 2	3.39	3.1E − 2	3.47
16	0.21	1.42	0.048	2.00	6.1E − 3	3.50	9.1E − 3	3.35
32	0.15	1.41	0.024	2.00	1.6E − 3	3.75	2.5E − 3	3.67
64	0.10	1.41	0.012	2.00	4.2E − 4	3.88	6.5E − 4	3.84

Table 1 shows the dependence of the convergence rate on the grid parameter r, when the trapezoidal rule (2.2) with $M=m=2$, $\eta_1=0$, $\eta_2=1$, $w_1=w_2=\frac{1}{2}$ is used. In this case the discrete Galerkin method coincides with the collocation method. From Theorem 3.1 it follows that for sufficiently large N

$$\varepsilon_N^{(2)} \approx \|u''-v_N\|_\infty \leqslant c \begin{cases} N^{-r/2} & \text{if } 1\leqslant r<4,\\ N^{-2} & \text{if } r\geqslant 4, \end{cases} \tag{5.3}$$

$$\max_{i=0,1} \varepsilon_N^{(i)} \approx \max_{i=0,1} \|u^{(i)}-u_N^{(i)}\|_\infty \leqslant c \begin{cases} N^{-3r/2} & \text{if } 1\leqslant r<4/3,\\ N^{-2} & \text{if } r>4/3. \end{cases} \tag{5.4}$$

Due to (5.3), the ratio $\varrho_N^{(2)}$ ought to be approximately $2^{r/2}=(N/2)^{-r/2}/N^{-r/2}$ for $1\leqslant r<4$ and 4 for $r\geqslant 4$, i.e., $\varrho_N^{(2)}\approx 1.41$ for $r=1$, $\varrho_N^{(2)}\approx 2$ for $r=2$ and $\varrho_N^{(2)}\approx 4$ for $r\geqslant 4$ is expected. Due to (5.4), $\max_{i=0,1}\varrho_N^{(i)}$ ougth to be approximately $2^{3/2}\approx 2.83$ for $r=1$ and 4 for $r\geqslant 1.4$. As we can see in Table 1, the observed errors $\varepsilon_N^{(2)}$ and $\max_{i=0,1}\varepsilon_N^{(i)}$ are in good accordance with the theoretical estimates (5.3) and (5.4), respectively. Actually, if $r=1$, then the decrease of $\varepsilon_N^{(0)}$ is faster than it is predicted by the right-hand side of estimate (5.4). However, there is no any improvement in the convergence rate if r is increasing and in all cases the maximal convergence rate is of order $O(N^{-2})$ as it is prescribed by (5.3) and (5.4).

The results in Table 2 correspond to the case when the Gaussian quadrature formula (2.2) with $M=m=2$, $\eta_1=(3-\sqrt{3})/6$, $\eta_2=1-\eta_1=(3+\sqrt{3})/6$, $w_1=w_2=\frac{1}{2}$ is used. As we can see from Table 2, the observed errors $\varepsilon_N^{(2)}\approx\|u''-v_N\|_\infty$ are in good agreement with the theoretical estimate (5.3). They are comparable to those in Table 1 and the use of quadrature rules of higher precision does not give any improvement in the convergence rate of v_N to u''. However, the convergence rate of J_1v_N and especially of J_0v_N is much better in this case (compared with Table 1). Since the Gaussian quadrature formula with two Gaussian points η_1 and η_2 is exact for polynomials up to the degree $2M-1=3$, it follows from Theorem 4.1 that for $r>4$ and for sufficiently large N

$$\varepsilon_N^{(0)} \approx \|u - J_0 v_N\|_\infty \leqslant cN^{-4}, \quad \varepsilon_N^{(1)} \approx \|u' - J_1 v_N\|_\infty \leqslant cN^{-3}. \tag{5.5}$$

Thus, for $r>4$ the ratios $\varrho_N^{(0)}$ and $\varrho_N^{(1)}$ ought to be approximately $2^4=16$ and $2^3=8$, respectively. From Table 2 we see that such convergence rates occur already by $r=2$. By larger values of r the convergence rate is the same: $\varepsilon_N^{(0)} = \mathrm{O}(N^{-4})$ and $\varepsilon_N^{(1)} = \mathrm{O}(N^{-3})$, as prescribed by (5.5). However, the corresponding convergence rate is achieved for smaller value of the parameter r than predicted by Theorem 4.1. The question regarding the optimal value of r in estimate (4.21) will be discussed elsewhere.

We finish with the remark that nearly the same convergence rates as in Table 2 will take place when instead of the two point Gaussian quadrature the three point Gaussian quadrature or the Simpson's rule is used.

References

[1] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge Monographs on Applied and Computational Mathematics, vol. 4, Cambridge University Press, Cambridge, UK, 1997.
[2] K. Atkinson, A. Bogomolny, The discrete Galerkin method for integral equations, Math. Comp. 48 (1987) 596–616.
[3] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, Cambridge Monographs on Applied and Computational Mathematics, vol. 15, Cambridge University Press, Cambridge, UK, 2004.
[4] H. Brunner, A. Pedas, G. Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations, Math. Comp. 68 (1999) 1079–1095.
[5] H. Brunner, A. Pedas, G. Vainikko, Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels, SIAM J. Numer. Anal. 39 (2001) 957–982.
[6] Z. Chen, Y. Xu, J. Zhao, The discrete Petrov–Galerkin method for weakly singular integral equations, J. Integral Equations Appl. 11 (1999) 1–35.
[7] M. Ganesh, R.D. Grigorieff, K. Mustapha, Discrete Petrov–Galerkin Scheme for boundary value differential and integral problems: theory and applications, Math. Comput. Modelling 40 (2004) 1323–1334.
[8] M. Ganesh, I.H. Sloan, Optimal order spline methods for nonlinear differential and integro-differential equations, Appl. Numer. Math. 29 (1999) 445–478.
[9] R.D. Grigorieff, I.H. Sloan, High-order spline Petrov–Galerkin methods with quadrature, Z. Angew. Math. Mech. 76 (1) (1996) 15–18.
[10] R.D. Grigorieff, I.H. Sloan, Stability of discrete orthogonal projections for continuous splines, Bull. Austral. Math. Soc. 58 (1998) 307–332.
[11] I. Parts, A. Pedas, E. Tamme, Piecewise polynomial collocation for Fredholm integro-differential equations with weakly singular kernels, SIAM J. Numer. Anal. 43 (2005) 1897–1911.
[12] A. Pedas, E. Tamme, Spline collocation method for integro-differential equations with weakly singular kernels, J. Comput. Appl. Math. 197 (2006) 253–269.
[13] A. Pedas, G. Vainikko, Smooting transformation and piecewise polynomial collocation for weakly singular Volterra integral equations, Computing 73 (2004) 271–293.
[14] A. Pedas, G. Vainikko, Smooting transformation and piecewise polynomial projection methods for weakly singular Fredholm integral equations, Comm. Pure Appl. Anal. 5 (2006) 395–413.
[15] G. Vainikko, Multidimensional Weakly Singular Integral Equations, Lecture Notes in Mathematics, vol. 1549, Springer, Berlin, 1993.